Advances in Pure Mathematics, 2022, 12, 600-613
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Structural Stability in 4-Dimensional Canards

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How to cite this paper: Kanagawa, S. and Tchizawa, K. (2022) Structural Stability in 4-Dimensional Canards. *Advances in Pure Mathematics*, **12**, 600-613.
https://doi.org/10.4236/apm.2022.1211046

Received: September 29, 2022
Accepted: November 1, 2022
Published: November 4, 2022

Abstract

Let us consider higher dimensional canards in a sow-fast system R^{2+2} with a bifurcation parameter. Then, the slow manifold sometimes shows various aspects due to the bifurcation. Introducing a key notion "symmetry" to the slow-fast system, it becomes clear when the pseudo singular point obtains the structural stability or not. It should be treated with a general case. Then, it will also be given about the sufficient conditions for the existence of the center manifold under being "symmetry". The higher dimensional canards in the sow-fast system are deeply related to Hilbert's 16th problem. Furthermore, computer simulations are done for the systems having Brownian motions. As a result, the rigidity for the system is confirmed.

Keywords

Canard Solution, Slow-Fast System, Nonstandard Analysis, Hilbert's 16th Problem, Brownian Motion, Stochastic Differential Equation

1. Introduction

In 4-dimensional canards, there are three formulas, that is, the slow-fast system in R^{2+2}, R^{1+3} and in R^{3+1}. Especially, the system in R^{2+2} is a standard formula. They have two kinds of vector field, *i.e.*, slow and fast one. Although precise reasons have already been described in [1], the rank condition on the linearized system of the slow and the fast equations is only applied to prove the existence of the canards. In other cases, it is complicated much more. A concrete system was first analyzed in 2002 [2], however, it is done by applying the indirect method.

In this paper, we take up the generalized system including a bifurcation parameter. When having the bifurcation, it is very complicated to analyze the system as it is. Setting up the system as the parameter depends on only slow vectors, it becomes easy to get a simple geometrical point of view.

Then, it is analyzed by using the direct method under a key notion of "symmetry". Sue Ann Campbell once pointed it out in the concrete system. It is now generalized and playing an important role of catching up the bifurcation structure, in Section 2. When and why the pseudo singular point is structurally stable or not? It becomes clear that if "pitch-fork bifurcation" causes on the invariant manifold, it is unstable, and if the pseudo singular point is on the orthogonal complement, there is no bifurcation, that is, it is structurally stable. In Section 3, the reason why it happens is described. Regarding near the singular point, it is also very effective to prove the existence of "center manifold". See [3] and [4]. Tracing a canard orbit along the vector field, the slow manifold should be connected with the center manifold. Furthermore, in reality it should be confirmed to keep the rigidity for a certain concrete system with random noise. Therefore, in Section 4, some computer simulations are presented for such systems with Brownian motions. As the concrete system is originally based on the coupled neuron systems, it is very important to know the rigidity. It is done by using a non-standard analysis developed in [5].

The higher dimensional canards in the sow-fast system are deeply related to Hilbert's 16th problem. The aspect of limit cycles including canard solutions links with polynomial systems. The reason why it links to is the following. When constructing canard solutions in order to get an exact solution, we take up a local model by bowing up. Then, the system is described by the polynomials as an approximation. See e.g. [6] [7] [8] [9] [10].

2. Slow-Fast System with Bifurcation Parameter

Consider the following system:

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}t} = h(x, y, \varepsilon) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, b) \end{cases} \tag{1}$$

where ε is infinitesimal, b is any constant and

$$x = (x_1, x_2) \in R^2,\ \ y = (y_1, y_2) \in R^2,$$

$$h = (h_1, h_2): R^4 \to R^2,\ \ g = (g_1, g_2): R^4 \to R^2.$$

Assume that $g(x, y, b) = g(x, by)$, for the simplicity, and the origin is a singular point.

Furthermore we assume that the system (1) satisfies the following conditions (A1)-(A6):

(A1) h is of class $\mathbf{C}^1$ and g is of class $\mathbf{C}^2$.

(A2) The slow manifold $S = \{(x, y) \in \mathbf{R}^4 \mid h(x, y, 0) = 0\}$ is a two-dimensional differential manifold and intersects the set

$$T = \left\{(x, y) \in \mathbf{R}^4 \mid \det\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 0\right\} \tag{2}$$

transversely, where

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} \tag{3}$$

Then, the pli set

$$PL = \{(x, y) \in S \cap T\} \tag{4}$$

is a one-dimensional differentiable manifold.

(A3) Either the value of g_1 or that of g_2 is nonzero at any point of *PL*.

Note that the pli set *PL* divides the slow manifolds *S**PL* into three parts depending on the signs of the two eigenvalues of $\frac{\partial h}{\partial x}(x, y, 0)$.

First consider the following reduced system which is obtained from (1) with $\varepsilon = 0$:

$$\begin{cases} 0 = h(x, y, 0) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, b) \end{cases} \tag{5}$$

By differentiating $h(x, y, 0)$ with respect to *t*, we have

$$\frac{\partial h}{\partial x}(x, y, 0)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial h}{\partial y}(x, y, 0)\, g(x, y, b) = 0 \tag{6}$$

Then (4) becomes the following:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -\left[\frac{\partial h}{\partial x}(x, y, 0)\right]^{-1} \frac{\partial h}{\partial y}(x, y, 0)\, g(x, y, b) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, b) \end{cases} \tag{7}$$

where $(x, y) \in S \setminus PL$. To avoid degeneracy in (6), we consider the time-scaled-reduced system:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \left\{-\det\left[\frac{\partial h}{\partial x}(x, y, 0)\right]^{-1}\right\}\left[\frac{\partial h}{\partial x}(x, y, 0)\right]^{-1} \frac{\partial h}{\partial y}(x, y, 0)\, g(x, y, b) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \left\{\det\left[\frac{\partial h}{\partial x}(x, y, 0)\right]^{-1}\right\} g(x, y, b) \end{cases} \tag{8}$$

The phase portrait of the system (8) is the same as that of (7) except the region where $\det\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 0$, but only the orientation of the orbit is different. The following definition is described in [1].

Definition 1. *A singular point of* (8), *which is on PL, is called a pseudo singular point of* (1). *The set of pseudo singular points is denoted by PS.*

(A4) $\operatorname{rank}\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 2$, $\operatorname{rank}\left[\frac{\partial h}{\partial y}(x, y, 0)\right] = 2$ for any $(x, y) \in S \setminus PL$.

From (A4), the implicit function theorem guarantees the existence of a unique function $y=\varphi(x)$ such that $h(x,\varphi(x),0)=0$. By using $y=\varphi(x)$, we obtain the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\left\{-\det\left[\frac{\partial h}{\partial x}(x,\varphi(x),0)\right]^{-1}\right\}\left[\frac{\partial h}{\partial x}(x,\varphi(x),0)\right]^{-1}\frac{\partial h}{\partial y}(x,\varphi(x),0)\,g(x,\varphi(x),b). \quad (9)$$

(A5) All singular points of (8) are non-degenerate, that is, the linearization of (8) at a singular point has two nonzero eigenvalues.

Now, let us introduce a definition of "symmetry". For example, see [2]. It is a key word through this paper.

Definition 2. *If* $h_1(x_1,x_2,y_1,y_2,\varepsilon)=h_2(x_2,x_1,y_2,y_1,\varepsilon)$, *and* $g_1(x_1,x_2,y_1,y_2,b)=g_2(x_2,x_1,y_2,y_1,b)$, *then the system is "symmetric" for the subspace* $I=\{(x_1,x_2,y_1,y_2)\mid x_1=x_2, y_1=y_2\}$.

(A6) I intersects PL transversely.

The following definition is also described in [1].

Definition 3. *Let* λ_1,λ_2 *be two eigenvalues of the linearization of* (8) *at a pseudo singular point. The pseudo singular point with real eigenvalues is called a pseudo singular saddle point if* $\lambda_1<0<\lambda_2$ *and a pseudo singular node point if* $\lambda_1<\lambda_2<0$ *or* $\lambda_1>\lambda_2>0$.

The following theorem is established (see, e.g. [1]).

Theorem 1. *Let* (x_0,y_0) *be a pseudo singular saddle or node point. If* $\operatorname{trace}\left[\frac{\partial h}{\partial x}(x_0,y_0,0)\right]<0$, *then there exists a solution which first follows the attractive part and the repulsive part after crossing PL near the pseudo singular point.*

Remark 1. *Using Theorem* 1, *if the condition that the psuedo singular point is only saddle or node, then there exist canards. Therefore, the following theorems are given under this condition.*

Remark 2. *The condition* $\operatorname{trace}\left[\frac{\partial h}{\partial x}(x_0,y_0,0)\right]<0$ *implies that one of eigenvalues of* $\left[\frac{\partial h}{\partial x}(x_0,y_0,0)\right]$ *is equal to zero and the other one is negative. Notice that the system has two kinds of vector fields: one is* 2*-dimensional slow and the other is* 2*-dimensional fast one. The condition provides the state of the fast vector field.*

Remark 3. *The singular solution in Theorem* 1 *is called a canard in* $\mathbf{R}^4$ *with* 2*-dimensional slow manifold. As a result, it causes a delayed jumping. The study of canards requires still more precise topological analysis on the slow vector field.*

Remark 4. *On the subspace I, the following system is established for some b. I is an invariant manifold.*

$$\begin{cases}\varepsilon\dfrac{\mathrm{d}x_1}{\mathrm{d}t}=h_1(x_1,y_1,\varepsilon)\\[2mm]\dfrac{\mathrm{d}y_1}{\mathrm{d}t}=g_1(x_1,y_1,b)\end{cases}\quad (10)$$

Remark 5. *On the set PL,* $\det\left[\frac{\partial h}{\partial x}\right]=0$ *is satisfied and at* $(x_0,y_0)\in PS$ *the following equation is established*:

$$\left\{\frac{\partial h_1}{\partial x_1}(x_0,\varphi(x_0),0)\frac{\partial h_2}{\partial x_2}(x_0,\varphi(x_0),0)-\frac{\partial h_1}{\partial x_2}(x_0,\varphi(x_0),0)\frac{\partial h_2}{\partial x_1}(x_0,\varphi(x_0),0)\right\}g_1(x_0,\varphi(x_0),b)=0. \quad (11)$$

Note that there exists $y=\phi(x)$ *because of assuming* $\operatorname{rank}\left[\frac{\partial h}{\partial y}\right]=2$.

3. Structural Stability

When and why the pseudo singular point has structural stability? A geometrical point of view to make it clear is shown in this section.

Lemma 1. *The matrix* $\left[\frac{\partial h}{\partial x}\right]$ *is symmetric.*

Proof. Because the system is symmetric for the set *I*, it is obvious from elementary calculus. □

From (A6), the subspace I intersects PL transversely. Lemma 1 ensures that I^c also intersects PL transversely, where I^c is the orthogonal complement of *I*. Since the matrix $\left[\frac{\partial h}{\partial y}\right]$ is also symmetric, for the sake of simplicity, suppose that $\left[\frac{\partial h}{\partial y}\right]$ is identity without loss of generality.

Lemma 2. *Let* $(x_0,y_0)\in PS$ *be on* $I\cap PL$, *then it depends on the parameter b. On the other hand, on* $I^c\cap PL$, *it is independent of the parameter.*

Proof. Since $(x_0,y_0)\in I\cap PL$, there exists a critical value $b=b_0$, which depends on the shape of $\phi(x)$ satisfying

$$\begin{cases} h_1(x_1,y_1,0)=0 \\ \frac{\mathrm{d}y_1}{\mathrm{d}t}=g_1(x_1,b_0y_1) \end{cases} \quad (12)$$

that is,

$$\frac{\partial h_1}{\partial y_1}(x_0,\phi(x_0),0)g_1(x_0,b_0\phi(x_0),0)=0. \quad (13)$$

On $(x_0,y_0)\in I^c\cap PL$, for any *b*, satisfying

$$\frac{\partial h_2}{\partial x_2}(x_0,\phi(x_0),0)g_1(x_0,\phi(x_0),b)-\frac{\partial h_1}{\partial x_2}(x_0,\phi(x_0),0)g_2(x_0,\phi(x_0),b)=0 \quad (14)$$

and

$$-\frac{\partial h_2}{\partial x_1}(x_0,\varphi(x_0),0)g_1(x_0,\varphi(x_0),b)+\frac{\partial h_1}{\partial x_1}(x_0,\varphi(x_0),0)g_2(x_0,\varphi(x_0),b)=0. \quad (15)$$

On the set *PL*, from Remark 5, the above equations are established for any *b*. □

Theorem 2. *Let* $(x_0,y_0)\in PS$ *be a saddle or node point. Then, if* $(x_0,y_0)\in I^c\cap PL$, *the pseudo singular point is structurally stable. If* $(x_0,y_0)\in I\cap PL$ *it is structurally unstable.*

Proof. If $(x_0, y_0) \in I^c \cap PL$, the pseudo singular point does not depend on the parameter *b*, from Lemma 2. If $(x_0, y_0) \in I \cap PL$, it depends on the parameter. □

Lemma 3. *There exists a pseudo singular point* $(x^0, y^0) \in PS$, *which is one of a coupled points near the subspace I.*

Proof. There exists $b = b_1 \approx b_0$ satisfying

$$\frac{\partial h_1}{\partial y_1}(x^0, y^0, 0) g_1(x^0, y^0, b_1) \approx 0, \tag{16}$$

where $(x^0, y^0) \notin I$ and $y^0 = \phi(x^0)$. As the system is symmetry, there exists another coupled pseudo singular point (x^1, y^1) satisfying

$$\frac{\partial h_1}{\partial y_1}(x^1, y^1, 0) g_1(x^1, y^1, b_1) \approx 0, \tag{17}$$

where $y^1 = \phi(x^1)$. □

Theorem 3. *Canards near the subspace I has a center manifold, if* $\det\left[\frac{\partial f}{\partial x}\right]_{x=0} = 0$ *and* $\operatorname{trace}\left[\frac{\partial f}{\partial x}\right]_{x=0} < 0$, *where*

$$f(x) = \left[\frac{\partial h}{\partial x}(x, \varphi(x), 0)\right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x), 0) g(x, \varphi(x), b). \tag{18}$$

Proof. One of eigenvalues is zero, corresponding eigenvector exists on the set *I*, and the other one is negative, corresponding eigenvector is on the set I^c. Since the matrix $\left[\frac{\partial f}{\partial x}\right]_{x=0}$ is symmetric, it is easily confirmed. Then, there are two possibilities. A canard orbit passing through between the pseudo singular point (x^0, y^0) and (x^1, y^1) is connected with the center manifold. The second case is that the orbit is connected to a limit cycle satisfying (10). □

4. Concrete Example

4.1. Modified Coupled FitzHugh-Nagumo Equations

Consider the following typical example of modified coupled FitzHugh-Nagumo equations. See [2] for more details.

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 + y_1 - \dfrac{x_1^3}{3} \\ \varepsilon \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 + y_2 - \dfrac{x_2^3}{3} \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = -\dfrac{1}{c}(x_1 + by_1) \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = -\dfrac{1}{c}(x_2 + by_2) \end{cases} \tag{19}$$

The next equation is the time-scaled-reduced system corresponding to (19).

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{pmatrix} -x_2^2 & -1 \\ -1 & -x_1^2 \end{pmatrix} \begin{pmatrix} -\dfrac{1}{c}(x_1 + by_1) \\ -\dfrac{1}{c}(x_2 + by_2) \end{pmatrix} \Delta = f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{20}$$

There exist pseudo singular points $(x_0, y_0) \in PS$ of the system (19) satisfying $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ in (20) which are obtained by the following. If $(x_0, y_0) = (x_{01}, x_{02}, y_{01}, y_{02})$ exists on neighborhood of $I \cap PL$, then $x_1 x_2 = 1$ holds. Therefore,

$$\begin{cases} x_{01} = \sqrt{\dfrac{3 \pm \sqrt{9-4b^2}}{2b}} \\ x_{02} = \sqrt{\dfrac{3 \mp \sqrt{9-4b^2}}{2b}} \\ y_{01} = \dfrac{x_{01}^3}{3} - x_{02} \\ y_{02} = \dfrac{x_{02}^3}{3} - x_{01} \end{cases} \tag{21}$$

and

$$\begin{cases} x_{01} = -\sqrt{\dfrac{3 - \sqrt{9 \pm 4b^2}}{2b}} \\ x_{02} = -\sqrt{\dfrac{3 + \sqrt{9 \mp 4b^2}}{2b}} \\ y_{01} = \dfrac{x_{01}^3}{3} - x_{02} \\ y_{02} = \dfrac{x_{02}^3}{3} - x_{01} \end{cases} \tag{22}$$

Remark 6. *Notice that in Lemma* 2, *if* $(x_0, y_0) \in I \cap PL$, *then the critical value* $b = b_0 = \frac{3}{2}$ *holds. When* $0 < b < \frac{3}{2} = b_0$, *the solutions of* (21) *and* (22) *are on neighborhood of I but not on I, respectively, like as being described in Lemma* 3.

If $(x_0, y_0) \in I^c \cap PL$, then $x_1 x_2 = -1$ holds. Therefore,

$$\begin{cases} x_{01} = \pm 1 \\ x_{02} = \mp 1 \\ y_{01} = \dfrac{x_{01}^3}{3} - x_{02} \\ y_{02} = \dfrac{x_{02}^3}{3} - x_{01} \end{cases} \tag{23}$$

Remark 7. *The solution of* (23) *is structurally stable. Then*

$$\left[\frac{\partial h}{\partial x}\right]_{(x_1, x_2) = (1,-1)} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{24}$$

and

$$\left[\frac{\partial f}{\partial x}\right]_{(x_1, x_2) = (1,-1)} = -\frac{1}{c}\begin{pmatrix} 1 & -1-\dfrac{8}{3}b \\ -1-\dfrac{8}{3}b & 1 \end{pmatrix}. \tag{25}$$

On the set I^c *, the orbit traces an attractive region before the pseudo singular point* $(x_1,x_2)=(1,-1)$ *, which is saddle when* $0<b<3/2$ *. Therefore there exists a canard by Theorem* 1.

Here, the slow vector field at the origin $x=(0,0)$ is as follows.

$$\left[\frac{\partial f}{\partial x}\right]_{x=(0,0)}=\frac{1}{c}\begin{pmatrix}-b & 1\\ 1 & -b\end{pmatrix}. \tag{26}$$

The characteristic equation of (24) is

$$\begin{vmatrix}\lambda+b & -1\\ -1 & \lambda+b\end{vmatrix}=\lambda^2+2\lambda b+b^2-1=0. \tag{27}$$

Therefore the eigenvalues are

$$\begin{cases}\lambda_1=-b+1\\ \lambda_2=-b-1\end{cases} \tag{28}$$

From (26) and Remark 6 we have the following results.

If $1<b<\frac{3}{2}$ then $\lambda_1<0$ and $\lambda_2<0$, the origin is node, that is stable.

If $0<b<1$ then $\lambda_1<0$ and $\lambda_2>0$, the origin is saddle, that is unstable.

If $b=1$ then $\lambda_1=0$ and $\lambda_2=-2$, the origin is center.

The fast vector field at the origin is as follows.

$$\left[\frac{\partial h}{\partial x}\right]_{x=(0,0)}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{29}$$

The characteristic equation is

$$\lambda^2-1=0. \tag{30}$$

If $\lambda=1$, the corresponding eigenvector (x_1,x_2) holds

$$x_2=x_1, \tag{31}$$

and if $\lambda=-1$,

$$x_2=-x_1. \tag{32}$$

4.2. Modified Coupled FitzHugh-Nagumo Equations with Brownian Motions

Now, let us consider a stochastic differential equation for a slow-fast system with Brownian motions $B_1(t)$ and $B_2(t)$ as the random noises modifying the slow-fast system (19): For $t\in[0,T], T>0$

$$\begin{cases}\varepsilon\dfrac{\mathrm{d}x_1}{\mathrm{d}t}=x_2+y_1-\dfrac{x_1^3}{3}\\ \varepsilon\dfrac{\mathrm{d}x_2}{\mathrm{d}t}=x_1+y_2-\dfrac{x_2^3}{3}\\ \mathrm{d}y_1=-\dfrac{1}{c}(x_1+by_1)\mathrm{d}t+\sigma_1\mathrm{d}B_1(t)\\ \mathrm{d}y_2=-\dfrac{1}{c}(x_2+by_2)\mathrm{d}t+\sigma_2\mathrm{d}B_2(t)\end{cases} \tag{33}$$

On the other hand, Anderson [11] showed that the Brownian motion is described by step functions using non-standard analysis on a hyper finite time line by the following definition. (See also [12]).

Definition 4. *Let* $N_t = \frac{t}{\Delta t}, 0 \le t \le T$ *and* $N = N_T$. *Assume that a sequence of i.i.d. random variables* $\{\Delta B_k, k = 1, \cdots, N\}$ *has the distribution*

$$P\left\{\Delta B_k = \sqrt{\Delta t}\right\} = P\left\{\Delta B_k = -\sqrt{\Delta t}\right\} = \frac{1}{2}$$

for each $k = 1, \cdots, N$. *An extended Wiener process* $\{B(t), t \ge 0\}$ *is defined by*

$$B(t) = \sum_{k=1}^{N_t} \Delta B_k, \quad 0 \le t \le T.$$

Rewriting the system (31) *via step functions on the hyper finite time line, the following system* (32) *is obtained.*

$$\begin{cases} \varepsilon\left\{x_1(t_k) - x_1(t_{k-1})\right\} = \left(x_2 + y_1 - \dfrac{x_1^3}{3}\right)\Delta t \\ \varepsilon\left\{x_2(t_k) - x_2(t_{k-1})\right\} = \left(x_1 + y_2 - \dfrac{x_2^3}{3}\right)\Delta t \\ y_1(t_k) - y_1(t_{k-1}) = -\dfrac{1}{c}(x_1 + by_1)\Delta t + \sigma_1 \Delta B_{1k} \\ y_2(t_k) - y_2(t_{k-1}) = -\dfrac{1}{c}(x_2 + by_2)\Delta t + \sigma_2 \Delta B_{2k} \end{cases} \tag{34}$$

where

$$B_1(t) = \sum_{k=1}^{Nt} \Delta B_{1k}, \quad B_2(t) = \sum_{k=1}^{Nt} \Delta B_{2k}$$

and $\sigma_1 > 0$ *and* $\sigma_2 > 0$ *are positive constants which give standard deviations for the Brownian motions* $B_1(t)$ *and* $B_2(t)$, *respectively.*

4.3. Simulation Results

In this section, let us provide computer simulations for the modified coupled FitzHugh-Nagumo Equations (33). In (33), we assume that two Brownian motions $B_1(t)$ and $B_2(t)$ are mutually independent and note that

$$B_1(t_k) - B_1(t_{k-1}) \sim N\left(0, \Delta t \sigma_1^2\right), \quad B_2(t_k) - B_2(t_{k-1}) \sim N\left(0, \Delta t \sigma_2^2\right), \tag{35}$$

for each $1 \le k \le \frac{T}{\Delta t}$.

In **Figures 1-8**, the line $x_1 = x_2$ is an invariant manifold and two red points are psueod singular points. Furthermore, $\varepsilon = 0.01$, $c = 1$ and $\Delta t = 0.0001$ in (34). The curves, which satisfy $x_1 x_2 = 1$ and $x_1 x_2 = -1$, respectively, are pli set.

Figure 1 $\sigma_1 = \sigma_2 = 0$ (non-random)

Figure 1 shows an orbit of $\{(x_1(t), x_2(t)), 0 \le t \le T = 7\}$ satisfying the equation (34) with $\sigma_1 = \sigma_2 = 0$, $b = 1$ and starting from $(1.0, 1.5)$ near the pseudo singular point $\left(\sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)},\sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)}\right)$. The orbit converges to the invariant manifold $x_1 = x_2$.

Figure 2 $\sigma_1 = \sigma_2 = 0.1$ (random)

Figure 2 shows an orbit of $\left\{\left(x_1(t), x_2(t)\right), 0 \le t \le T = 7\right\}$ satisfying the equation (34) with $\sigma_1 = \sigma_2 = 0.1$, $b = 1$ and starting from $(1.0, 1.5)$ near the pseudo singular point $\left(\sqrt{\frac{1}{2}\left(3-\sqrt{5}\right)},\sqrt{\frac{1}{2}\left(3+\sqrt{5}\right)}\right)$. From **Figure 2** we observe that the orbit dose not converge to the invariant manifold $x_1 = x_2$, but it moves around a neighborhood of $x_1 = x_2$ by the effect of random noises.

Figure 3 $\sigma_1 = \sigma_2 = 0$

Figure 3 shows an orbit of $\left\{\left(x_1(t), x_2(t)\right), 0 \le t \le T = 7\right\}$ satisfying the equation (34) with $\sigma_1 = \sigma_2 = 0$, $b = 1.4$ and starting from $(0.8, 1.5)$ near the pseudo singular point $(0.828718, 1.20668)$. The orbit converges to the invariant manifold $x_1 = x_2$.

Figure 4 $\sigma_1 = \sigma_2 = 0.1$

Figure 4 shows an orbit of $\left\{\left(x_1(t), x_2(t)\right), 0 \le t \le T = 7\right\}$ satisfying the equation (34) with $\sigma_1 = \sigma_2 = 0.1$, $b = 1.4$ and starting from $(0.8, 1.5)$ near the pseudo singular point $(0.828718, 1.20668)$. From **Figure 4** we observe that the orbit dose not converge to the invariant manifold $x_1 = x_2$, but it moves around a neighborhood of $x_1 = x_2$ by random noises.

Figure 5 $\sigma_1 = \sigma_2 = 0$

Figure 5 shows an orbit of $\left\{\left(x_1(t), x_2(t)\right), 0 \le t \le T = 7\right\}$ satisfying the equation (34) with $\sigma_1 = \sigma_2 = 0$, $b = 1.48$ and starting from $(1.2, 0.8)$ near the pseudo singular point $(1.08557, 0.921173)$. The orbit converges to the invariant manifold $x_1 = x_2$, but it moves in small steps compared with **Figure 3** with $b = 1.4$.

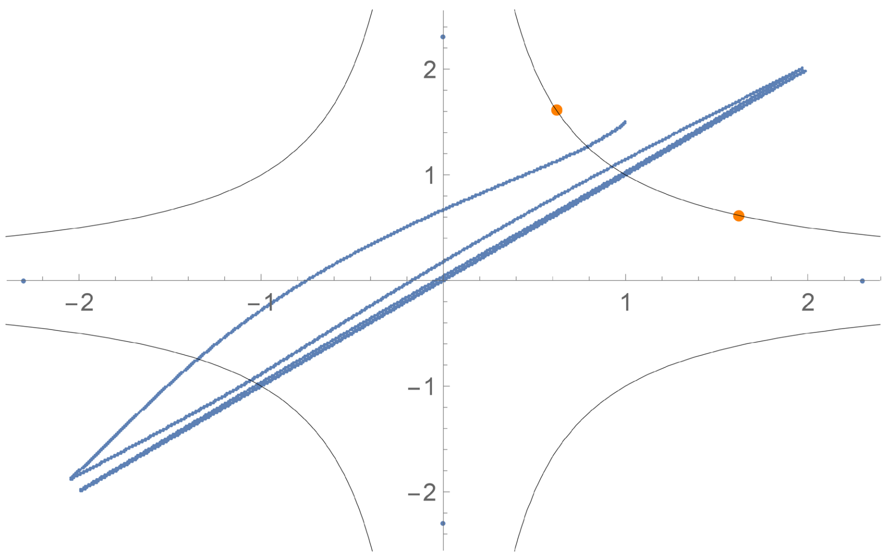

Figure 1. $b = 1$, $\left(x_1(0), x_2(0)\right) = (1.0, 1.5)$, $\sigma_1 = \sigma_2 = 0$.

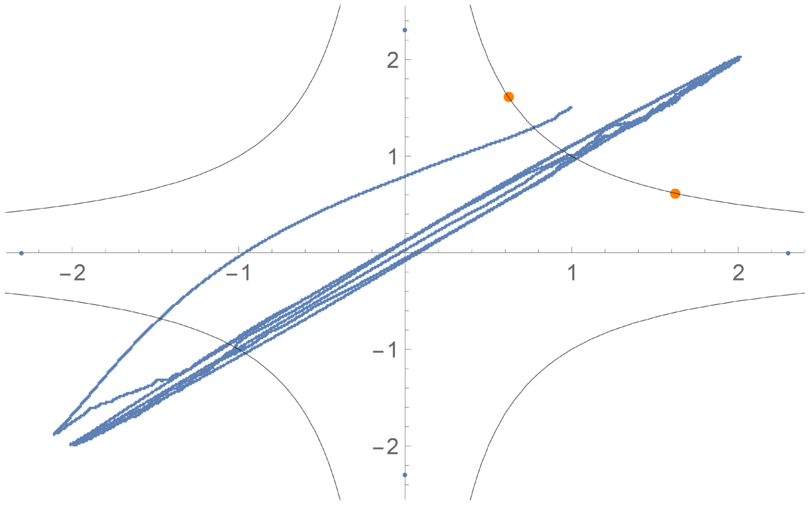

Figure 2. $b=1$, $(x_1(0), x_2(0))=(1.0, 1.5)$, $\sigma_1=\sigma_2=0.2$.

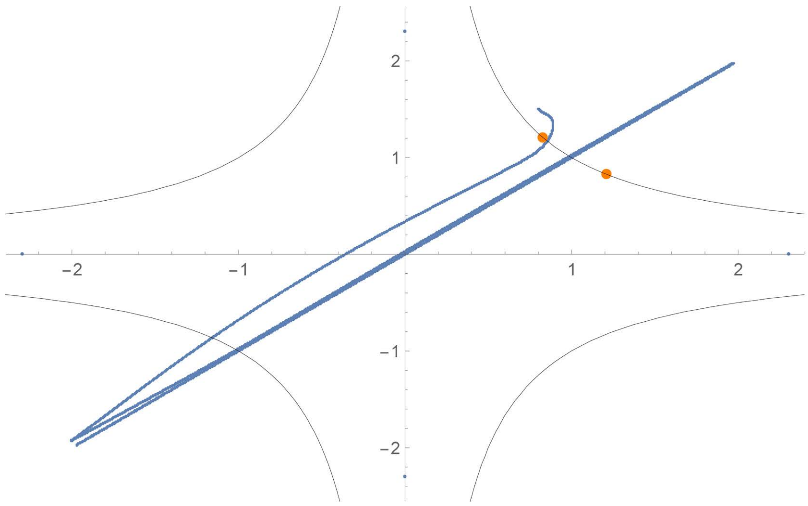

Figure 3. $b=1.4$, $(x_1(0), x_2(0))=(1.2, 0.8)$, $\sigma_1=\sigma_2=0$.

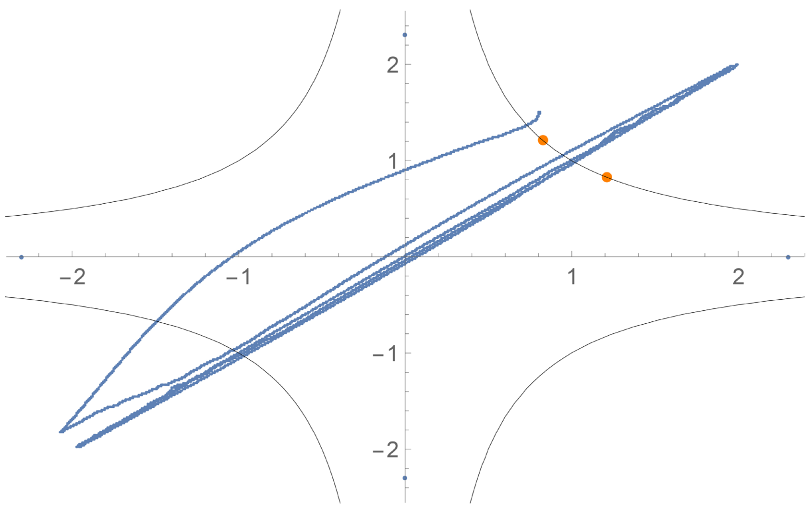

Figure 4. $b=1.4$, $(x_1(0), x_2(0))=(1.2, 0.8)$, $\sigma_1=\sigma_2=0.1$.

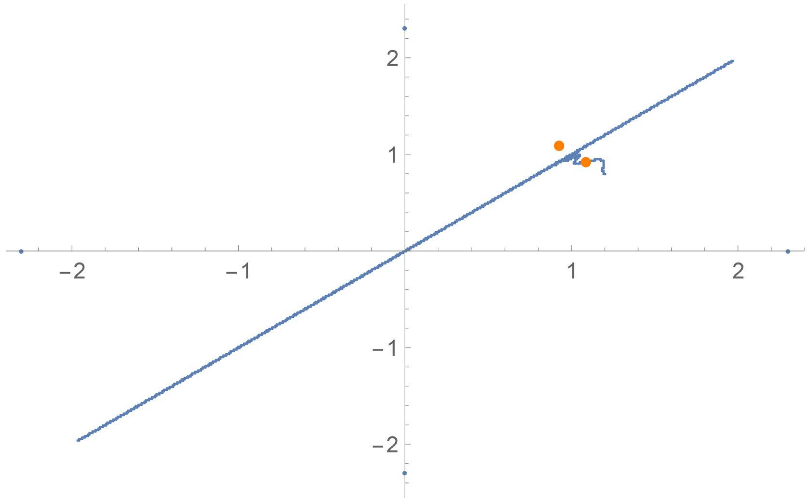

Figure 5. $b = 1.48$, $(x_1(0), x_2(0)) = (1.2, 0.8)$, $\sigma_1 = \sigma_2 = 0$.

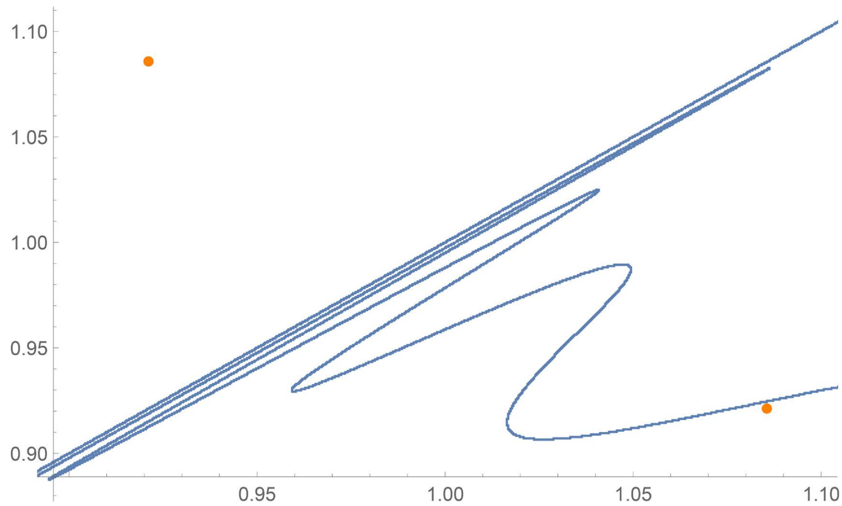

Figure 6. Enlarged orbit of **Figure 5**.

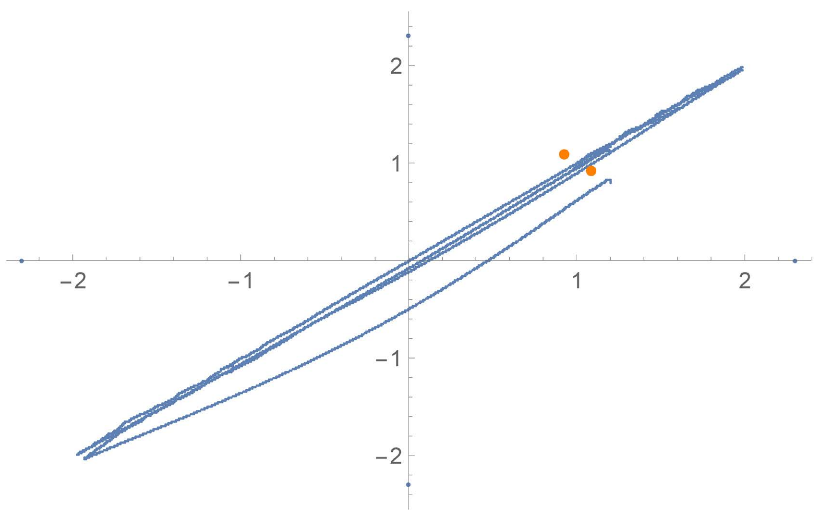

Figure 7. $b = 1.48$, $(x_1(0), x_2(0)) = (1.2, 0.8)$, $\sigma_1 = \sigma_2 = 0.1$.

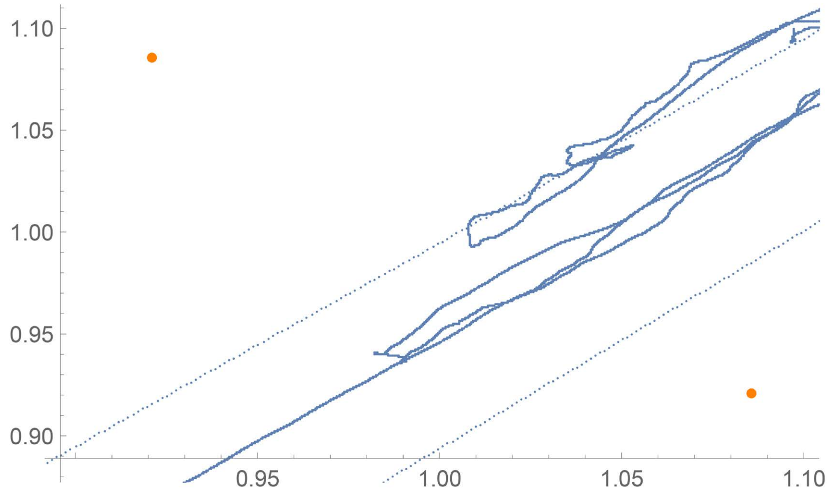

Figure 8. Enlarged orbit of Figure 7.

Figure 6 $\sigma_1 = \sigma_2 = 0$

Figure 6 shows an enlarged orbit of Figure 5. The oscillation after passing through the pseudo singular point is due to the shape of $\varphi(x_1, x_2)$, which causes jumping by short canards.

Figure 7 $\sigma_1 = \sigma_2 = 0.1$

Figure 7 shows an orbit of $\{(x_1(t), x_2(t)), 0 \le t \le T = 7\}$ satisfying the equation (34) with $\sigma_1 = \sigma_2 = 0.1$, $b = 1.48$ and starting from $(1.2, 0.8)$ near the pseudo singular point $(1.08557, 0.921173)$. The orbit moves in large steps compared with Figure 5 without noise. From Figure 7 we observe that the orbit dose not converge the invariant manifold $x_1 = x_2$, but it moves around a neighborhood of $x_1 = x_2$ by random noises.

Figure 8 $\sigma_1 = \sigma_2 = 0.1$

Figure 8 shows an enlarged orbit of Figure 7.

5. Conclusion

In general, 4-dimensional canards with a bifurcation parameter *b* have a complicated structure. The system (1) taken up in Section 2 brings us geometrical new point of view on structural stability near the pseudo singular points. Because of constructing the system geometrically simplified, it becomes easy to catch up the bifircation structure. "Symmetry" given in this paper makes it clear that the slow manifold depends on the parameter *b*. Especially, near the singular point there exists a center manifold. Note that the concrete model in Section 4 is basically composed of coupled neuron systems. Using a nonstandard method developed in [5], the rigidity for the system having Brownian motions is confirmed. They are observed in computer simulations. It means processing of quantum computing under the standard method.

Acknowledgements

The authors would like to express their sincere gratitude to the anonymous referees for their useful comments. The first author is supported in part by Grant-in-Aid Scientific Research (C), No. 18K03431, Ministry of Education, Science and Culture, Japan.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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