Advances in Pure Mathematics, 2022, 12, 565-586
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

A Basic Topological Approach to the Continuity of the Size Function

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How to cite this paper: Dubeau, F. (2022) A Basic Topological Approach to the Continuity of the Size Function. *Advances in Pure Mathematics*, **12**, 565-586.
https://doi.org/10.4236/apm.2022.1210044

Received: August 27, 2022
Accepted: October 23, 2022
Published: October 26, 2022

Abstract

We present old and new results about the size function of a set providing simple and complete proofs using basic tools of general topology. For instance, the decomposition of the size function is given and, under the calmness property of a set, the right continuity of the size function with respect to both arguments is established. Finally, a classification of its points of discontinuity is given.

Keywords

Size Function, Connexity, Path, Calm Set, Measure Function

1. Introduction

Size functions are shape descriptors, in a geometrical/topological sense. They are functions counting certain connected components of a topological space. In this paper, we revisit the very first basic results about the continuity of the size function. The main contribution is to present simple and complete proofs of old and new results using only basic topological tools. For example, the fundamental result on the decomposition of the size function is established in Theorem 9 of Section 0. Moreover, the introduction of the ***calmness*** property for a set allows us to, not only simplify some proofs of old results, but also present simple proofs of the continuity to the right for the size function with respect to its two arguments, given in Theorem 16 and Theorem 17 of Section 7. To complete the paper, we present a classification of the discontinuity points of the size function. Several papers have been dedicated to the study of size functions. For the purpose of this survey, and because we use only basic topology, it is enough to consider [1] [2] [3] [4] and references therein. One of the motivations for studying these functions is the possibility to use them in topology, image analysis, and pattern recognition, see for example [5] [6].

2. Path Connectivity and Equivalence Relation

Let $\mathcal{C}$ be a subset of $\mathbb{R}^n$. A ***path in*** $\mathcal{C}$ joining a point p to a point q in $\mathcal{C}$ is a continuous function $h(\cdot)$ defined on $[0,1]$ with values in $\mathcal{C}$ such that $h(0)=p$ and $h(1)=q$. The notion of a path in $\mathcal{C}$ induces an ***equivalence relation*** on any subset $\mathcal{E}$ of $\mathbb{R}^n$. Any two points p and q of $\mathcal{E}$ are said to be $\mathcal{C}$ ***-equivalent***, and we use the notation $p \equiv_{\mathcal{C}} q$, if and only if either $p=q$ or there is a path joining p and q in $\mathcal{C}$, in other words

$$p \equiv_{\mathcal{C}} q \quad \text{iff} \quad \begin{cases} p = q \\ \text{or} \\ \exists h:[0,1] \mapsto \mathcal{C}, \text{continuous,} \\ \text{such that } h(0)=p \text{ and } h(1)=q. \end{cases}$$

To each $p \in \mathbb{R}^n$ we associate a non empty subset

$$[p]_{\mathcal{C}} = \left\{ q \in \mathbb{R}^n \mid p \equiv_{\mathcal{C}} q \right\} \neq \varnothing.$$

For any subset $\mathcal{E} \subseteq \mathbb{R}^n$ and to each $p \in \mathbb{R}^n$ we define the set

$$[p]_{\mathcal{C}}^{\mathcal{E}} = [p]_{\mathcal{C}} \cap \mathcal{E} = \left\{ q \in \mathcal{E} \mid p \equiv_{\mathcal{C}} q \right\}.$$

Hence we have

$$[p]_{\mathcal{C}}^{\mathcal{E}} \begin{cases} \neq \varnothing & \text{if } p \in \mathcal{E}, \\ = \varnothing & \text{if } p \notin \mathcal{E}. \end{cases}$$

For $\mathcal{E}_1 \subset \mathcal{E}_2$, we have $[p]_{\mathcal{C}}^{\mathcal{E}_1} = [p]_{\mathcal{C}} \cap \mathcal{E}_1 \subseteq [p]_{\mathcal{C}} \cap \mathcal{E}_2 = [p]_{\mathcal{C}}^{\mathcal{E}_2}$, and we can write $[p]_{\mathcal{C}}^{\mathcal{E}_1} = [p]_{\mathcal{C}}^{\mathcal{E}_2} \cap \mathcal{E}_1$. Also, for $\mathcal{C}_1 \subset \mathcal{C}_2$, we have $[p]_{\mathcal{C}_1}^{\mathcal{E}} \subset [p]_{\mathcal{C}_2}^{\mathcal{E}}$.

The family of sets $\left\{ P \subseteq \mathbb{R}^n \mid \exists p \in \mathcal{E} \text{ such that } P = [p]_{\mathcal{C}}^{\mathcal{E}} \right\}$ is a partition of $\mathcal{E}$, a family of non empty and disjoint sets called ***quotient set*** of $\mathcal{E}$ with respect to $\equiv_{\mathcal{C}}$, and noted $\mathcal{E}_{/\equiv_{\mathcal{C}}}$. Elements of $\mathcal{E}_{/\equiv_{\mathcal{C}}}$ are called equivalence classes. Hence two elements of $\mathcal{E}$ are members of the same set P of $\mathcal{E}_{/\equiv_{\mathcal{C}}}$ if and only if they are $\mathcal{C}$-equivalent. Each element p of one $P \in \mathcal{E}_{/\equiv_{\mathcal{C}}}$ is a ***representative*** of the equivalence class P. So if p_1 and $p_2 \in P \in \mathcal{E}_{/\equiv_{\mathcal{C}}}$ then $[p_1]_{\mathcal{C}}^{\mathcal{E}} = P = [p_2]_{\mathcal{C}}^{\mathcal{E}}$. The application $\pi_{\mathcal{C}}^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}_{/\equiv_{\mathcal{C}}}$, defined by $\pi_{\mathcal{C}}^{\mathcal{E}}(p) = [p]_{\mathcal{C}}^{\mathcal{E}}$, is called the ***projection*** of $\mathcal{E}$ on $\mathcal{E}_{/\equiv_{\mathcal{C}}}$, it is a surjection.

For any set $\mathcal{A}$ we define the cardinality of this set, noted $\text{Card}(\mathcal{A})$, by

$$\text{Card}(\mathcal{A}) = \begin{cases} \text{the number of elements of } \mathcal{A}, & \text{if this number is finite,} \\ +\infty, & \text{else.} \end{cases}$$

The first two results follow easily.

Theorem 1. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two non empty sets such that $\mathcal{E}_1 \subset \mathcal{E}_2$. The application

$$[p]_{\mathcal{C}}^{\mathcal{E}_1} \mapsto [p]_{\mathcal{C}}^{\mathcal{E}_2},$$

defined for all $p \in \mathcal{E}_1$, induces an injection $\mathcal{I}_{\mathcal{C}}(\cdot)$ of $\mathcal{E}_{1/\equiv_{\mathcal{C}}}$ to $\mathcal{E}_{2/\equiv_{\mathcal{C}}}$. Hence

$$\mathrm{Card}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right) \le \mathrm{Card}\left(\mathcal{E}_{2/\equiv_{\mathcal{C}}}\right),$$

and

$$\mathcal{E}_{2/\equiv_{\mathcal{C}}} \setminus \mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right) = \left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right)\right)_{/\equiv_{\mathcal{C}}},$$

or equivalently

$$\mathcal{E}_{2/\equiv_{\mathcal{C}}} = \mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right) \cup \left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right)\right)_{/\equiv_{\mathcal{C}}}.$$

Theorem 2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ two non empty sets such that $\mathcal{C}_1 \subset \mathcal{C}_2$. The application

$$[p]_{\mathcal{C}_1}^{\mathcal{E}} \mapsto [p]_{\mathcal{C}_2}^{\mathcal{E}},$$

defined for any p in $\mathcal{E}$, induces a surjection $\mathcal{J}_{\mathcal{E}}(\cdot)$ from $\mathcal{E}_{/\equiv_{\mathcal{C}_1}}$ to $\mathcal{E}_{/\equiv_{\mathcal{C}_2}}$. Hence

$$\mathrm{Card}\left(\mathcal{E}_{/\equiv_{\mathcal{C}_1}}\right) \ge \mathrm{Card}\left(\mathcal{E}_{/\equiv_{\mathcal{C}_2}}\right).$$

Moreover, for all $P_2 \in \mathcal{E}_{/\equiv_{\mathcal{C}_2}}$, $\mathcal{J}_{\mathcal{E}}^{-1}(P_2) = \left\{P_1 \in \mathcal{E}_{/\equiv_{\mathcal{C}_1}} \mid \mathcal{J}_{\mathcal{E}}(P_1) = P_2\right\}$ is a partition of P_2.

We combine now the first two results we get the next one.

Theorem 3. Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{C}_1, \mathcal{C}_2$ be non empty sets such that $\mathcal{E}_1 \subset \mathcal{E}_2$ and $\mathcal{C}_1 \subset \mathcal{C}_2$. For the injections

$$\mathcal{I}_{\mathcal{C}_j} : \mathcal{E}_{1/\equiv_{\mathcal{C}_j}} \to \mathcal{E}_{2/\equiv_{\mathcal{C}_j}} \quad (j = 1, 2)$$

defined in Theorem 1 we have

$$\mathrm{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_1}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_1}}\right)\right)_{/\equiv_{\mathcal{C}_1}}\right) \ge \mathrm{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_2}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_2}}\right)\right)_{/\equiv_{\mathcal{C}_2}}\right)$$

Proof. For $\mathcal{E}_1 \subset \mathcal{E}_2$ and $\mathcal{C}$, let us consider the injection

$$\mathcal{I}_{\mathcal{C}} : \mathcal{E}_{1/\equiv_{\mathcal{C}}} \to \mathcal{E}_{2/\equiv_{\mathcal{C}}}.$$

The subset $\bigcup \mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right)$ of $\mathcal{E}_2$ increase when $\mathcal{C}$ increase. Indeed, for $p \in \mathcal{E}_1$, the set $[p]_{\mathcal{C}}^{\mathcal{E}_2}$ increase with $\mathcal{C}$. Because the elements of $\mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right)$ are of the form $[p]_{\mathcal{C}}^{\mathcal{E}_2}$ for $p \in \mathcal{E}_1$, we deduce that $\bigcup \mathcal{I}_{\mathcal{C}}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}}}\right)$ increase with $\mathcal{C}$. This fact implies that

$$\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_1}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_1}}\right) \supseteq \mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_2}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_2}}\right).$$

From Theorem 1 we have

$$\mathrm{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_1}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_1}}\right)\right)_{/\equiv_{\mathcal{C}_j}}\right) \ge \mathrm{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_2}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_2}}\right)\right)_{/\equiv_{\mathcal{C}_j}}\right)$$

for $j = 1, 2$. Otherwise from Theorem 2 we have

$$\operatorname{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_j}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_j}}\right)\right)_{/\equiv_{\mathcal{C}_1}}\right) \geq \operatorname{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_j}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_j}}\right)\right)_{/\equiv_{\mathcal{C}_2}}\right)$$

for $j = 1, 2$. Hence

$$\begin{aligned}\operatorname{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_1}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_1}}\right)\right)_{/\equiv_{\mathcal{C}_1}}\right) &\geq \operatorname{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_2}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_2}}\right)\right)_{/\equiv_{\mathcal{C}_1}}\right) \\ &\geq \operatorname{Card}\left(\left(\mathcal{E}_2 \setminus \bigcup \mathcal{I}_{\mathcal{C}_2}\left(\mathcal{E}_{1/\equiv_{\mathcal{C}_2}}\right)\right)_{/\equiv_{\mathcal{C}_2}}\right),\end{aligned}$$

and the result follows. □

We obtain directly the next result which will allow us to get a decomposition the size function.

Theorem 4. Let $\{\mathcal{E}_i\}_{i\in I}$ a partition of $\mathcal{E}$ and $\{\mathcal{C}_i\}_{i\in I}$ a partition of $\mathcal{C}$. We have

(a) If, for all $p \in \mathcal{E}$, there exists a set $\mathcal{E}_i$ such that $[p]_{\mathcal{C}}^{\mathcal{E}} \subset \mathcal{E}_i$, in other words $[p]_{\mathcal{C}}^{\mathcal{E}} = [p]_{\mathcal{C}}^{\mathcal{E}_i}$, then $\mathcal{E}_{/\equiv_{\mathcal{C}}} = \bigcup_{i\in I} \mathcal{E}_{i/\equiv_{\mathcal{C}}}$.

(b) If, for all p and $q \in \mathcal{E}_i$, we have $p \equiv_{\mathcal{C}} q \Leftrightarrow p \equiv_{\mathcal{C}_i} q$, then $\mathcal{E}_{i/\equiv_{\mathcal{C}}} = \mathcal{E}_{i/\equiv_{\mathcal{C}_i}}$.

(c) Under the preceding assumptions in (a) and (b), we have $\mathcal{E}_{/\equiv_{\mathcal{C}}} = \bigcup_{i\in I} \mathcal{E}_{i/\equiv_{\mathcal{C}_i}}$.

Finally, let us state a classical result for a path connected set [7].

Theorem 5. The quotient set $\mathcal{E}_{/\equiv_{\mathcal{C}}}$ is the set of path connected components of $\mathcal{E}$.

3. Size Function

The size function of a set $\mathcal{M}$ is defined from a ***measure function***, noted φ, which is simply a well defined and continuous application on $\mathbb{R}^n$. For any subset $\mathcal{D}$ of $\mathbb{R}^n$ the following notation will be used

$$\mathcal{D}(\varphi \leq x) = \{p \in \mathcal{D} \mid \varphi(p) \leq x\},$$

which can be extended to subsets $\mathcal{D}(\varphi = x)$, $\mathcal{D}(\varphi \geq x)$, $\mathcal{D}(\varphi < x)$ and $\mathcal{D}(\varphi > x)$.

We consider the quotient set $\mathcal{M}(\varphi \leq x)_{/\equiv_{\mathcal{M}(\varphi\leq y)}}$ of $\mathcal{M}(\varphi \leq x)$ obtained from the equivalence relation $\equiv_{\mathcal{M}(\varphi\leq y)}$ induced by continuous paths in $\mathcal{M}(\varphi \leq y)$. The ***size function*** associated with the set $\mathcal{M}$ and the measure function φ, noted $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\cdot)$, is defined as the number of elements in this quotient set. Using the function $\operatorname{Card}(\cdot)$, we have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x, y) = \operatorname{Card}\left(\mathcal{M}(\varphi \leq x)_{/\equiv_{\mathcal{M}(\varphi\leq y)}}\right).$$

Hence

1) $\mathcal{L}_{(\mathcal{M},\varphi)}(x, y) = 0$ if $\mathcal{M}(\varphi \leq x)_{/\equiv_{\mathcal{M}(\varphi\leq y)}}$ is empty;

2) $\mathcal{L}_{(\mathcal{M},\varphi)}(x, y) \in \mathbb{N} = \{1, 2, 3, \cdots\}$ if $\mathcal{M}(\varphi \leq x)_{/\equiv_{\mathcal{M}(\varphi\leq y)}}$ has a finite number of elements;

3) $\mathcal{L}_{(\mathcal{M},\varphi)}(x, y) = +\infty$ if $\mathcal{M}(\varphi \leq x)_{/\equiv_{\mathcal{M}(\varphi\leq y)}}$ has infinitely many elements.

Example 6. The simplest example, with no interest for the sequel of the paper, is for $\mathcal{M} = \{a\}$. Let $\varphi(a) = s$. Hence

$$\mathcal{M}(\varphi \le x) = \begin{cases} \mathcal{M} & \text{if } x \ge s, \\ \varnothing & \text{if } x < s. \end{cases}$$

For any $y \in \mathbb{R}$, we have

$$\mathcal{M}(\varphi \le x)_{/\equiv_{\mathcal{M}(\varphi \le y)}} = \begin{cases} \mathcal{M}_{/\equiv_{\mathcal{M}(\varphi \le y)}} = \{\mathcal{M}\} & \text{if } x \ge s, \\ \varnothing & \text{if } x < s, \end{cases}$$

hence

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x, y) = \begin{cases} 1 & \text{if } x \ge s, \\ 0 & \text{if } x < s. \end{cases} \quad \square$$

Since $\mathcal{M}(\varphi \le \zeta)$ increases with respect to ζ, an increasing/decreasing property of the size function follows directly from Theorem 1 and Theorem 2.

Theorem 7. [2] [4] $\mathcal{L}_{(\mathcal{M},\varphi)}(x, y)$ increases with respect to *x* and decreases with respect to *y*.

4. Decomposition of the Size Function

Under some assumptions, the set $\mathcal{M}$ can be decomposed into disjoint subsets. The size function is then decomposable into a sum of size functions of its subsets.

Theorem 8. If $\mathcal{M}$ is compact and locally path connected, then path connected components of $\mathcal{M}$ are in finite number and form $\mathcal{M}_{/\equiv_{\mathcal{M}}}$.

Proof. As $\mathcal{M}$ is locally path connected, for any point $p \in \mathcal{M}$ and any neighborhood V_p of *p* in $\mathcal{M}$ there is a sub-neighborhood U_p in $\mathcal{M}$ which is path connected. The elements of $\mathcal{M}_{/\equiv_{\mathcal{M}}}$ are then path connected components of $\mathcal{M}$. These related components are both open and closed in $\mathcal{M}$, and therefore also compact. As we have $\mathcal{M} = \bigcup \mathcal{M}_{/\equiv_{\mathcal{M}}}$, it is a covering of the compact set $\mathcal{M}$ by disjoint open sets. Then, there is a finite sub-covering of $\mathcal{M}$, but this sub-covering may only be the covering itself and therefore $\mathcal{M}_{/\equiv_{\mathcal{M}}}$ has a finite number of elements, hence the result follows. □

Let us set

$$\mathcal{M}_{/\equiv_{\mathcal{M}}} = \{\mathcal{M}_n \mid n = 1, \cdots, N(\mathcal{M})\},$$

where $N(\mathcal{M})$ represents the number of non empty and disjoint elements of $\mathcal{M}_{/\equiv_{\mathcal{M}}}$ in such a way that

$$\mathcal{M} = \bigcup_{n=1}^{N(\mathcal{M})} \mathcal{M}_n.$$

Moreover, we have

$$\mathcal{M}(\varphi \le x) = \bigcup_{n=1}^{N(\mathcal{M})} \mathcal{M}_n(\varphi \le x).$$

Theorem 9. If $\mathcal{M}$ is compact and locally path connected, we have

$$\mathcal{M}(\varphi \le x)_{/\equiv_{\mathcal{M}(\varphi\le y)}} = \bigcup_{n=1}^{N(\mathcal{M})} \mathcal{M}_n(\varphi \le x)_{/\equiv_{\mathcal{M}_n(\varphi\le y)}},$$

and then

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x,y) = \sum_{n=1}^{N(\mathcal{M})} \mathcal{L}_{(\mathcal{M}_n,\varphi)}(x,y).$$

Proof. We use Theorem 4 with $\mathcal{E} = \mathcal{M}(\varphi \le x)$ and $\mathcal{C} = \mathcal{M}(\varphi \le y)$. Based on (a) of Theorem 4, we have

$$\mathcal{M}(\varphi \le x)_{/\equiv_{\mathcal{M}(\varphi\le y)}} = \bigcup_{n=1}^{N(\mathcal{M})} \mathcal{M}_n(\varphi \le x)_{/\equiv_{\mathcal{M}(\varphi\le y)}}.$$

Based on (b) of Theorem 4, since for any p and $q \in \mathcal{M}_n(\varphi \le x)$ we have

$$p \equiv_{\mathcal{M}(\varphi\le y)} q \text{ if and only if } p \equiv_{\mathcal{M}_n(\varphi\le y)} q,$$

so we obtain

$$\mathcal{M}(\varphi \le x)_{/\equiv_{\mathcal{M}(\varphi\le y)}} = \mathcal{M}_n(\varphi \le x)_{/\equiv_{\mathcal{M}_n(\varphi\le y)}}.$$

The result follows from (c) of Theorem 4. □

Using this result, it is enough now to consider a compact, connected, and locally path connected set $\mathcal{M}$, *i.e.*, a compact set $\mathcal{M}$ with only one path connected component. Let us note this family by

$$\mathbb{K}_{\mathrm{loc}}^{\mathrm{c}} = \left\{ \mathcal{M} \subseteq \mathbb{R}^n \;\middle|\; \begin{array}{ll} \text{(i)} & \mathcal{M} \text{ is compact,} \\ \text{(ii)} & \mathcal{M} \text{ is connected,} \\ \text{(iii)} & \mathcal{M} \text{ is locally path connected,} \\ \text{(iv)} & \mathcal{M} \text{ contains more than one point.} \end{array} \right\}.$$

Condition (iv) implies that the set $\mathcal{M}$ contains at least two points, so consequently it contains infinitely many points and

$$\mathrm{Card}(\mathcal{M}) = +\infty.$$

The set $\mathcal{M} \in \mathbb{K}_{\mathrm{loc}}^{\mathrm{c}}$ will be said ***calm*** if the number of connected components of $\mathcal{M}(\varphi \le s)$ is always finite, that is to say

$$\mathrm{Card}\left(\mathcal{M}(\varphi \le s)_{/\equiv_{\mathcal{M}(\varphi\le s)}}\right) < +\infty$$

for all s.

5. General Results

Let $\mathcal{M}$ be a *calm* element of $\mathbb{K}_{\mathrm{loc}}^{\mathrm{c}}$. From the continuity of $\varphi(\cdot)$ on the compact set $\mathcal{M}$, let us define $\underline{s}$ and $\overline{s}$, and the set $\Delta(\underline{s},\overline{s})$ by

$$\underline{s} = \min\{\varphi(p) \mid p \in \mathcal{M}\} \le \max\{\varphi(p) \mid p \in \mathcal{M}\} = \overline{s},$$

and

$$\Delta(\underline{s},\overline{s}) = \{(x,y) \in \mathbb{R}^n \mid \underline{s} \le x \le y \le \overline{s}\}.$$

We start by establishing basic general results for $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ defined on $\mathbb{R}^2$ which are illustrated in Figure 1.

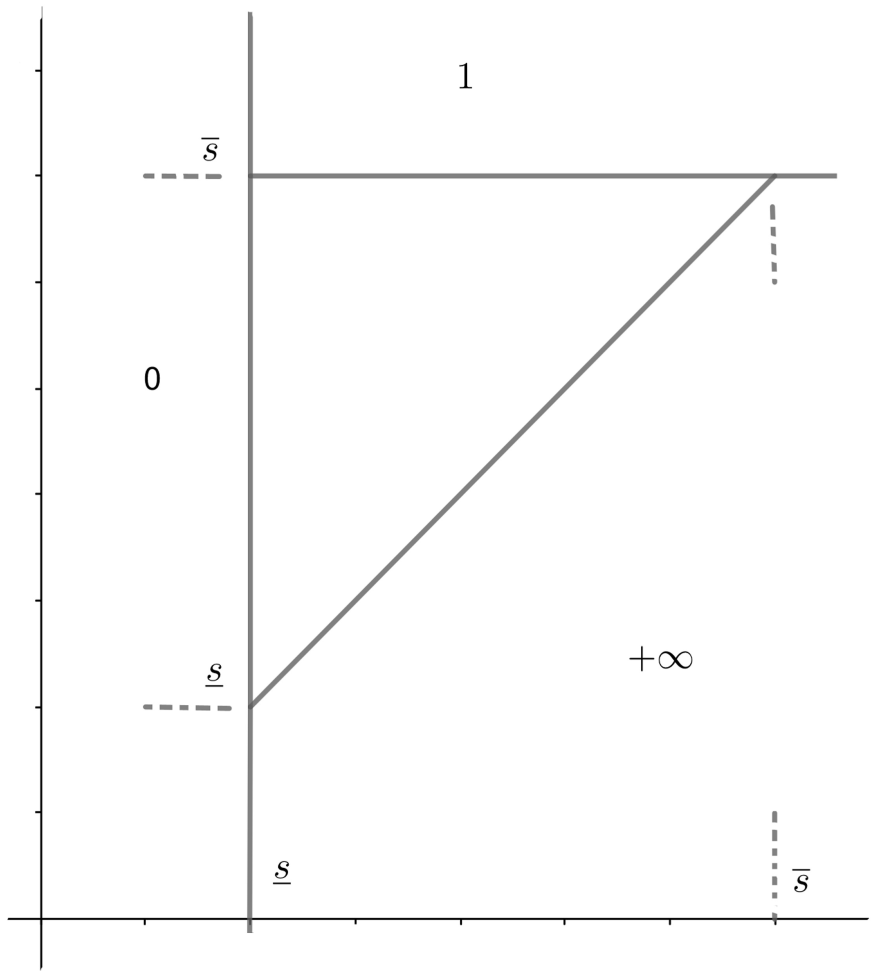

Figure 1. The function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$.

Theorem 10. [2] [4] Suppose $\underline{s} \le \overline{s}$. We have

i) $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)=0$ for all $(x,y)\in R_0=\{(x,y)\in\mathbb{R}^2 \mid x<\underline{s}\}$;

ii) $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)=1$ for all $(x,y)\in R_1$ where

$$R_1=\{(x,y)\in\mathbb{R}^2 \mid x\ge\underline{s} \text{ and } y\ge\overline{s}\};$$

iii) $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)=+\infty$ for all $(x,y)\in R_\infty$ where

$$R_\infty=\{(x,y)\in\mathbb{R}^2 \mid x\ge\underline{s} \text{ and } y<\min\{x,\overline{s}\}\};$$

iv) $1\le\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)<+\infty$ for all $s\in\Delta(\underline{s},\overline{s})$.

Proof. i) In this case $\mathcal{M}(\varphi\le x)=\varnothing$.

ii) Here we have

$$\mathcal{M}(\varphi\le x)_{/\equiv_{\mathcal{M}(\varphi\le y)}}=\mathcal{M}(\varphi\le x)_{/\equiv_{\mathcal{M}}}=\{\mathcal{M}(\varphi\le x)\}.$$

iii) There exists a non isolated point $p\in\mathcal{M}$ such that $y<\varphi(p)\le x$. Hence from continuity of $\varphi(\cdot)$, $\mathcal{M}(\varphi\le x)_{/\equiv_{\mathcal{M}(\varphi\le y)}}$ contains an infinity of singletons, the result follows.

iv) Direct consequence of the definition of the size function for a calm set

$\mathcal{M}$, and the increasing/decreasing property of Theorem 7. □

Now, since $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ is an integer valued function, we get directly the next result.

Theorem 11. [2] [4] At any point of continuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ there is a neighborhood of this point where the function is constant.

6. Examples

6.1. Generalities

We present three simple examples of the size function based on two different measure functions $\varphi(\cdot)$. In each example, the set $\mathcal{M}$ is a simple continuous curve of finite length. It is a compact, connected, and locally path connected set.

6.2. First Example

Let $\mathcal{M} \subset \mathbb{R}^2$ given in **Figure 2**. The measure function is

$$\varphi(p) = \left(u^2 + v^2\right)^{1/2}$$

which is the euclidean distance from the origin $O = (0,0)$ to $p = (u,v) \in \mathbb{R}^2$. We have $\underline{s} = 1$ and $\overline{s} = 6$. **Figure 3** presents the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ on $\Delta(\underline{s},\overline{s})$ for the set $\mathcal{M}$.

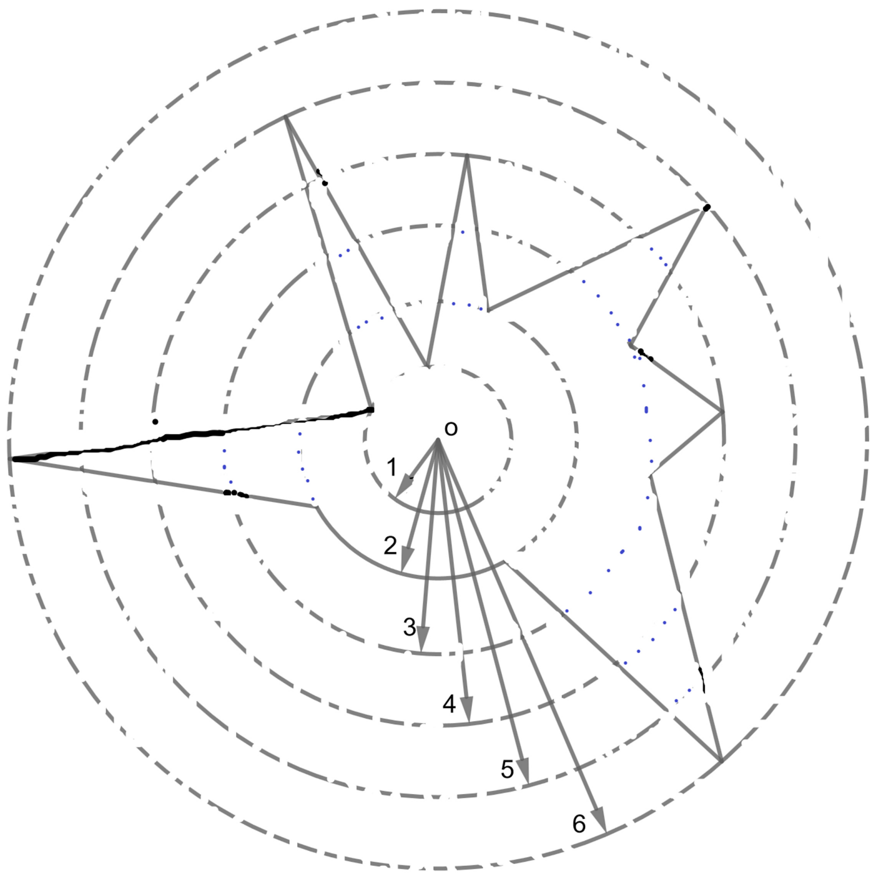

Figure 2. The set $\mathcal{M}$.

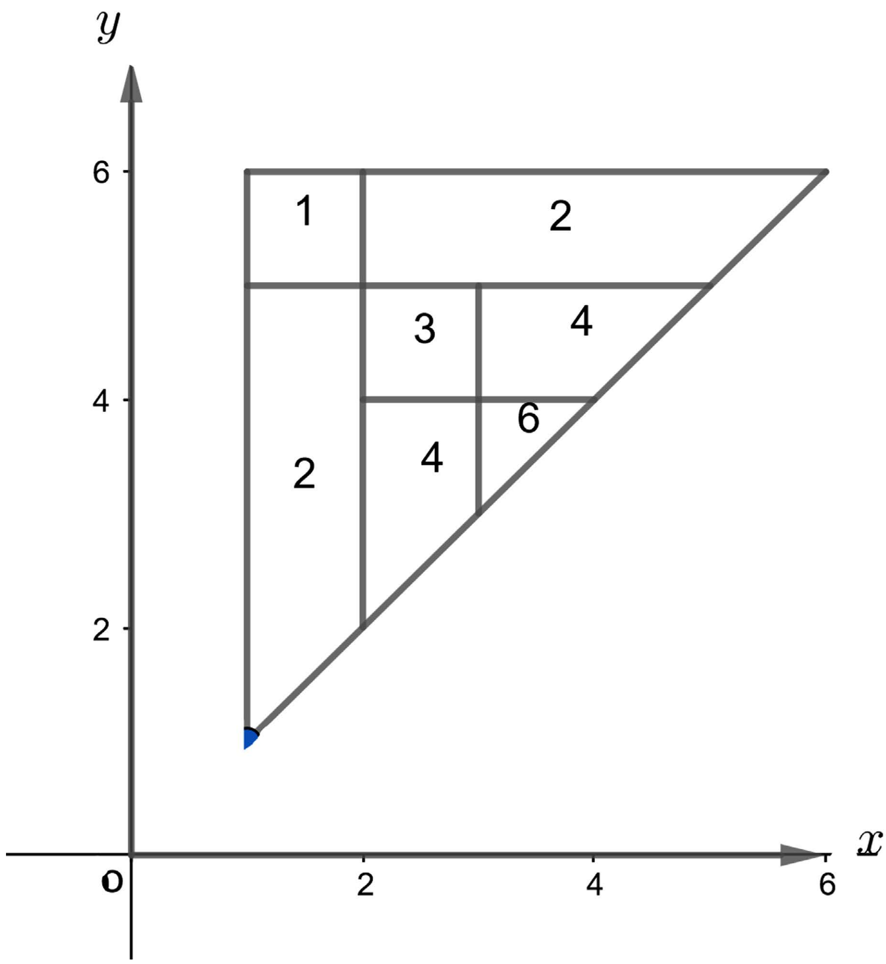

Figure 3. Values of the function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$.

$\mathcal{M}$ is a calm set. On the diagonal of $\Delta(\underline{s},\overline{s})$, namely $(x,y)=(s,s)$ for $s\in[1,6]$, we have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(s,s)=\begin{cases}2 & \text{for } s\in[1,2),\\ 4 & \text{for } s\in[2,3),\\ 6 & \text{for } s\in[3,4),\\ 4 & \text{for } s\in[4,5),\\ 2 & \text{for } s\in[5,6),\\ 1 & \text{for } s=6.\end{cases}$$

6.3. Second Example

The set $\mathcal{M}$ of **Figure 4** is a curve of finite length. The measure function is the distance of a point $p=(u,v)$ to the Ou axis, so we have

$$\varphi(p)=|v|.$$

The size function is given in **Figure 5**. We have $\underline{s}=1$ and $\overline{s}=6$.

$\mathcal{M}$ is also a clam set. On the diagonal of $\Delta(\underline{s},\overline{s})$, namely $(x,y)=(s,s)$ for $s\in[1,6]$, we have

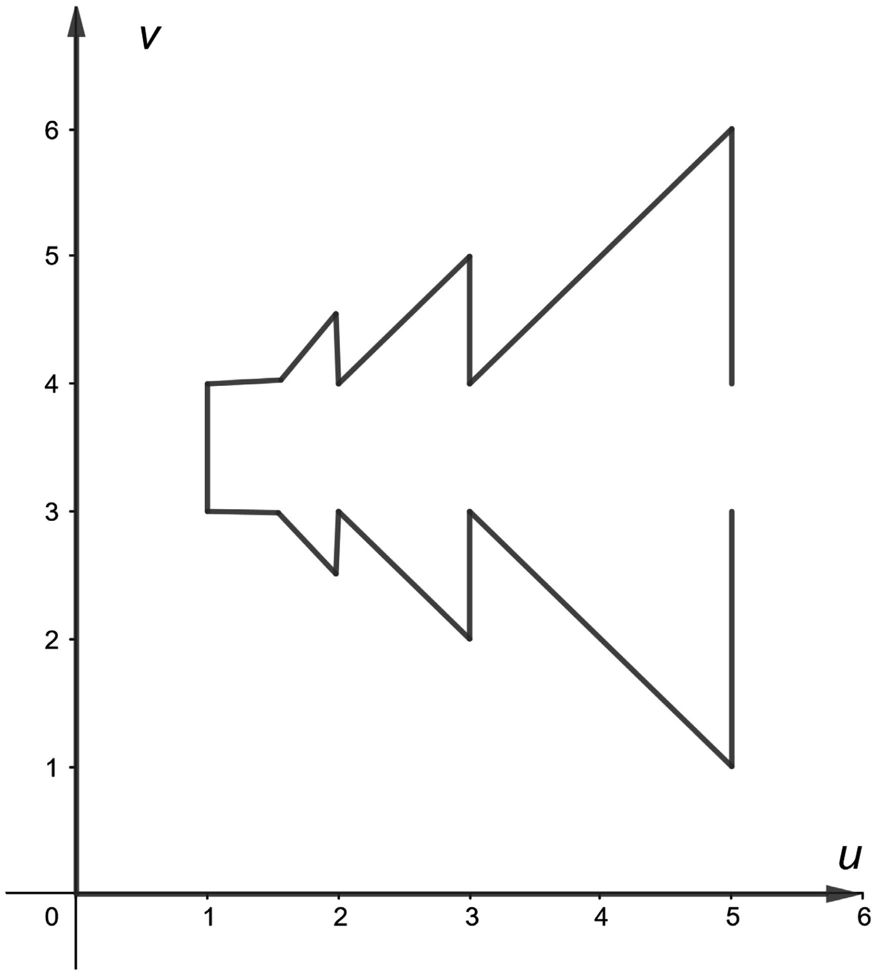

Figure 4. The set $\mathcal{M}$.

$$\mathcal{L}_{(\mathcal{M},\varphi)}(s,s)=\begin{cases}1 & \text{for } s\in[1,2),\\ 2 & \text{for } s\in[2,2.5),\\ 3 & \text{for } s\in[2.5,3),\\ 1 & \text{for } s\in[3,4),\\ 4 & \text{for } s\in[4,4.5),\\ 3 & \text{for } s\in[4.5,5),\\ 2 & \text{for } s\in[5,6),\\ 1 & \text{for } s=6.\end{cases}$$

6.4. Third Example

For the third example, we slightly modify the preceding example as indicated in **Figure 6**. In this set, two sequences of triangles with decreasing height go to the points A and B. For the point A, the coordinates of the highest vertex of each triangle are

$$(u_n,v_n)=\left(1+4\left(\frac{1}{2}\right)^n,4+2\left(\frac{1}{2}\right)^n\right),$$

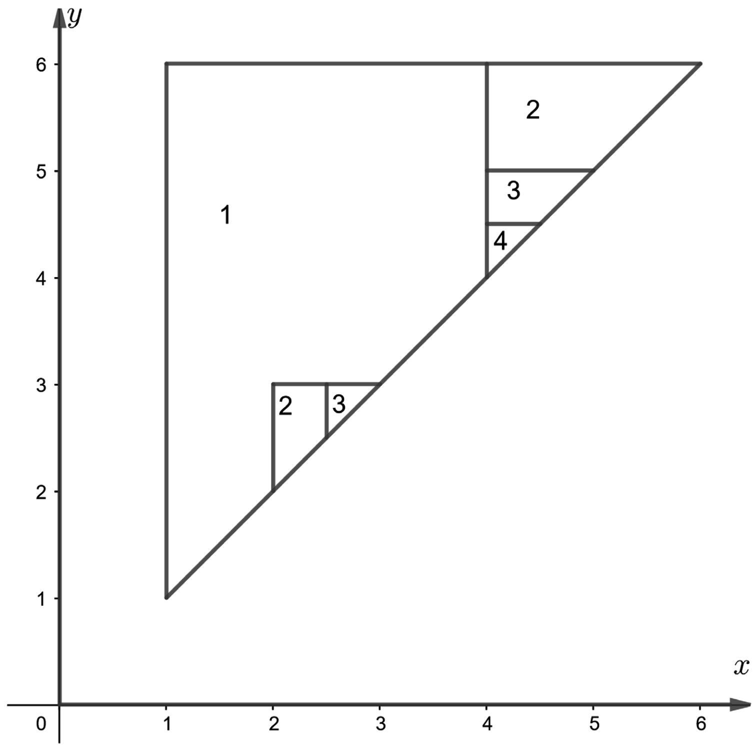

Figure 5. The function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$.

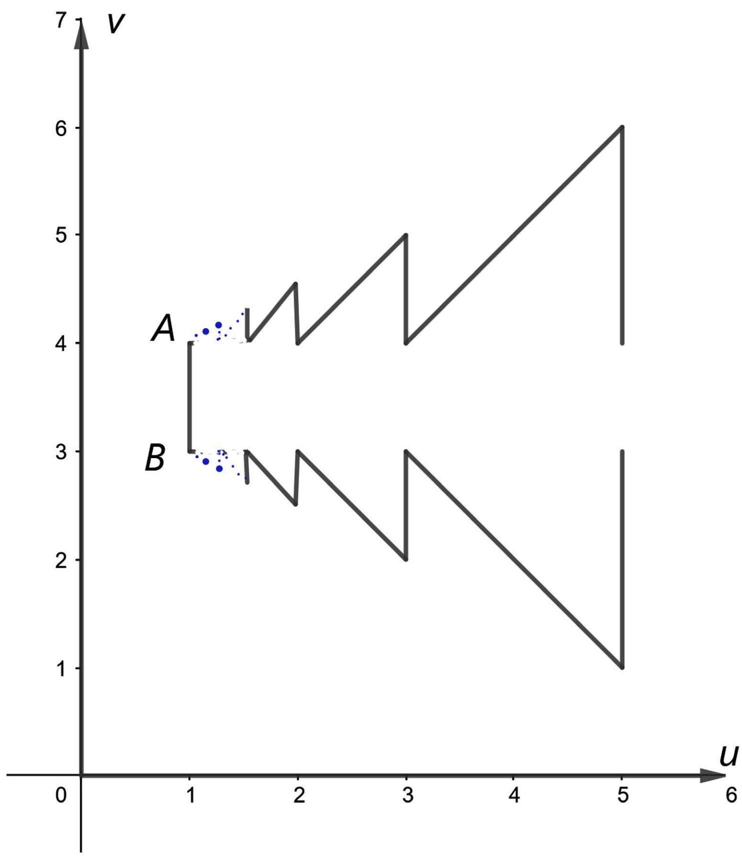

Figure 6. The set $\mathcal{M}$.

for $n=0,1,2,\cdots$. We have $\lim_{n\to\infty}(u_n,v_n)=(1,4)=A$. For the point *B*, the coordinates of the lowest vertex of each triangle are

$$(u_n,v_n)=\left(1+4\left(\frac{1}{2}\right)^n,3-2\left(\frac{1}{2}\right)^n\right).$$

for $n=0,1,2,\cdots$. We have $\lim_{n\to\infty}(u_n,v_n)=(1,3)=B$.

Figure 7 contains the graph of the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ for the set $\mathcal{M}$ of Figure 6. We have $\underline{s}=1$ and $\overline{s}=6$. On the diagonal of $\Delta(\underline{s},\overline{s})$, namely $(x,y)=(s,s)$ for $s\in[1,6]$, we have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(s,s)$$

$$\mathcal{L}_{(\mathcal{M},\varphi)}(s,s)=\begin{cases} n+1 & \text{for } s\in\left[3-2\left(\frac{1}{2}\right)^n,3-2\left(\frac{1}{2}\right)^{(n+1)}\right), \quad n=0,1,2,\cdots,\\ 1 & \text{for } s\in[3,4),\\ +\infty & \text{for } s=4,\\ n+1 & \text{for } s\in\left[4+2\left(\frac{1}{2}\right)^{(n+1)},4+2\left(\frac{1}{2}\right)^n\right), \quad n=0,1,2,\cdots,\\ 1 & \text{for } s=6.\end{cases}$$

So, $\mathcal{M}$ is not a calm set, the problem is around the point *A*.

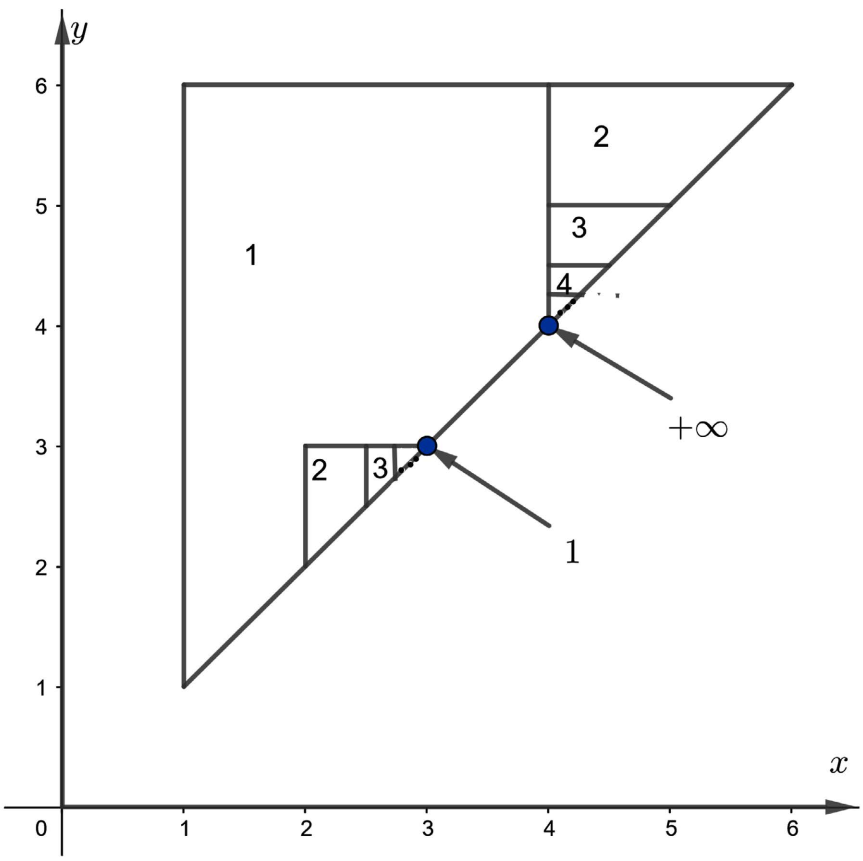

Figure 7. The function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$.

7. Continuity Properties of Size Function

In this section we present continuity results of the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ for a particular family of sets, namely *calm* set $\mathcal{M}$ in $\mathbb{K}^{c}_{loc}$. New results will complete old ones that can be found in [2] [4]. We present simple proofs of all results. In particular, the assumption of calmness of a set, which might simplify proofs of old results, is mainly introduced to prove, using elementary topological tools, the right continuity of the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ with respect to the variable *y*.

Let us now consider the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ defined on the set $\Delta(\underline{s},\overline{s})$ for $\underline{s}<\overline{s}$.

Theorem 12. For $(x,y)\in\Delta(\underline{s},\overline{s})$,

1) there exists $\varepsilon(x,y)>0$ such that the function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,y)$ is constant on the segments $(x-\varepsilon(x,y),x)\times\{y\}$ and $(x,x+\varepsilon(x,y))\times\{y\}$ in $\Delta(\underline{s},\overline{s})$;

2) there exists $\varepsilon(x,y)>0$ such that the function $\mathcal{L}_{(\mathcal{M},\varphi)}(x,\cdot)$ is constant on the segments $\{x\}\times(y-\varepsilon(x,y),y)$ and $\{x\}\times(y,y+\varepsilon(x,y))$ in $\Delta(\underline{s},\overline{s})$.

Proof. We know that $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)<+\infty$ and take only integer values. From the increasing property with respect to *x*, and the decreasing property with respect to *y*, there exists only a finite number of discontinuities on the parallel segments to the axes in $\Delta(\underline{s},\overline{s})$. The result follows. □

Theorem 13. [4] Let x_1, x_2, y_1 and y_2 be real numbers such that $\underline{s}\le x_1\le x_2\le y_1\le y_2<\overline{s}$. Then we have the following inequality

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x_2,y_1)-\mathcal{L}_{(\mathcal{M},\varphi)}(x_1,y_1)\ge\mathcal{L}_{(\mathcal{M},\varphi)}(x_2,y_2)-\mathcal{L}_{(\mathcal{M},\varphi)}(x_1,y_2)\ge 0,$$

which can be written as

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x_2,y_1)-\mathcal{L}_{(\mathcal{M},\varphi)}(x_2,y_2)\ge\mathcal{L}_{(\mathcal{M},\varphi)}(x_1,y_1)-\mathcal{L}_{(\mathcal{M},\varphi)}(x_1,y_2)\ge 0.$$

Proof. Using notation of Section 2, set $\mathcal{E}_i=\mathcal{M}(\varphi\le x_i)$ for $i=1,2$, and $\mathcal{C}_j=\mathcal{M}(\varphi\le y_j)$ for $j=1,2$. So, we consider the injections

$$\mathcal{I}_{\mathcal{C}_j}:\mathcal{E}_{1/\equiv_{\mathcal{C}_j}}\to\mathcal{E}_{2/\equiv_{\mathcal{C}_j}}.$$

for $j=1,2$, and the result follows from Theorem 1 and Theorem 3. □

Corollary 1. Let x_1, x_2, y_1 and y_2 be real numbers such that $\underline{s}\le x_1\le x_2\le y_1\le y_2<\overline{s}$. If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x_2,y_1)=\mathcal{L}_{(\mathcal{M},\varphi)}(x_1,y_2),$$

then $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ is constant on the rectangle with vertices (x_1,y_1), (x_2,y_1), (x_2,y_2), and (x_1,y_2).

Remember that $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ is increasing with respect to *x* and decreasing with respect to *y*, and let $(\overline{x},\overline{y})\in\Delta(\underline{s},\overline{s})$. We will consider the following definitions.

1) We will say that $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\overline{y})$ is *left discontinuous* at $\overline{x}$ if

$$\lim_{x\uparrow\overline{x}}\mathcal{L}_{(\mathcal{M},\varphi)}(x,\overline{y})<\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},\overline{y}),$$

and is *right discontinuous* at $\overline{x}$ if

$$\lim_{x\downarrow\bar{x}} \mathcal{L}_{(\mathcal{M},\varphi)}(x,\bar{y}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}).$$

2) We will say that $\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\cdot)$ is *left discontinuous* at $\bar{y}$ if

$$\lim_{y\uparrow\bar{y}} \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},y) > \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}),$$

and *right discontinuous* at $\bar{y}$ if

$$\lim_{y\downarrow\bar{y}} \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},y) < \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}).$$

Theorem 14. [4] Let $(\bar{x},\bar{y}) \in \Delta(\underline{s},\bar{s})$. We have the following two results.

1) If $\bar{x}$ is a point of left discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\bar{y})$, then $\bar{x}$ is also a point of left discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,y)$ for all y in the interval $[\bar{x},\bar{y}]$.

2) If $\bar{x}$ is a point of right discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\bar{y})$, then $\bar{x}$ is also a point of right discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,y)$ for all y in the interval $[\bar{x},\bar{y}]$.

Proof. 1) Let us assume the contrary, *i.e.*, the function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,y)$ is left continuous at $\bar{x}$ for a y in the interval $[\bar{x},\bar{y})$. We have

$$\lim_{x\uparrow\bar{x}} \mathcal{L}_{(\mathcal{M},\varphi)}(x,y) = \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},y).$$

But we know that $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ increases with respect to x. So, from Theorem 13, for all $x < \bar{x} \le y < \bar{y}$ we have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},y) - \mathcal{L}_{(\mathcal{M},\varphi)}(x,y) \ge \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}) - \mathcal{L}_{(\mathcal{M},\varphi)}(x,\bar{y}) \ge 0.$$

Taking the limit for $x \uparrow \bar{x}$, and using the left continuity at $(\bar{x},y)$, we get

$$\lim_{x\uparrow\bar{x}} \mathcal{L}_{(\mathcal{M},\varphi)}(x,\bar{y}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}),$$

which contradict the left discontinuity at $(\bar{x},\bar{y})$.

2) A similar proof holds for this result. Let us assume the contrary, i.e., the function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,y)$ is right continuous at $\bar{x}$ for a y in the interval $(\bar{x},\bar{y})$. We have

$$\lim_{x\downarrow\bar{x}} \mathcal{L}_{(\mathcal{M},\varphi)}(x,y) = \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},y).$$

But we know that $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ increases with respect to x. So, from Theorem 13, for all $\bar{x} < x \le y < \bar{y}$ we have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(x,y) - \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},y) \ge \mathcal{L}_{(\mathcal{M},\varphi)}(x,\bar{y}) - \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}) \ge 0.$$

Taking the limit for $x \downarrow \bar{x}$, and using the right continuity at $(\bar{x},y)$, we get

$$\lim_{x\downarrow\bar{x}} \mathcal{L}_{(\mathcal{M},\varphi)}(x,\bar{y}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\bar{y}),$$

which contradict the right discontinuity at $(\bar{x},\bar{y})$. □

We have a similar proof for the next result.

Theorem 15. [4] Let $(\bar{x},\bar{y}) \in \Delta(\underline{s},\bar{s})$. We have the following two results.

1) If $\bar{y}$ is a point of left discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\cdot)$, then $\bar{y}$ is also a point of left discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(x,\cdot)$ for all x in the interval $[\bar{x},\bar{y}]$.

2) If $\bar{y}$ is a point of right discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x},\cdot)$, then $\bar{y}$ is also a point of right discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(x,\cdot)$ for all x in the interval $[\bar{x},\bar{y}]$.

Theorem 16. [4] At every point $(\bar{x},\bar{y}) \in \Delta(\underline{s},\bar{s})$ such that $\bar{x} < \bar{y}$, the function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\overline{y})$ is right continuous with respect to *x*, *i.e.*,

$$\lim_{x\downarrow\overline{x}}\mathcal{L}_{(\mathcal{M},\varphi)}(x,\overline{y})=\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},\overline{y}).$$

Proof. Since the function is constant to the right, there is a finite number of equivalence classes. When x decreases to $\overline{x}$, by continuity of $\varphi(\cdot)$ and compactness of the sets, all the equivalence classes decrease and remain non empty. The limit equivalence classes are non empty and no new equivalence class is created, so the size function remains constant, and hence is right continuous with respect to x. □

Theorem 17. At any point $(\overline{x},\overline{y})\in\Delta(\underline{s},\overline{s})$ such that $\overline{x}\le\overline{y}<\overline{s}$, the function $\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},\cdot)$ is right continuous with respect to *y*, *i.e.*,

$$\lim_{y\uparrow\overline{y}}\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},y)=\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},\overline{y}).$$

Proof. The proof of this result proceeds in several steps.

Step 1. Suppose $\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},\cdot)$ is not right continuous with respect to y at $\overline{y}$, so

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},\overline{y})>\lim_{y\downarrow\overline{y}}\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{x},y).$$

From Theorem 15, the discontinuity is extended to the point $(\overline{y},\overline{y})$ on the diagonal of $\Delta(\underline{s},\overline{s})$ and we have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{y},\overline{y})>\lim_{y\downarrow\overline{y}}\mathcal{L}_{(\mathcal{M},\varphi)}(y,\overline{y})$$

Also, from Theorem 12 there exists $\varepsilon(\overline{y},\overline{y})>0$ such that $\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{y},y)$ is constant for $y\in(\overline{y},\overline{y}+\varepsilon(\overline{x},\overline{y}))$.

Step 2. From the calmness assumption, $\mathcal{L}_{(\mathcal{M},\varphi)}(\overline{y},\overline{y})=L<+\infty$, so there are L equivalence classes in $\mathcal{M}(\varphi\le\overline{y})_{/\equiv_{\mathcal{M}(\varphi\le\overline{y})}}$ which are disjoint compact subsets of $\mathbb{R}^n$. Let $d>0$ be less than the distance between any pair of equivalence classes of $\mathcal{M}(\varphi\le\overline{y})_{/\equiv_{\mathcal{M}(\varphi\le\overline{y})}}$. This d is well defined because there is a finite number L of compact subsets of $\mathbb{R}^n$ to consider. Each element $P_l\in\mathcal{M}(\varphi\le\overline{y})_{/\equiv_{\mathcal{M}(\varphi\le\overline{y})}}$ can be covered by the open set

$$P_l\subseteq U_l=\bigcup_{p\in P_l}B(p;d/4).$$

From the definition of d, those U_l's are disjoint open sets and we have

$$\mathcal{M}(\varphi\le\overline{y})\subseteq U=\bigcup_{l}^{L}U_l.$$

Step 3. Equivalence classes of $\mathcal{M}(\varphi\le\overline{y})_{/\equiv_{\mathcal{M}(\varphi\le\overline{y}+\varepsilon)}}$, for any ε such that $\overline{y}<\overline{y}+\varepsilon\le\overline{s}$ are union of a finite number of those of $\mathcal{M}(\varphi\le\overline{y})_{/\equiv_{\mathcal{M}(\varphi\le\overline{y})}}$, so they are also compact subsets of $\mathbb{R}^n$. Our assumption on the right discontinuity implies that we can join two equivalence classes, say P_1 and P_2, of $\mathcal{M}(\varphi\le\overline{y})_{/\equiv_{\mathcal{M}(\varphi\le\overline{y})}}$ in $\mathcal{M}(\varphi\le\overline{y}+\varepsilon)$ but we cannot join them in $\mathcal{M}(\varphi\le\overline{y})$.

Step 4. Let us built now an open covering of $\mathcal{M}(\varphi>\overline{y})$. Take

$\varepsilon_0 = \varepsilon(\bar{x}, \bar{y})/2$ and take a path in $\mathcal{M}(\varphi \le \bar{y} + \varepsilon_0)$ to join the two equivalence classes P_1 and P_2 that we cannot join in $\mathcal{M}(\varphi \le \bar{y})$. Let us suppose that this path has (at least) one point q_1 in $\mathcal{M} \setminus U$ with $\bar{y} < \varphi(q_1) \le \bar{y} + \varepsilon_0$. Take $\varepsilon_1 = (\varphi(q_1) - \bar{y})/2 \le \varepsilon_0/2$ and $V_1 = \mathcal{M}(\varphi > \bar{y} + \varepsilon_1)$. Take a path in $\mathcal{M}(\varphi \le \bar{y} + \varepsilon_1)$ which join the two equivalent classes that we cannot join in $\mathcal{M}(\varphi \le \bar{y})$. Let us suppose that this path has (at least) one point q_2 in $\mathcal{M} \setminus U$ with $\bar{y} < \varphi(q_2) \le \bar{y} + \varepsilon_1$. Take $\varepsilon_2 = (\varphi(q_2) - \bar{y})/2 \le \varepsilon_1/2$ and $V_2 = \mathcal{M}(\varphi > \bar{y} + \varepsilon_2)$. And so on, we construct a sequence of open sets $\{V_k\}_{k=1}^{\infty}$, and a sequence of points $\{q_k\}_{k=1}^{\infty}$, such that the open set $\bigcup_{k=1}^{\infty} V_k$ cover $\mathcal{M}(\varphi > \bar{y})$.

Step 5. So $U \cup \bigcup_{k=1}^{\infty} V_k$ covers $\mathcal{M}$. Any finite subcovering of this covering is included in an open covering of the form $U \cup \bigcup_{k=1}^{K} V_k$ for a certain *K*. But this open set does not contain $q_{K+1} \in \mathcal{M}$. Since $\mathcal{M}$ is compact, it means that we cannot construct the sequence of open sets $\{V_k\}_{k=1}^{\infty}$.

Step 6. If we cannot build the sequence of open sets $\{V_k\}_{k=1}^{\infty}$, there is $0 < \varepsilon < \varepsilon_0 = \varepsilon(\bar{x}, \bar{y})/2$ such that for each path joining the two equivalence classes P_1 and P_2 in $\mathcal{M}(\varphi \le \bar{y} + \varepsilon)$, either 1) there exists *p* on the path such that $\varphi(p) > \bar{y}$ and then $p \in U$, or else 2) any point *p* on the path is such that $\varphi(p) \le \bar{y}$.

1) In the first case it means that all points of the path are in *U*, and the path is covered by a finite union of non empty disjoint open sets, at least the two sets U_1 and U_2, which is not possible because the path is a connected set.

2) In the second case P_1 and P_2 are joined in $\mathcal{M}(\varphi \le \bar{y})$, which is contrary to the assumption.

In both cases we get a contradiction and the result follows. □

Now we can establish the next result.

Theorem 18. [4] Any open ball around a point of discontinuity $(\bar{x}, \bar{y})$ of the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\cdot)$ in $\Delta(\underline{s}, \bar{s})$ contains at least one point of discontinuity with respect to *x* or to *y* in $\Delta(\underline{s}, \bar{s})$, and this point is not $(\bar{x}, \bar{y})$.

Proof. Let $(\bar{x}, \bar{y}) \in \Delta(\underline{s}, \bar{s})$ be a point of discontinuity of $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\cdot)$. Then, any open ball in $\Delta(\underline{s}, \bar{s})$ around $(\bar{x}, \bar{y})$ contains one point $(\hat{x}, \hat{y}) \in \Delta(\underline{s}, \bar{s})$ such that

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}) \ne \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{x}, \hat{y}).$$

We can connect $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$ with a path which belongs to the open ball composed of at least two parallel segments to the axes *Ox* and *Oy*. We can arrive to the point $(\bar{x}, \bar{y})$ from the right, in *x* or in *y*, and from the right continuity, on this segment the value is $\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y})$. It is therefore necessary that $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\cdot)$ be discontinuous along at least one of these segments, *i.e.*, that $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\cdot)$ admits at least one point of discontinuity, either with respect to *x* or with respect to *y*. □

Theorem 19. [4] For any $\bar{p} = (\bar{x}, \bar{y}) \in \Delta(\underline{s}, \bar{s})$, with $\bar{x} < \bar{y}$, there is $\varepsilon(\bar{p}) > 0$ such that the open set

$$W_{\varepsilon(\bar{p})}(\bar{p})=\{(x,y)\in\Delta(\underline{s},\bar{s}):|x-\bar{x}|<\varepsilon(\bar{p}),|y-\bar{y}|<\varepsilon(\bar{p}),x\neq\bar{x},y\neq\bar{y}\}$$

contains no point of discontinuity of the size function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\cdot)$.

Proof. If $\bar{p}$ is a point of continuity, the result follows. So let us suppose that $\bar{p}$ is a point of discontinuity and for any integer $n>0$ there is a point of discontinuity $p_n=(x_n,y_n)$ in the neighborhood $W_{1/n}(\bar{p})$. From Theorem 18, the points p_n can be supposed to be points of discontinuity with respect to *x* or to *y*. We can extract a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$, of $\{p_n\}_{n=1}^{\infty}$ such that p_{n_k} are all points of discontinuity with respect to *x* or all points of discontinuity with respect to *y*. Let us select the case where the p_{n_k} are points of discontinuity with respect to *x* (we could repeat the same step in case of discontinuity with respect to *y*). Fixing the integer *N* such that $\bar{x}+1/N<\bar{y}-1/N$, and let us consider the function

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\bar{y}-1/N):(\bar{x}-1/N,\bar{x}+1/N)\to\mathbb{N}.$$

We know that the discontinuity of the function $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\bar{y}-1/N)$ with respect to *x* is repeated for inferior values towards the diagonal. Consequently $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\bar{y}-1/N)$ as an infinite number of point of discontinuity. Otherwise, $\mathcal{L}_{(\mathcal{M},\varphi)}(\cdot,\bar{y}-1/N)$ is increasing with respect to *x*. Since each discontinuity implies an increasing value of at least 1, it follows that $\mathcal{L}_{(\mathcal{M},\varphi)}(\bar{x}+1/N,\bar{y}-1/N)=+\infty$. But this contradict the fact that $\mathcal{L}_{(\mathcal{M},\varphi)}(x,y)$ is finite for all $x<y$ (Theorem 10), hence the result follows. □

Let us look now at the closed subset of $\Delta(\underline{s},\bar{s})$ well defined for $\varepsilon>0$ by

$$\Delta_{\varepsilon}(\underline{s},\bar{s})=\{(x,y)\in\Delta(\underline{s},\bar{s})\mid x+\varepsilon\leq y\}$$

The diagonal of this set is given by $\{(x,y)\in\Delta(\underline{s},\bar{s})\mid x+\varepsilon=y\}$.

Theorem 20. For all $\varepsilon>0$, $\Delta_{\varepsilon}(\underline{s},\bar{s})$ contains a finite number of vertical lines having discontinuities with respect to *x* and a finite number of horizontal lines having discontinuities with respect to *y*. Consequently, $\Delta_{\varepsilon}(\underline{s},\bar{s})$ contains a finite number of intersections of vertical and horizontal lines which contain the discontinuities.

Proof. We cover the compact set $\Delta_{\varepsilon}(\underline{s},\bar{s})$ with the family $\{W_{\varepsilon(\bar{p})}(\bar{p})\}_{\bar{p}\in\Delta_{\varepsilon}(\underline{s},\bar{s})}$ for which we can extract a finite subcovering. Each element of this subcovering has a vertical segment and a horizontal segment which can both be extended in $\Delta(\underline{s},\bar{s})$ to the boundary. So there is a finite number of vertical and horizontal lines, and consequently of intersections, in $\Delta_{\varepsilon}(\underline{s},\bar{s})$ which contains all the discontinuities. □

Thanks to Theorems 13, 14, 15, and 19, we can now classify the intersections. To any intersection points of these lines $\hat{p}=(\hat{x},\hat{y})$, let us choose the neighborhood

$$W_{\varepsilon(\hat{p})}(\hat{p})=\{(x,y)\in\Delta(\underline{s},\bar{s}):|x-\hat{x}|<\varepsilon(\hat{p}),|y-\hat{y}|<\varepsilon(\hat{p}),x\neq\hat{x},y\neq\hat{y}\}$$

for an $\varepsilon(\hat{p})>0$ small enough such that this neighborhood contains no other vertical or horizontal lines of the preceding result except $x=\hat{x}$ and $y=\hat{y}$.

Consequently, around this point, the discontinuities will be on the horizontal and vertical segments passing through that intersection point.

We will discuss discontinuity on horizontal segments (increasing x and decreasing x, for $y=\hat{y}$) and vertical segments (increasing y and decreasing y, for $x=\hat{x}$) from $\hat{p}$. Let us consider the values of $\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{x},\hat{y})$ for the 4 following points in this neighborhood:

$$\hat{p}_{NE}=(\hat{x}_E,\hat{y}_N)=\left(\hat{x}+\frac{1}{2}\varepsilon(\hat{p}),\hat{y}+\frac{1}{2}\varepsilon(\hat{p})\right)$$

$$\hat{p}_{NO}=(\hat{x}_O,\hat{y}_N)=\left(\hat{x}-\frac{1}{2}\varepsilon(\hat{p}),\hat{y}+\frac{1}{2}\varepsilon(\hat{p})\right)$$

$$\hat{p}_{SO}=(\hat{x}_O,\hat{y}_S)=\left(\hat{x}-\frac{1}{2}\varepsilon(\hat{p}),\hat{y}-\frac{1}{2}\varepsilon(\hat{p})\right)$$

$$\hat{p}_{SE}=(\hat{x}_E,\hat{y}_S)=\left(\hat{x}+\frac{1}{2}\varepsilon(\hat{p}),\hat{y}-\frac{1}{2}\varepsilon(\hat{p})\right).$$

Figure 8 presents those points.

There are 7 cases to analyze. The first case corresponds to a point of continuity, and the six remaining cases correspond to points of discontinuity. Figure 9 presents each case.

Case (C). We have

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE})=\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE})=\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO})$$

and

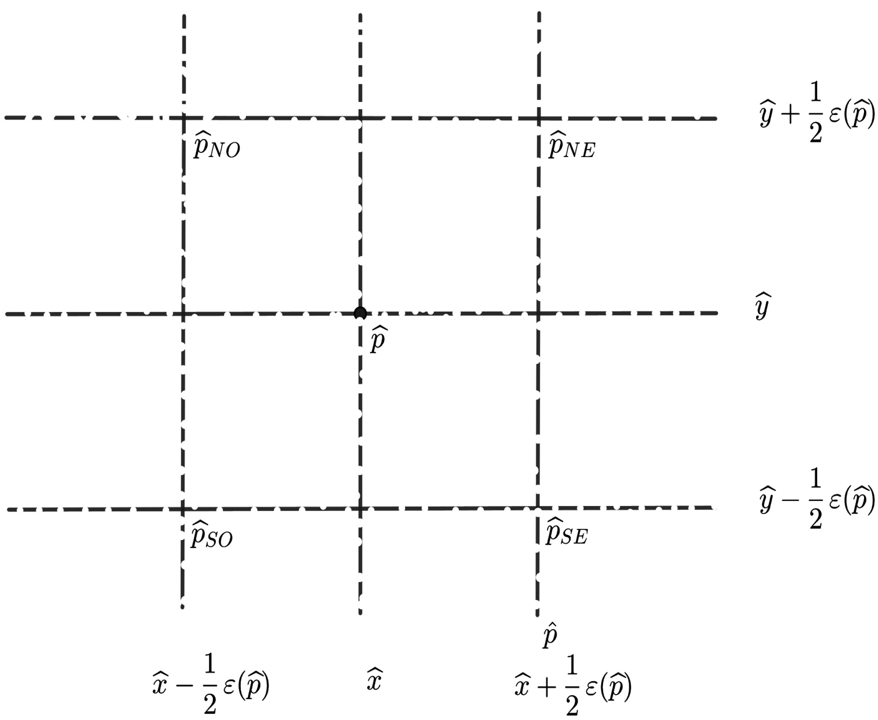

Figure 8. $W_{\varepsilon(\hat{p})}(\hat{p})$ neighborhood of a point $\hat{p}$.

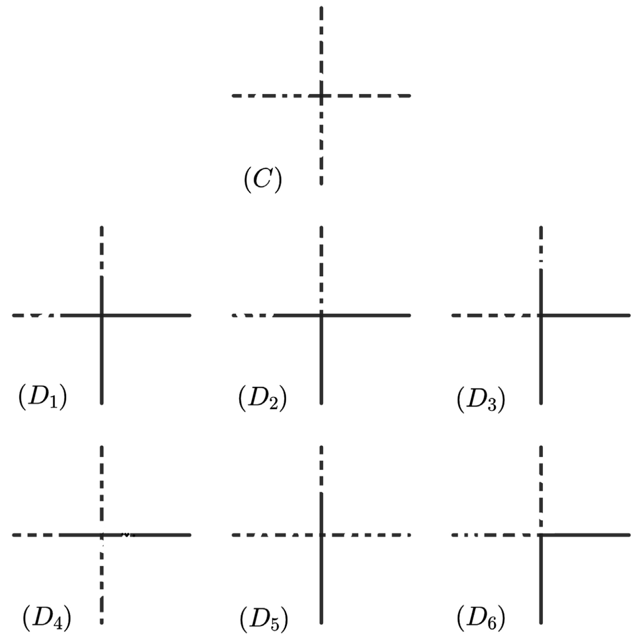

Figure 9. Possible configurations of an intersection point $\hat{p}$.

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

Decreasing x and increasing y segments contain no discontinuity up to the vertical boundary $x = \underline{s}$ or the horizontal boundary $y = \overline{s}$ of $\Delta(\underline{s},\overline{s})$. The increasing x and decreasing y segments contain no discontinuity up to the next intersection point on these segments or up to the diagonal boundary of $\Delta_\varepsilon(\underline{s},\overline{s})$.

Case (D_1). If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}),$$

and

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

The increasing x and decreasing y segments contain discontinuities up to the diagonal boundary $x = y$ of $\Delta(\underline{s},\overline{s})$. Decreasing x and increasing y segments contain discontinuities at least until the next intersection point on each of these segments or up to the vertical boundary $x = \underline{s}$ or the horizontal boundary $y = \overline{s}$ of $\Delta(\underline{s},\overline{s})$.

Case (D_2). If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}),$$

and

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

The increasing y segment contains no discontinuity up to the horizontal boundary $y = \overline{s}$ of $\Delta(\underline{s},\overline{s})$. For the other segments we have the same conclusions as in the Case (D_1).

Case (D_3). If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}),$$

and

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

The decreasing x segment contains no discontinuity up to the vertical boundary $x = \underline{s}$ of $\Delta(\underline{s}, \overline{s})$. For the other segments we have the same conclusions as in the Case (D_1).

Case (D_4). If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}),$$

and

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

The increasing y segment contains no discontinuity up to the horizontal boundary $y = \overline{s}$ of $\Delta(\underline{s}, \overline{s})$. The decreasing y segments contain no discontinuity up to the next intersection point on these segments or up to the diagonal boundary of $\Delta_\varepsilon(\underline{s}, \overline{s})$. The decreasing x segment contains discontinuities up to the next intersection point or up to the vertical boundary $x = \underline{s}$ of $\Delta(\underline{s}, \overline{s})$. The increasing x segment contains discontinuities up to the diagonal of $\Delta(\underline{s}, \overline{s})$.

Case (D_5). If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}),$$

and

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

The decreasing x segment contains no discontinuity up to the vertical boundary $x = \underline{s}$ of $\Delta(\underline{s}, \overline{s})$. The increasing x segment contains no discontinuity up to the next intersection point or up to the diagonal of $\Delta_\varepsilon(\underline{s}, \overline{s})$. The increasing y segment contains discontinuities up to the next intersection point or up to the horizontal boundary $y = \overline{s}$ of $\Delta(\underline{s}, \overline{s})$. The decreasing y segment contains discontinuities up to the diagonal of $\Delta(\underline{s}, \overline{s})$.

Case (D_6). If

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NE}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}),$$

and

$$\mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SE}) > \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{SO}) = \mathcal{L}_{(\mathcal{M},\varphi)}(\hat{p}_{NO}).$$

The decreasing x segment contains no discontinuity up to the vertical boundary $x = \underline{s}$ of $\Delta(\underline{s}, \overline{s})$. The increasing x segment contains discontinuities up to the diagonal of $\Delta(\underline{s}, \overline{s})$. The increasing y segment contains no discontinuity up to the horizontal boundary $y = \overline{s}$ of $\Delta(\underline{s}, \overline{s})$. The decreasing y segment contains discontinuities up to the diagonal of $\Delta(\underline{s}, \overline{s})$.

8. Conclusion

We did an overview of the main properties of the size functions by giving basic demonstrations of each of the results using elementary topology. In particular, we presented the decomposition of the size function (Theorem 9) and established results of continuity to the right of the size function (Theorem 16 and Theorem 17) under an assumption that the set $\mathcal{M}$ is calm. This assumption could be removed but a proof of the right continuity with respect to the y variable would be much more difficult to establish [8]. An interesting problem would be to find a way to approximate a set by a calm set and study the difference between the two size functions. The reader interested by a survey using more advanced tools of topology, like Morse theory, could consider [9]. A modern approach used in topological data analysis is to use persistent homology and Betti numbers [10].

Acknowledgements

This work has been financially supported by an individual discovery grant from the Natural Sciences and Engineering Research Council of Canada.

Conflicts of Interest

Author declares no competing interests.

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