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New Necessary Conditions for a Fixed-Point of Maps in Non-Metric Spaces

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Abstract

Our purpose is to introduce new necessary conditions for a fixed point of maps on non-metric spaces. We use a contraction map on a metric topological space and a lately published definition of limit of a function between the metric topological space and the non-metric topological space. Then we show that we can create a function h on the non-metric space Y, $h: Y \to Y$ and present necessary conditions for a fixed point of this map on this map on Y. Therefore, this gives an opportunity to take a best conclusion in some sense, when non-metrizable matter is under consideration.

Keywords

Topological Space, Compact Metric Space, Fixed Point, Contraction Map, Non-Metric Space

1. Introduction

Classification in non-metric spaces is considered before (ref. [1]). Fixed point sets of non-metric spaces were also under interest (ref. [2]).

With this work, we introduce new necessary conditions for a fixed point of maps on non-metric spaces. We use a contraction map on a metric topological space and a lately published definition of limit of a function between the metric topological space and the non-metric topological space. Then we show that we can create a function h on the non-metric space Y, $h: Y \to Y$ and present necessary conditions for a fixed point of this map on Y.

For that purpose, we denote by X a compact metric topological space and $f: X \to X$ a contraction map of X onto X. We suppose that Y is a bounded closed non-metric space and $g: X \to Y$ is a map from X to Y satisfying Definition 3.

We remind next basic definitions and theorems:

Definition 1. Contraction Mapping

Let (*X*, *d*) be a complete metric space. Then the map $T: X \to X$ is called a contraction map on *X* if there exists $q \in [0,1)$ such that

$$d(T(x), T(y)) \le qd(x, y)$$

for all $x, y \in X$ (ref. [3], ref. [4], ref. [5], ref [6], ref. [7], ref. [8], ref. [9]).

We remind that Banach contraction principle for multivalued maps is valid and also the next.

Theorem, proved by H. Covitz and S. B. Nadler Jr. (ref. [9]).

Theorem 1. Let (*X*, *d*) be a complete metric space and $F: X \to B(X)$ a contraction map. (*B*(*X*) denotes the family of all nonempty closed bounded (compact) subsets of *X*.) Then there exists $x \in X$ such that $x \in F(x)$.

Definition 2. Attracting Fixed Points

An *attracting fixed point* of a function *f* is a fixed point x_0 of *f* such that for any value of *x* in the domain that is close enough to x_0, the iterated function sequence

$$x, f(x), f(f(x)), f(f(f(x))), \cdots$$

converges to x_0 (ref. [9]).

Theorem 2. Banach Fixed Point Theorem.

Let (*X*, *d*) be a non-empty complete metric space with a contraction mapping $T: X \to X$. Then T admits a unique fixed-point x^* in *X* (*i.e.* $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_n\}_{n \in N}$ by $x_n = T(x_{n-1})$ for $n \ge 1$. Then $\lim_{n \to \infty} x_n = x^*$ (ref. [3], ref. [4], ref. [5], ref. [6], ref. [7], ref. [8], ref. [9]).

Definition 3. Let $g: X \to Y$ be a function between a metric topological space *X* and non-metric topological space *Y*. We say that the limit of *g* at a point $x \in X$ is the point $y \in Y$ if for all neighborhoods *N* of *y* in *Y*, there exists a neighborhood *M* of *x* such that $g(M) \subset N$ (ref. [10]).

2. Main Result

We consider now the next theorem:

Theorem 3. Let *X* denote a non-empty compact metric topological space with a contraction set-valued map $f: X \to X$.

Let *Y* is a bounded closed non-metric topological space.

We suppose also that the map:

$g: X \to Y$ exists and satisfies Definition 3.

Then we can construct a fixed-point of map in *Y*, $h: Y \to Y$.

Proof. If $x^* \in X$ is a fixed-point for *f* (*i.e.* $f(x^*) = x^*$), $I \subset X$ is a neighborhood close enough of x^*. Let $x_0 \in I$ close enough to x^* and we suppose that that the contracting map *f* will satisfy Banach Fixed Point Theorem and the iterated function sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \cdots$$

will satisfy Definition 2 and will converge to x^*. Therefore x^* is an attracting fixed point of *f*. Let us denote $x_1 \in f(x_0)$, $x_2 \in f(x_1) = f(f(x_0))$, $x_3 \in f(x_2) = f(f(f(x_0)))$, and so on, or $x_{i+1} \in f(x_i), i = 0,1,2,3,\cdots$. Hence we created a sequence $\{x_i\}$ such that $\lim_{i\to\infty} x_i = x^*$ and $f(x^*) = x^*$.

We suppose now that a function $g: X \to Y$ exists and satisfies Definition 3 and the limit of $g(x)$ at the point $x^* \in X$ is the point $y^* \in Y$. According to Definition 3, a corresponding neighborhood M_0 of x^* to a neighborhood $N_0 \subset Y$ of $y^* \in Y$, $g(M_0) \subset N_0$, can be chosen such that it will contain the sequence $\{x_i\}_{i=0}^{\infty}$. We can find also a neighborhood $M_1 \subset M_0$ of x^* containing only the sequence $\{x_i\}_{i=1}^{\infty}$, such that $g(M_0 \setminus M_1) \subset N_0$ and $x_0 \in M_0 \setminus M_1$, and also a neighborhood $M_2 \subset M_1$ of x^* containing only the sequence $\{x_i\}_{i=2}^{\infty}$, such that $g(M_1 \setminus M_2) \subset N_0$, where $x_1 \in M_1 \setminus M_2$. This process of creating neighborhoods M_k of x^* can continue such that each M_k will contain only the corresponding sequence $\{x_i\}_{i=k}^{\infty}$, $x_{i-1} \in M_{i-1} \setminus M_i$, $g(M_{i-1} \setminus M_i) \subset N_0$, and so on. We created a sequence $\{M_i\}$ of neighborhoods of x^*. According to their construction neighborhoods M_i are closer and closer to x^* when *i* is larger and larger.

A correspondent sequence of neighborhoods $\{N_i\}$ of $y^* \in Y$ can be created also such that $g(M_i) \subset N_i$.

We can choose $N_{i+1} \subset N_i$ according to Definition 3, because by construction $M_{i+1} \subset M_i$ and *g*(*x*) has the limit the $y^* \in Y$ at the point $x^* \in X$, and therefore $g(M_{i+1}) \subset g(M_i)$.

Therefore, we can choose a sequence of neighborhoods $\{N_i\}$ of $y^* \in Y$ such that $g(M_i) \subset N_i$. Because the function *g*(*x*) has a limit $y^* \in Y$ as *x* approaches $x^* \in X$ then N_i from the correspondent sequence of neighborhoods $\{N_i\}$ becomes smaller and smaller and closer to $y^* \in Y$. By construction $y_i \in g(x_i)$, $x_i \in M_i \setminus M_{i+1}$, and therefore $y_i \in N_i \setminus N_{i+1}$.

It follows from Definition 3 that:

$\lim_{x_i \to x^*} g(x_i) = g(x^*) = y^* = \lim_{x_i \to x^*} y_i = y^*$. It means that when N^* is the only point y^* then M^* will be only the point x^* and then $g(x^*) = y^*$.

Therefore, by using the sequence $\{y_i\}$, we can introduce the function $h: Y \to Y$, where $y_0, h(y_0), h(h(y_0)), h(h(h(y_0))), \cdots$.

If we denote $y_1 \in h(y_0)$, $y_2 \in h(y_1) = h(h(y_0))$, $y_3 \in h(y_2) = h(h(h(y_0)))$, and so on, or $y_{i+1} \in h(y_i), i = 0,1,2,3,\cdots$, for which $h(y_i) \to y^*$. Therefore the iterated function sequence $\{h(y_i)\}$ will have a fixed point y^*, or $h(y^*) = y^*$, if N^* contains the only point y^*.

Because every sequence $\{y_i\}$ constructed by this way will have the same limit y^* then y^* will be the fixed point of the so constructed function $h(y)$, $h(y^*) = y^*$. □

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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