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The Proof and Application of a Summation Formula

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Abstract

In this paper, some conclusions related to the prime number theorem, such as the Mertens formula are improved by the improved Abelian summation formula, and some problems such as "Dirichlet" function and "$W(n)$" function are studied.

Keywords

Abel Summation Formula, Mertens a Prime Number Theorem, Dirichlet Function, Conclusion Improvement

1. The Main Conclusions to Be Used in This Paper

1.1. Theorem A (Be Summation Formula) [1]

$b(n)(n=1,2,\cdots)$ is a plural column, The and function $B(n)=\sum_{n\le u} b(n)$, to set up $0\le u_1<u_2$, $f(u)$ interval $[u_1,u_2]$ continuous differentiable function of, so there are

$$\sum_{u_1<n\le u_2} b(n)f(n)=B(u_2)f(u_2)-B(u_1)f(u_1)-\int_{u_1}^{u_2} B(u)f'(u)\mathrm{d}u \qquad (1)$$

special: if $u_1=1,u_2=u>1$, Have a type:

$$\sum_{1\le n\le u} b(n)f(n)=B(u)f(u)-\int_1^u B(t)f'(t)\mathrm{d}t \qquad (2)$$

1.2. Theorem B (Prime Number Theorem) [2]

1) A. Walfsz the results of the:

$$\theta(x)=x+O\left(x\exp\left(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$\pi(x)=Lix+O\left(x\exp\left(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right)$$

2) In "Riemann" The prime number theorem under the condition that the conjecture is true (Vonkock the results of the).

$$\pi(x)=Lix+O\left(x^{\frac{1}{2}}\log x\right)$$

$$\theta(x)=x+O\left(x^{\frac{1}{2}}\log^2 x\right)$$

3) Theorem C (Siegel-Walfisz) [3]

Set l, k is suitable for $(l,k)=1$ and $3\le k\le(\log x)^{k_0}$. The natural Numbers, Among them, k_0 is any normal number, then:

$$\pi(x,k,l)=\frac{1}{\varphi(k)}lix+O\left(xe^{-a_0\sqrt{\log x}}\right),$$

Here $a_0>0$, And with the "O" the relevant constants depend only on k_0.

4) Theorem D (Mertens 1874)

If $L(1,x)\ne 0$, for any model q Non-principal features hold, so for any of these $(a,q)=1$ the a there are:

$$\sum_{\substack{p\le x\\ p\equiv a(\bmod q)}}\frac{1}{p}=\frac{\log\log x}{\varphi(q)}+c(a,q)+O\left(\frac{1}{\log x}\right)(x\to\infty),$$

Among them $c(a,q)=\frac{1}{\varphi(q)}\left\{\gamma-\sum_p\left(\log\left(\frac{1}{1-\frac{1}{p}}\right)-\frac{1}{p}\right)+\sum_{x\ne x_0}\bar{x}(a)\sum_p\frac{x(p)}{p}\right\}.$

2. The Proof of Summation Formula

2.1. Theorem 1

Set $\alpha(n)$, $f(n)$, for the real function, $A(x)=\sum_{n\le x}a(n)=g(x)+O(g_0(x))$,

Meet the conditions:

1) $g(x),g_0(x),f(x),f'(x)$ is interval $t\in[y,\infty)$, continuous function of;

2) For any small positive number ε, in $t\in[y,\infty)$ Satisfy condition on: $|g_0(t)f'(t)|\ll\left(t(\log t)^{1+\varepsilon}\right)^{-1}$.

2.1.1. When $1\le y\lll 1$ When

$$\sum_{y<n\le x}a(n)f(n)=g(x)f(x)-\int_y^x g(t)f'(t)\mathrm{d}t-A(y)f(y)$$
$$-\int_y^\infty(A(t)-g(t))f'(t)\mathrm{d}t+O\left(\int_x^\infty|g_0(t)f'(t)|\mathrm{d}t+|g_0(x)f(x)|\right) \tag{3}$$

Among them $A(y)f(y)+\int_y^\infty(A(t)-g(t))f'(t)\mathrm{d}t$, for only with "$y$" related constants.

2.1.2. When *y* Sufficiently Large

$$\sum_{y<n\le x} a(n) f(n) = g(x) f(x) - \int_y^x g(t) f'(t)\mathrm{d}t - g(y) f(y) + O\left(\int_y^x |g_0(t) f'(t)|\mathrm{d}t + |g_0(x) f(x)| + |g_0(y) f(y)|\right) \tag{4}$$

Prove:

1) When $1<y\ll 1<x$ when, by "(1) type" available

$$\sum_{y<n\le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\mathrm{d}t$$
$$= g(x) f(x) + O(|g_0(x) f(x)|) - A(y) f(y)$$
$$-\int_y^x (A(t)-g(t)) f'(t)\mathrm{d}t - \int_y^x g(t) f'(t)\mathrm{d}t$$
$$= g(x) f(x) - \int_y^x g(t) f'(t)\mathrm{d}t - A(y) f(y) - \int_y^\infty (A(t)-g(t)) f'(t)\mathrm{d}t$$
$$+\int_x^\infty (A(t)-g(t)) f'(t)\mathrm{d}t + O(|g_0(x) f(x)|)$$
$$= g(x) f(x) - \int_y^x g(t) f'(t)\mathrm{d}t - A(y) f(y) - \int_y^\infty (A(t)-g(t)) f'(t)\mathrm{d}t$$
$$+O\left(\int_x^\infty |g_0(t) f'(t)|\mathrm{d}t + |g_0(x) f(x)|\right)$$

Due to the $1<y\ll 1$, due to the

$$A(y) f(y) + \int_y^\infty (A(t)-g(t)) f'(t)\mathrm{d}t \ll 1 + \int_y^\infty |(A(t)-g(t)) f'(t)|\mathrm{d}t$$
$$\ll 1 + \int_y^\infty |g_0(t) f'(t)|\mathrm{d}t \ll 1 + \int_y^\infty \left(t(\log t)^{1+\varepsilon}\right)^{-1}\mathrm{d}t \ll 1 + \log^{-\varepsilon} y \ll 1$$

Namely: $A(y) f(y) + \int_y^\infty (A(t)-g(t)) f'(t)\mathrm{d}t$ For only with "*y*" Related constants.

2) When $1<y<x$, and y Sufficiently large, by "(1) type" known:

$$\sum_{1<y<n\le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\mathrm{d}t$$
$$= g(x) f(x) - g(y) f(y) - \int_y^x g(t) f'(t)\mathrm{d}t - \int_y^x (A(t)-g(t)) f'(t)\mathrm{d}t$$
$$+O(|g_0(x) f(x)| + |g_0(y) f(y)|)$$
$$= g(x) f(x) - g(y) f(y) - \int_y^x g(t) f'(t)\mathrm{d}t$$
$$+O\left(\int_y^x |g_0(t) f'(t)|\mathrm{d}t + |g_0(x) f(x)| + |g_0(y) f(y)|\right)$$

Notes: Set some strings attached, Such as when $x\ge t\ge T$ (Among them:

$T > y$) when, $\left|g_0(t) f'(t)\right| \ll \left(t(\log t)^{1+\varepsilon}\right)^{-1}$ To set up, At this time

" $A(y) f(y)+\int_y^{\infty}(A(t)-g(t)) f'(t) \mathrm{d} t$ " As with the "y" "T" Related constants.

For not meeting the conditions $\left|g_0(t) f'(t)\right| \ll \left(t(\log t)^{1+\varepsilon}\right)^{-1}$ can be summed as follows:

2.2. Theorem1'

Set $a(n)$ It's an arithmetic function, $A(x)=\sum_{n \leq x} a(n)=g(x)+O\left(g_0(x)\right)$,

Meet the conditions: $g(x), g_0(x), g_0'(x), f(x), f'(x)$ interval $[y, \infty)$.
Is a continuous function of, then

$$\begin{aligned} & \sum_{1<y<n \leq x} a(n) f(n) \\ & =\int_y^x g'(t) f(t) \mathrm{d} t+O\left(\int_y^x\left|g_0(t) f'(t)\right| \mathrm{d} t+\left|g_0(x) f(x)\right|+\left|g_0(y) f(y)\right|\right) \end{aligned} \tag{5}$$

Prove: by Abel the summation formula is easy to know:

$$\sum_{1<y<n \leq x} a(n) f(n)=\int_y^x f(t) \mathrm{d} A(t)=\int_y^x f(t) \mathrm{d}\left(g(t)+O\left(g_0(t)\right)\right)$$

$\Rightarrow$ conclusion.

3. The Proof of Several Common Conclusions about "Prime Number Theorem" [3]

Theorem 2 (Mertens Improvement of Prime Number Theorem) [4]

Verification: 1) $\sum_{p \leq x} \frac{\log p}{p}=\log x+A_1+O\left(\exp \left(-c_1 \log^{\frac{3}{5}} x(\log \log x)^{-\frac{1}{5}}\right)\right)$,

2) $\sum_{p \leq x} \frac{1}{p}=\log x+A_2+O\left(\exp \left(-c_2 \log^{\frac{3}{5}} x(\log \log x)^{-\frac{1}{5}}\right)\right)$,

3) $\prod_{p \leq x}\left(1-\frac{1}{p}\right)=\frac{\mathrm{e}^{-Y}}{\log x}+O\left(\exp \left(-c_3 \log^{\frac{3}{5}} x(\log \log x)^{-\frac{1}{5}}\right)\right)$,

Among them A_1, A_2. For constant, $c_i > 0$ ($i = 1, 2, 3$), Y for Euler constant.

Prove: From the prime number theorem:

$$\theta(x)=\sum_{p \leq x} \log p=x+O\left(x \exp \left(-c \log^{\frac{3}{5}} x(\log \log x)^{-\frac{1}{5}}\right)\right),$$

Make $A(x)=\theta(x)$, $g(x)=x$, $g_0(x)=x \exp \left(-c \log^{\frac{3}{5}} x(\log \log x)^{-\frac{1}{5}}\right)$,

$f(x)=\frac{1}{x}$, $f'(x)=-\frac{1}{x^2}$, Easy card with it.

The foot "(3) type" all conditions of, by "(3) type" available:

$$\sum_{1<p\le x}\frac{\log p}{p}=1+\int_1^x\frac{1}{t}\mathrm{d}t+\int_1^\infty\frac{\theta(t)-t}{t^2}\mathrm{d}t$$
$$+O\left(\int_x^\infty\frac{\exp\left(-c\log^{\frac{3}{5}}t(\log\log t)^{-\frac{1}{5}}\right)}{t}\mathrm{d}t+\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

(when $c>c_1>0$ when, Easy card:

$$\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\log^2 x\ll\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right))$$

$$=1+\log x+\int_1^\infty\frac{\theta(t)-t}{t^2}\mathrm{d}t$$
$$+O\left(\int_x^\infty\frac{\exp\left(-c_1\log^{\frac{3}{5}}t(\log\log t)^{-\frac{1}{5}}\right)}{t\log^2 t}\mathrm{d}t+\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$=1+\log x+\int_1^\infty\frac{\theta(t)-t}{t^2}\mathrm{d}t$$
$$+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\int_x^\infty\frac{\mathrm{d}t}{t\log^2 t}+\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$
$$=1+\log x+\int_1^\infty\frac{\theta(t)-t}{t^2}\mathrm{d}t+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$
$$=\log x+\left(\int_1^\infty\frac{\theta(t)-t}{t^2}\mathrm{d}t+1\right)+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

If the:

$$A_1=\int_1^\infty\frac{\theta(t)-t}{t^2}\mathrm{d}t+1,$$

If the A_1 For constant.

2) $A(x)=\theta(x)$, $g(x)=x$, $g_0(x)=x\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)$,

$f(x)=\frac{1}{x\log x}$, $f'(x)=-\frac{\log x+1}{x^2\log^2 x}$,

It is easy to prove that it satisfies "(3) type" all conditions of, by "(3) type" available:

$$\sum_{1<P\le x}\frac{1}{p}=\frac{1}{2}+\sum_{2<p\le x}\frac{1}{p}=\frac{1}{2}+\frac{1}{\log x}+\int_2^x\frac{\log t+1}{t\log^2 t}\mathrm{d}t-\frac{1}{2}+\int_2^\infty\frac{(\theta(t)-t)(\log t+1)}{t^2\log^2 t}\mathrm{d}t$$
$$O\left(\int_x^\omega\frac{\exp\left(-c\log^{\frac{3}{5}}t(\log\log t)^{-\frac{1}{5}}\right)(\log t+1)}{t\log^2 t}\mathrm{d}t+\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$= \log\log x + \left(\frac{1}{\log 2} - \log\log 2 + \int_2^{\infty} \frac{(\theta(t)-t)(\log t + 1)}{t^2 \log^2 t} \mathrm{d}t \right)$$

$$+ O\left(\exp\left(-c_2 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right)$$

Among them: $A_2 = \frac{1}{\log 2} - \log\log 2 + \int_2^{\infty} \frac{(\theta(t)-t)(\log t + 1)}{t^2 \log^2 t} dt$, for constant.

3) By "Mertens" Formula to know: $\sum_{P \le x} \frac{1}{p} = \log\log x + B_1 + O\left(\frac{1}{\log x} \right)$, among them $B_1 = Y + \sum_P \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right)$.

Knowing from the evidence before:

$$\sum_{p \le x} \frac{1}{p} = \log\log x + A_2 + O\left(\exp\left(-c_2 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right)$$

$$\Rightarrow B_1 - A_2 = O\left(\frac{1}{\log x} \right) = 0 \quad (x \to \infty \quad \text{when})$$

$$\Rightarrow \text{constant } A_2 = B_1 = Y + \sum_P \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right)$$

$$\Rightarrow \sum_{p \le x} \frac{1}{p} = \log\log x + Y + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right)$$

$$+ O\left(\exp\left(-c_2 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right)$$

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \exp\left(\sum_{p \le x} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) - \sum_{p \le x} \frac{1}{p} \right)$$

$$\Rightarrow \prod_{p \le x} \left(1 - \frac{1}{p}\right) = \exp\left(-\log\log x - Y - \sum_{p > x} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) \right.$$

$$\left. + O\left(\exp\left(-c_2 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right) \right)$$

$$= \frac{e^{-Y}}{\log x} \exp\left(\sum_{p > x} O\left(\frac{1}{p^2} \right) + O\left(\exp\left(-c_2 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right) \right)$$

$$= \frac{e^{-Y}}{\log x} \exp\left(O\left(\exp\left(-c_2 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right) \right)$$

$$= \frac{e^{-Y}}{\log x} + O\left(\exp\left(-c_3 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}} \right) \right)$$

Inference 1:

Verification: 1) $\sum_{y < p \le x} \frac{\log p}{p} = \log\left(\frac{x}{y} \right) + O\left(\exp\left(-c_1 \log^{\frac{3}{5}} y (\log\log y)^{-\frac{1}{5}} \right) \right)$,

2) $\sum_{y<p\le x}\frac{1}{p}=\log\left(\frac{\log x}{\log y}\right)+O\left(\exp\left(-c_2\log^{\frac{3}{5}}y\left(\log\log y\right)^{-\frac{1}{5}}\right)\right)$,

3) $\prod_{y<p\le x}\left(1-\frac{1}{p}\right)=\frac{\log y}{\log x}\left(1+O\left(-c_3\log^{\frac{3}{5}}y\left(\log\log y\right)^{-\frac{1}{5}}\right)\right)$.

Prove: It can be derived directly from theorem two.

Inference 2: Two identity relations:

1) $\int_2^{\infty}\frac{\theta(t)-\pi(t)(\log t-1)}{t^2}\mathrm{d}t=0$,

2) $\int_2^{\infty}\frac{\theta(t)(\log t-1)-\pi(t)\log^2 t}{t^2\log^2 t}\mathrm{d}t=0$.

Proof: if so

$$A(x)=\pi(x)=\frac{x}{\log x}+\frac{x}{\log^2 x}+O\left(\frac{x}{\log^3 x}\right),$$

$$g(x)=\frac{x}{\log x}+\frac{x}{\log^2 x},g_0(x)=\frac{x}{\log^3 x}$$

1) Take $f(x)=\frac{\log x}{x}$, using "(3) type" $\sum_{1\le p\le x}\frac{\log p}{p}$, available

$$A_1=\int_1^{\infty}\frac{(\log t-1)\pi(t)-t}{t^2}\mathrm{d}t+1.$$

2) Take $f(x)=\frac{1}{x}$, using "(3) type" $\sum_{1\le p\le x}\frac{1}{p}$, available

$$A_2=\int_2^{\infty}\frac{\pi(t)-\frac{t}{\log t}}{t^2}\mathrm{d}t-\log\log 2.$$

Take the above A_1, A_2 is compared with the corresponding value in "Therorem 2".

Inference 3: A_1, A_2 other forms of values:

1) Take $f(x)$,

$$A_1=\int_1^{\infty}\frac{\theta(t)-t}{t^2}\mathrm{d}t+1=\int_1^{\infty}\frac{(\log t-1)\pi(t)-t}{t^2}\mathrm{d}t+1=-Y-\sum_p\frac{\log p}{p(p-1)}\approx-1.332,$$

2)

$$A_2=\int_2^{\infty}\frac{(\theta(t)-t)(\log t+1)}{t^2\log^2 t}\mathrm{d}t+\frac{1}{\log 2}-\log\log 2$$

$$=\int_2^{\infty}\frac{\pi(t)-\frac{t}{\log t}}{t^2}\mathrm{d}t-\log\log 2=Y+\sum_p\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\approx 0.216$$

Proof: a little.

Note: For A_1, A_2, the various expressions of value are derived from different methods used to prove "Theorem two", some of which we have proved, some of which we will prove later, but it is worth mentioning that many times we just need to know that it is a constant.

Inference 4: An improvement on theorem D, Make $f(t)=\frac{1}{t}, g(t)=\frac{lit}{\varphi(q)}, g_0(t)=te^{-a_0\sqrt{\log t}}$,

Easy card: $\left|g_0(t)f'(t)\right| \ll \left(t(\log t)^{1+\varepsilon}\right)^{-1}$ ($\varepsilon>0$ when),

If the conditions of "Theorem 1" are met, the following can be obtained from "Theorem 1" and "Theorem C":

$$\sum_{\substack{p\le x\\ p\equiv a(\bmod q)}}\frac{1}{p}=\sum_{\substack{2<p\le x\\ p\equiv a(\bmod q)}}\frac{1}{p}+\frac{\pi(2,q,a)}{2}$$

$$=\frac{lix}{\varphi(q)}\cdot\frac{1}{x}-\int_2^x\frac{lit}{\varphi(q)}\cdot\left(-\frac{1}{t^2}\right)\mathrm{d}t-\int_2^\infty\left(\pi(t,q,a)-\frac{lit}{\varphi(q)}\right)\left(-\frac{1}{t^2}\right)\mathrm{d}t$$

$$+O\left(\int_x^\infty\left|te^{-a_0\sqrt{\log t}}\cdot\frac{1}{t^2}\right|\mathrm{d}t+\left|xe^{-a_0\sqrt{\log t}}\cdot\frac{1}{x}\right|\right)$$

$$=\frac{\log\log x}{\varphi(q)}+\frac{li2}{2\varphi(q)}-\frac{\log\log 2}{\varphi(q)}+\int_2^\infty\frac{\pi(t,q,a)-\frac{lit}{\varphi(q)}}{t^2}\mathrm{d}t+O\left(e^{-a_0'\sqrt{\log x}}\right)(a_0'>0)$$

It is easy to know from the above formula and "Theorem D":

$$c(a,q)=\frac{li2}{2\varphi(q)}-\frac{\log\log 2}{\varphi(q)}+\int_2^\infty\frac{\pi(t,q,a)-\frac{lit}{\varphi(q)}}{t^2}\mathrm{d}t,$$

Namely:

$$\sum_{\substack{p\le x\\ p\equiv a(\bmod q)}}\frac{1}{p}=\frac{\log\log x}{\varphi(q)}+c(a,q)+O\left(e^{-a_0'\sqrt{\log x}}\right)(a_0'>0).$$

Theorem 2: if "Riemann" If the guess is true, then:

1) $\sum_{p\le x}\frac{\log p}{p}=\log x+A_1+O\left(x^{-\frac{1}{2}}\log^2 x\right)$,

2) $\sum_{p\le x}\frac{1}{p}=\log\log x+A_2+O\left(x^{-\frac{1}{2}}\log x\right)$,

3) $\prod_{p\le x}\left(1-\frac{1}{p}\right)=\frac{e^{-Y}}{\log x}+O\left(x^{-\frac{1}{2}}\right)$.

Prove: using "Riemann" If the guess is true, then "$\theta(x)=x+O\left(x^{\frac{1}{2}}\log^2 x\right)$".

This conclusion, the proof process is the same as "Theorem 2", so it is omitted.

Theorem 3: 1) $\sum_{p\le x}\frac{\wedge(n)}{n}=\log x-Y+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$,

2) $\sum_{p\le x}\frac{\log p}{p-1}=\log x-Y+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$.

Prove: 1) The first $A_1=-Y-\sum_p\frac{\log p}{p(p-1)}$,

$$\sum_{1\le n\le x}\log n=\sum_{1\le p\le x}\left[\frac{x}{p}\right]\log p+\sum_{1\le p\le\sqrt{x}}\left(\left[\frac{x}{p^2}\right]+\left[\frac{x}{p^3}\right]+\cdots+\left[\frac{x}{p^{[\log_p x]}}\right]\right)\log p$$

$$=\sum_{1\le p\le x}\left[\frac{x}{p}\right]\log p+\sum_{1\le p\le\sqrt{x}}\left(\frac{x}{p^2}+\frac{x}{p^3}+\cdots+\frac{x}{p^{[\log_p x]}}+O(\log_p x)\right)\log p$$

$$=\sum_{1\le p\le x}\left[\frac{x}{p}\right]\log p+\sum_{1\le p\le\sqrt{x}}\left(\frac{x}{p(p-1)}\log p+O(\log x)\right)$$

$$=\sum_{1\le p\le x}\left[\frac{x}{p}\right]\log p+\sum_{1\le p\le\sqrt{x}}\frac{x\log p}{p(p-1)}+O(\sqrt{x}) \tag{6}$$

$$=\sum_{1\le p\le\sqrt{x}}\left[\frac{x}{p}\right]\log p+\sum_{1\le n\le\sqrt{x}}\left(\theta\left(\frac{x}{n}\right)-\theta(\sqrt{x})\right)+\sum_{1\le p\le\sqrt{x}}\frac{x\log p}{p(p-1)}+O(\sqrt{x})$$

$$=x\sum_{1\le p\le\sqrt{x}}\frac{\log p}{p}+\sum_{1\le n\le\sqrt{x}}\theta\left(\frac{x}{n}\right)-\sqrt{x}\theta(\sqrt{x})+x\sum_{p}\frac{\log p}{p(p-1)}+O(\sqrt{x})$$

A prime number theorem:

$$\theta(x)=x+O(x^{\theta})=x+O\left(x\exp\left(-c\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$\Rightarrow\sum_{1\le n\le x}\log x=x\sum_{1\le p\le\sqrt{x}}\frac{\log p}{p}+x\sum_{1\le n\le\sqrt{x}}\frac{1}{n}-x+x\sum_{p}\frac{\log p}{p(p-1)}$$
$$+O\left(\sqrt{x}+x^{\frac{1+\theta}{2}}+\sum_{1\le n\le\sqrt{x}}\left(\frac{x}{n}\right)^{\theta}\right)$$

$$=x\sum_{1\le p\le\sqrt{x}}\frac{\log p}{p}+x\left(\log\sqrt{x}+Y+O\left(\frac{1}{\sqrt{x}}\right)\right)-x+x\sum_{p}\frac{\log p}{p(p-1)}$$
$$+O\left(x\exp\left(-c_4'\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$=x\sum_{1\le p\le\sqrt{x}}\frac{\log p}{p}+x\left(\frac{1}{2}\log x+Y-1\right)+x\sum_{p}\frac{\log p}{p(p-1)}$$
$$+O\left(x\exp\left(-c_4'\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$\sum_{1\le n\le x}\log n=x\log x-x+O(\log x)$$

$$\Rightarrow\sum_{1\le p\le\sqrt{x}}\frac{\log p}{p}=\frac{1}{2}\log x-Y-\sum_{p}\frac{\log p}{p(p-1)}+O\left(\exp\left(-c_4'\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$\Rightarrow\sum_{1\le p\le x}\frac{\log p}{p}=\log x-Y-\sum_{p}\frac{\log p}{p(p-1)}+O\left(\exp\left(-c_4'\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

Theorem 2

$$\Rightarrow A_1=-Y-\sum_{p}\frac{\log p}{p(p-1)}$$

2) $$\sum_{1\le p\le x}\frac{\wedge(n)}{n}=\sum_{1\le p\le x}\frac{\log p}{p}+\sum_{1<p\le\sqrt{x}}\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots+\frac{1}{p^{[\log_p x]}}\right)\log p$$

$$=\sum_{1\le p\le x}\frac{\log p}{p}+\sum_{1\le p\le\sqrt{x}}\left(\frac{\log p}{p(p-1)}+O\left(\frac{\log p}{x}\right)\right)$$

$$=\sum_{1\le p\le x}\frac{\log p}{p}+\sum_{1\le p\le\sqrt{x}}\frac{\log p}{p(p-1)}+O\left(\frac{1}{\sqrt{x}}\right)$$

Theorem 2

$$\Rightarrow\sum_{1\le p\le x}\frac{\wedge(n)}{n}=\log x-Y-\sum_{p}\frac{\log p}{p(p-1)}+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$+\sum_{1<p\le\sqrt{x}}\frac{\log p}{p(p-1)}+O\left(\frac{1}{\sqrt{x}}\right)$$

$$=\log x-Y-\sum_{p>\sqrt{x}}\frac{\log p}{p(p-1)}+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$=\log x-Y+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

3) $$\sum_{1\le p\le x}\frac{\log p}{p-1}=\sum_{1\le p\le x}\frac{\log p}{p}+\sum_{1\le p\le x}\frac{\log p}{p(p-1)}$$

Theorem 2

$$\Rightarrow\sum_{1\le p\le x}\frac{\log p}{p-1}=\log x-Y-\sum_{p}\frac{\log p}{p(p-1)}+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$+\sum_{1<p\le x}\frac{\log p}{p(p-1)}$$

$$=\log x-Y-\sum_{p>x}\frac{\log p}{p(p-1)}+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

$$=\log x-Y+O\left(\exp\left(-c_1\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right)$$

Note: The paper proves "$A_1=-Y-\sum_{p}\frac{\log p}{p(p-1)}$". In fact, it is another proof method of theorem 2.

Theorem 3: If "Riemann" conjecture is true, then

1) $$\sum_{1\le n\le x}\frac{\wedge(n)}{n}=\log x-Y+O\left(x^{-\frac{1}{2}}\log^2 x\right).$$

2) $$\sum_{1\le n\le x}\frac{\log p}{p-1}=\log x-Y+O\left(x^{-\frac{1}{2}}\log^2 x\right).$$

Proof: using the "(1)" process in "Theorem 2" is the same as the proof of "Theorem 3".

Corollary 1: Let N be even,

$$\omega(p)=\frac{P}{P-1},(P,N)=1,$$

the

$$\sum_{\omega\le p<z}\frac{\omega(p)\log p}{p}$$

$$=\begin{cases}\log\left(\frac{z}{\omega}\right)+O(\log\log N);\ \text{when } \omega\le\frac{\log N}{\log\log N}\\ \log\left(\frac{z}{\omega}\right)+O\left(\frac{\log N}{\omega}\right);\ \text{when } \frac{\log N}{\log\log N}<\omega\le\log N\exp\left(c_1\left(\log^{\frac{2}{5}}\omega(\log\log\omega)^{-\frac{1}{5}}\right)\right)\\ \log\left(\frac{z}{\omega}\right)+O\left(\exp\left(-c_1\log^{\frac{1}{5}}\omega(\log\log\omega)^{-\frac{1}{5}}\right)\right);\ \text{when } \log N<\omega\exp\left(-c_1\log^{\frac{3}{5}}\omega(\log\log\omega)^{-\frac{1}{5}}\right)\end{cases}$$

Prove: Club meets the condition $1<\omega\le p<z$, $p\mid N$ In the "p" If the number of is k_0, then we know:

$$N=p_1^{l_1}p_2^{l_2}\cdots p_{k_0}^{l_{k_0}}\Rightarrow k_0\le k=\sum_{m=1}^{k_0}l_m\Rightarrow$$

$$\omega^k\le N\Rightarrow k\le\frac{\log N}{\log\omega},$$

$$\sum_{\substack{\omega\le p<Z\\ p\mid N}}\frac{\log p}{p-1}\ll\sum_{\substack{\omega\le p<z\\ p\mid N}}\frac{\log\omega}{\omega}\le k\frac{\log\omega}{\omega},$$

$$\Rightarrow\sum_{\substack{\omega\le p<z\\ p\mid N}}\frac{\log p}{p-1}\ll\frac{\log N}{\omega}.$$

Easy to prove:

$$\sum_{\substack{\omega\le p<z\\ p\mid N}}\frac{\log p}{p-1}\ll\log\log N\Rightarrow\sum_{\substack{\omega\le p<z\\ p\mid N}}\frac{\log P}{p-1}\ll\min\left(\frac{\log N}{\omega};\log\log N\right),$$

$$\sum_{\omega\le p<z}\frac{\omega(p)\log p}{P}=\sum_{\omega\le P<z}\frac{\log P}{P-1}-\sum_{\substack{\omega\le p<z\\ p\mid N}}\frac{\log p}{p-1}.$$

Theorem 3

$\Rightarrow$ Conclusion.

Note: The result we have obtained is actually an improvement on an important result cited in the screening method to prove goldbach's conjecture. Of course, there are several related formulas that can also be improved, which will not be described here.

4. The Application of Summation Formula in "Dirichlet" Function [5] [6]

Theorem 4: If the $\sum_{k\le x}d(k)=x(\log x+2y-1)+O(x^{\theta})\left(\frac{1}{4}\le\theta<\frac{1}{3}\right)$, the

1) $\sum_{k\le x}\frac{d(k)}{k}=\frac{1}{2}\log^2 x+2Y\log x+c_4+O\left(x^{\theta-1}\right)$ (c_4 For a constant),

2) $\sum_{k\le x}\left\{\frac{x}{k}\right\}=(1-Y)x+O\left(x^{\theta}\right)$.

Prove: 1) make $A(x)=\sum_{n\le x}d(n)$, $g(x)=x(\log x+2Y-1)$, $g_0(x)=x^{\theta}$, $f(x)=\frac{1}{x}$, $f'(x)=-\frac{1}{x^2}$, It is easy to prove that it satisfies "(3) type" All conditions of, by "(3) type" Have to:

$$\sum_{1\le k\le x}\frac{d(k)}{k}=1+\sum_{1<k\le x}\frac{d(k)}{k}$$

$$=1+(\log x+2Y-1)+\int_1^x\frac{\log t+2Y-1}{t}\mathrm{d}t-1$$

$$+\int_1^{\infty}\frac{\sum_{1\le k\le t}d(k)-t(\log t+2Y-1)}{t^2}\mathrm{d}t+O\left(\int_x^{\infty}t^{\theta-2}\mathrm{d}t+x^{\theta-1}\right)$$

$$=\frac{1}{2}\log^2 x+2Y\log x-1+2Y+\int_1^{\infty}\frac{\sum_{1\le k\le t}d(k)-t(\log t+2Y-1)}{t^2}\mathrm{d}t+O\left(x^{\theta-1}\right)$$

$$=\frac{1}{2}\log^2 x+2Y\log x+\left(\int_1^{\infty}\frac{\sum_{1\le k\le t}d(k)-t(\log t+2Y-1)}{t^2}\mathrm{d}t+2Y-1\right)+O\left(x^{\theta-1}\right)$$

The constant $c_4=\int_1^{\infty}\frac{\sum_{1\le k\le t}d(k)-t(\log t+2Y-1)}{t^2}\mathrm{d}t+2Y-1$.

2) $\sum_{k\le x}\left\{\frac{x}{k}\right\}=x\sum_{k\le x}\frac{1}{k}-\sum_{k\le x}d(k)$

$$=x\left(\log x+Y+O\left(\frac{1}{x}\right)\right)-\left(x(\log x+2Y-1)+O\left(x^{\theta}\right)\right)$$

$$=(1-Y)x+O\left(x^{\theta}\right)$$

Theorem 5:

If the $\sum_{k\le x}d(k)=x(\log x+2Y-1)+\Delta(x)$, the:

1)
$$\sum_{k=1}^{n}\frac{d(k)}{k}=\frac{1}{2}\log^2 n+2Y\log n+\left(Y^2-2\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t\right)+\sum_{k=1}^{\sqrt{n}}\frac{1-2\left\{\frac{n}{k}\right\}}{n}+O\left(\frac{\log n}{n}\right),$$

The constant $c_4=Y^2-2\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t$.

2) $\int_1^x\frac{\Delta(t)}{t^2}\mathrm{d}t=(1-Y)^2-2\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t+O\left(\frac{\log x}{x}\right)$,

3) $\int_1^x\Delta(t)\mathrm{d}t=O(x\log x)$,

Prove:

$$1)\quad \sum_{k=1}^{n}\frac{d(k)}{k}=\sum_{k=1}^{n}\frac{1}{k}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left[\frac{n}{k}\right]}\right)$$

$$=2\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left[\frac{n}{k}\right]}\right)-\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\left[\sqrt{n}\right]}\right)$$

$$=2\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\left(\log\left[\frac{n}{k}\right]+Y+\frac{1}{2\left[\frac{n}{k}\right]}+O\left(\frac{k^2}{n^2}\right)\right)-\left(\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\right)^2$$

$$=2\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\left(\log\left(\frac{n}{k}\right)+Y-\frac{\left\{\frac{n}{k}\right\}}{\frac{n}{k}}+\frac{1}{2(n/k)}+O\left(\frac{k^2}{n^2}\right)\right)-\left(\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\right)^2$$

$$=\left(2\log n+2Y-\sum_{k=1}^{\sqrt{n}}\frac{1}{k}\right)\cdot\sum_{k=1}^{\sqrt{n}}\frac{1}{k}-2\sum_{k=1}^{\sqrt{n}}\frac{\log k}{k}+\sum_{k=1}^{\sqrt{n}}\frac{1-2\left\{\frac{n}{k}\right\}}{n}+O\left(\frac{1}{n}\right)$$

$$=\left(\frac{3}{2}\log n+Y+\frac{\left\{\sqrt{n}\right\}-\frac{1}{2}}{\sqrt{n}}+O\left(\frac{1}{n}\right)\right)\left(\frac{1}{2}\log n-\frac{\left\{\sqrt{n}\right\}-\frac{1}{2}}{\sqrt{n}}+Y+O\left(\frac{1}{n}\right)\right)$$

$$-2\sum_{k=1}^{\sqrt{n}}\frac{\log k}{k}+\sum_{k=1}^{\sqrt{n}}\frac{1-2\left\{\frac{n}{k}\right\}}{n}+O\left(\frac{1}{n}\right)$$

$$=\left(\frac{3}{2}\log n+Y+\frac{\left\{\sqrt{n}\right\}-\frac{1}{2}}{\sqrt{n}}\right)\cdot\left(\frac{1}{2}\log n+Y-\frac{\left\{\sqrt{n}\right\}-\frac{1}{2}}{\sqrt{n}}\right)$$

$$-2\sum_{k=1}^{\sqrt{n}}\frac{\log k}{k}+\sum_{k=1}^{\sqrt{n}}\frac{1-2\left\{\frac{n}{k}\right\}}{n}+O\left(\frac{\log n}{n}\right)$$

$$=\frac{3}{4}\log^2 n+2Y\log n+Y^2-\frac{\left(\left\{\sqrt{n}\right\}-\frac{1}{2}\right)\log n}{\sqrt{n}}+\sum_{k=1}^{\sqrt{n}}\frac{1-2\left\{\frac{n}{k}\right\}}{n}$$
$$-2\sum_{k=1}^{\sqrt{n}}\frac{\log k}{k}+O\left(\frac{\log k}{n}\right)\tag{7}$$

while

$$\sum_{k=1}^{\sqrt{n}}\frac{\log k}{k}=\int_1^{\left[\sqrt{n}\right]+1}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t+\int_1^{\left[\sqrt{n}\right]+1}\frac{\log t}{t}\mathrm{d}t$$

$$=\frac{1}{2}\log^2\left(\left[\sqrt{n}\right]+1\right)+\int_1^{\left[\sqrt{n}\right]+1}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t$$

$$=\frac{1}{2}\log^2\left(\left[\sqrt{n}\right]+1\right)+\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t-\int_{\left[\sqrt{n}\right]+1}^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t \quad (8)$$

1) $\frac{1}{2}\log^2\left(\left[\sqrt{n}\right]+1\right)=\frac{1}{2}\left(\log\sqrt{n}+\frac{1-\left\{\sqrt{n}\right\}}{\sqrt{n}}+O\left(\frac{1}{n}\right)\right)^2$

$$=\frac{1}{8}\log^2 n+\frac{\left(1-\left\{\sqrt{n}\right\}\right)\log\sqrt{n}}{\sqrt{n}}+O\left(\frac{\log n}{n}\right)$$

2) $\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t \ll \int_1^{\infty}\left|\frac{\log t}{[t]}-\frac{\log t}{t}\right|\mathrm{d}t<\int_1^{\infty}\frac{\log t}{[t]t}\mathrm{d}t \ll \int_1^{\infty}\frac{\log t}{t^2}\mathrm{d}t \ll 1$

$$\Rightarrow \int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t$$

For constant

3) $\int_{\left\{\sqrt{n}\right\}+1}^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t=\int_{\left\{\sqrt{n}\right\}+1}^{\infty}\left(\frac{\log t-\frac{\{t\}}{t}+O\left(\frac{1}{t^2}\right)}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t$

$$=\int_{\left\{\sqrt{n}\right\}+1}^{\infty}\left(\frac{\{t\}(\log t-1)}{[t]t}+O\left(\frac{1}{t^3}\right)\right)\mathrm{d}t$$

$$=\int_{\sqrt{n}}^{\infty}\frac{\{t\}(\log t-1)}{[t]t}\mathrm{d}t+O\left(\frac{\log n}{n}\right)$$

$$=\int_{\sqrt{n}}^{\infty}\frac{\{t\}(\log t-1)}{t^2}\mathrm{d}t+O\left(\frac{\log n}{n}\right)$$

$$=\frac{1}{2}\int_{\sqrt{n}}^{\infty}\frac{\log t-1}{t^2}\mathrm{d}t+O\left(\frac{\log n}{n}\right)$$

$$=\frac{\log\sqrt{n}}{2\sqrt{n}}+O\left(\frac{\log n}{n}\right)$$

will (1), (2), (3) Can be substituted into Equation (8), available:

$$\sum_{k=1}^{\sqrt{n}}\frac{\log k}{k}=\frac{1}{8}\log^2 n+\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t+\frac{\left(\frac{1}{2}-\left\{\sqrt{n}\right\}\right)\log\sqrt{n}}{\sqrt{n}}+O\left(\frac{\log n}{n}\right)$$

Then substitute the above formula into "Formula (7)" and sort it out

$$\sum_{k=1}^{\sqrt{n}}\frac{d(k)}{k}=\frac{1}{2}\log^2 n+2Y\log n+\left(Y^2-2\int_1^{\infty}\left(\frac{\log[t]}{[t]}-\frac{\log t}{t}\right)\mathrm{d}t\right)+\sum_{k=1}^{\sqrt{n}}\frac{1-2\left\{\frac{n}{k}\right\}}{n}+O\left(\frac{\log n}{n}\right) \quad (9)$$

2) By

$$\sum_{k=1}^{n}d(k)=n(\log n+2Y-1)+\sqrt{n}-2\sum_{1\le k\le\sqrt{n}}\left\{\frac{x}{n}\right\}+O(1)$$

$$= n(\log n + 2Y - 1) + \sum_{1\le k\le\sqrt{n}} \left(1 - 2\left\{\frac{x}{n}\right\}\right) + O(1)$$

$$= \sum_{k=1}^{n} \frac{d(k)}{n} = \log n + 2Y - 1 + \sum_{1\le k\le\sqrt{n}} \frac{1-2\left\{\frac{n}{k}\right\}}{n} + O\left(\frac{1}{n}\right) \tag{10}$$

From "Formula (2)" in "Theorem A" $\Rightarrow \sum_{k=1}^{n}\frac{d(k)}{k} - \sum_{k=1}^{n}\frac{d(k)}{n} = \int_1^n \frac{\sum_{1\le k\le t} d(k)}{t^2}\mathrm{d}t$.

(9)

(10)

$$\Rightarrow \int_1^n \frac{\sum_{1\le k\le t} d(k)}{t^2} = \left(\frac{1}{2}\log^2 n + 2Y\log n + Y^2 - 2\int_1^\infty \left(\frac{\log[t]}{[t]} - \frac{\log t}{t}\right)\mathrm{d}t\right.$$

$$\left. + \sum_{k=1}^{\sqrt{n}} \frac{1-2\left\{\frac{n}{k}\right\}}{n} + O\left(\frac{\log n}{n}\right)\right) - \left(\log n + 2Y - 1 + \sum_{k=1}^{\sqrt{n}} \frac{1-2\left\{\frac{n}{k}\right\}}{n} + O\left(\frac{1}{n}\right)\right)$$

$$\Rightarrow \int_1^n \frac{\sum_{1\le k\le t} d(k)}{t^2}\mathrm{d}t = \frac{1}{2}\log^2 n + (2Y-1)\log n + (1-Y)^2$$

$$-2\int_1^\infty \left(\frac{\log[t]}{[t]} - \frac{\log t}{t}\right)\mathrm{d}t + O\left(\frac{\log n}{n}\right)$$

$$\Rightarrow \int_1^n \frac{\Delta(t)}{t^2}\mathrm{d}t = \int_1^n \frac{\sum_{1\le k\le t} d(k) - t(\log t + 2Y - 1)}{t^2}\mathrm{d}t$$

$$= (1-Y)^2 - 2\int_1^\infty \left(\frac{\log[t]}{[t]} - \frac{\log t}{t}\right)\mathrm{d}t + O\left(\frac{\log n}{n}\right).$$

namely,

$$\int_1^n \frac{\Delta(t)}{t^2}\mathrm{d}t = (1-Y)^2 - 2\int_1^\infty \left(\frac{\log[t]}{[t]} - \frac{\log t}{t}\right)\mathrm{d}t + O\left(\frac{\log n}{n}\right) \tag{11}$$

3) (1) by

$$\sum_{1\le t\le x}\frac{\Delta(t)}{t^2} - \int_1^x \frac{\Delta(t)}{t^2}\mathrm{d}t = \int_1^\infty \left(\frac{\Delta([t])}{[t]^2} - \frac{\Delta(t)}{t^2}\right)\mathrm{d}t - \int_x^\infty \left(\frac{\Delta([t])}{[t]^2} - \frac{\Delta(t)}{t^2}\right)\mathrm{d}t + O\left(\frac{1}{x}\right)$$

$$= \int_1^\infty \left(\frac{\Delta([t])}{[t]^2} - \frac{\Delta(t)}{t^2}\right)\mathrm{d}t - \int_x^\infty \left(\left(\frac{1}{[t]^2} - \frac{1}{t^2}\right)\cdot\sum_{k\le t} d(k) - \left(\frac{[t](\log[t] + 2Y - 1)}{[t]^2}\right.\right.$$

$$\left.\left. - \frac{t(\log t + 2Y - 1)}{t^2}\right)\right)\mathrm{d}t + O\left(\frac{1}{x}\right)$$

$$= \int_1^\infty \left(\frac{\Delta([t])}{[t]^2} - \frac{\Delta(t)}{t^2}\right)\mathrm{d}t + O\left(\int_x^\infty \left(t\log t\cdot\frac{1}{t^3} + \frac{\log t}{t^2}\right)\mathrm{d}t\right) + O\left(\frac{1}{x}\right)$$

$$= \int_1^\infty \left(\frac{\Delta([t])}{[t]^2} - \frac{\Delta(t)}{t^2} \right) \mathrm{d}t + O\left(\frac{\log x}{x} \right)$$

$$\Rightarrow \sum_{1 \le t \le x} \frac{\Delta(t)}{t^2} = \int_1^x \frac{\Delta(t)}{t^2} \mathrm{d}t + \int_1^\infty \left(\frac{\Delta([t])}{[t]^2} - \frac{\Delta(t)}{t^2} \right) \mathrm{d}t + O\left(\frac{\log x}{x} \right) \quad (12)$$

2) $\sum_{1 \le t \le x} \Delta(t) - \int_1^x \Delta(t) \mathrm{d}t = \int_1^x \left(\Delta([t]) - \Delta(t) \right) \mathrm{d}t + O(\Delta(x))$

$$= \int_1^x \left(t(\log t + 2Y - 1) - [t](\log[t] + 2Y - 1) \right) \mathrm{d}t + O(x) \ll x \log x$$

$$\Rightarrow \int_1^x \Delta(t) \mathrm{d}t = \sum_{1 \le t \le x} \Delta(t) + O(x \log x)$$

$$= \int_1^x t^2 \mathrm{d}\left(\sum_{1 \le m \le t} \frac{\Delta(m)}{m^2} \right) + O(x \log x).$$

(12)

$$\Rightarrow \int_1^x \Delta(t) \mathrm{d}t = \int_1^x t^2 \mathrm{d}\left(\int_1^t \frac{\Delta(m)}{m^2} \mathrm{d}m + \int_1^\infty \left(\frac{\Delta([m])}{[m]^2} - \frac{\Delta(m)}{m^2} \right) \mathrm{d}m + O\left(\frac{\log t}{t} \right) \right)$$

$$+ O(x \log x)$$

$$= \int_1^x t^2 \mathrm{d}\left(\int_1^t \frac{\Delta(m)}{m^2} \mathrm{d}m + O\left(\frac{\log t}{t} \right) \right) + O(x \log x).$$

(11)

$$\Rightarrow \int_1^x \Delta(t) \mathrm{d}t = \int_1^x t^2 \mathrm{d}\left((1-Y)^2 - 2\int_1^\infty \left(\frac{\log[t]}{[t]} - \frac{\log t}{t} \right) \mathrm{d}t + O\left(\frac{\log t}{t} \right) \right) + O(x \log x)$$

$$= \int_1^x t^2 \mathrm{d}\left(O\left(\frac{\log t}{t} \right) \right) + O(x \log x).$$

$$\ll x \log x$$

Type $\int_1^x \Delta(t) \mathrm{d}t = O(x \log x)$.

Note: 1) This paper proves that $\int_1^x \Delta(t) = O(x \log x)$, while $\int_1^x \Delta^2(t) \mathrm{d}t = c' x^{\frac{3}{2}} + O(x \log^5 x)$ (Dong Guangchang, 1956) [7], if you combine these two equations, you can understand it better "$\Delta(t)$". Some variation rules of.

2) We can further prove this in other ways, $\sum_{1 \le t \le x} \Delta(t) = \frac{1}{2} x \log x + O(x)$, and then get $\int_1^x \Delta(t) \mathrm{d}t = O(x)$, I won't go into details here for lack of space.

Theorem 6: The function $w(n)$ (or $\Omega(n)$) improvement of relevant conclusions [8] [9]:

1) $\sum_{n \le x} w(n) = x \log\log + A_2 x - (1-Y) \frac{x}{\log x} + O\left(\frac{x}{\log^2 x} \right)$.

2) $\sum_{n \le x} w^2(n) = x(\log\log x)^2 + (1 + 2A_2) x \log\log x + O(x)$.

3) $\sum_{n\le x}\left(w(n)-\log\log x\right)^2 = x\log\log x + O(x)$.

Prove: 1)

$$\sum_{n\le x} w(n) = \sum_{p\le x}\left[\frac{x}{p}\right] = \left[\frac{x}{2}\right] + \sum_{2<p\le x}\left[\frac{x}{p}\right]$$

$$= \left[\frac{x}{2}\right] + \int_2^x \frac{1}{\log t}\,\mathrm{d}\left(\sum_{p\le t}\left[\frac{x}{p}\right]\log p\right)$$

$$= \left[\frac{x}{2}\right] + \frac{\sum_{p\le t}\left[\frac{x}{p}\right]\log p}{\log t}\Bigg|_2^x + \int_2^x \frac{\sum_{p\le t}\left[\frac{x}{p}\right]\log p}{t\log^2 t}\mathrm{d}t$$

$$= \frac{\sum_{p\le x}\left[\frac{x}{p}\right]\log p}{\log x} + \int_2^x \frac{\sum_{p\le t}\frac{x}{p}\log p + O\left(\sum_{p\le t}\log p\right)}{t\log^2 t}\mathrm{d}t$$

$$= x\int_2^x \frac{\log t + A_1}{t\log^2 t}\mathrm{d}t + x\int_2^\infty \frac{\sum_{p\le t}\frac{\log p}{p} - \left(\log t + A_1\right)}{t\log^2 t}\mathrm{d}t$$

$$-x\int_x^\infty \frac{\sum_{p\le t}\frac{\log p}{p} - \left(\log t + A_1\right)}{t\log^2 t}\mathrm{d}t + \frac{\sum_{p\le x}\left[\frac{x}{p}\right]\log p}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{13}$$

1) Theorem two is easy to prove:

$$\int_2^\infty \frac{\sum_{p\le t}\frac{\log p}{p} - \left(\log t + A_1\right)}{t\log^2 t}\mathrm{d}t,$$

For constant

$$\int_x^\infty \frac{\sum_{p\le t}\frac{\log p}{p} - \left(\log t + A_1\right)}{t\log^2 t}\mathrm{d}t = O\left(\frac{1}{\log^2 x}\right).$$

2) By "(6) type" Easy card:

$$\sum_{p\le x}\left[\frac{x}{p}\right]\log p = x\log x - \left(1 + \sum_p \frac{\log p}{p(p-1)}\right)x + O\left(\sqrt{x}\right).$$

3) Theorem two: $A_1 = -Y - \sum_p \frac{\log p}{p(p-1)}$.

4) By

$$\sum_{p\le x}\left[\frac{x}{p}\right] = x\sum_{p\le x}\frac{1}{p} + O\left(\pi(x)\right).$$

Theorem 2

$$\Rightarrow \sum_{p\le x} w(n) = x\log\log x + A_2 X + O\left(\frac{x}{\log x}\right).$$

Will (1), (2), (3) (4) Can be substituted into equation (13), available:

Will

$$\sum_{n \le x} w(n) = x \log\log x + A_2 x + (Y-1)\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \tag{14}$$

2) $w(n)(w(n)-1) = \sum_{pq|n} 1 - \sum_{p^2|n} 1$

$$\Rightarrow \sum_{n \le x} w^2(n) - \sum_{n \le x} w(n) = \sum_{pq \le x}\left[\frac{x}{pq}\right] - \sum_{p^2 \le x}\left[\frac{x}{p^2}\right]$$

$$= \sum_{pq \le x} \frac{x}{pq} + O(x)$$

$$= 2x \sum_{\substack{pq \le x \\ p \le \sqrt{x}}} \frac{1}{pq} - x\left(\sum_{\substack{p \le \sqrt{x} \\ q \le \sqrt{x}}} \frac{1}{p}\right) + O(x)$$

$$= 2x \sum_{p \le \sqrt{x}} \frac{1}{p} \sum_{q \le \frac{x}{p}} \frac{1}{q} - x\left(\sum_{p \le \sqrt{x}} \frac{1}{p}\right)^2 + O(x)$$

Theorem 2

$$\Rightarrow \sum_{n \le x} w^2(n) - \sum_{n \le x} w(n) = 2x \sum_{p \le \sqrt{x}} \frac{\log\log \frac{x}{p}}{p} - x(\log\log x)^2$$
$$+ 2\log 2(x \log\log x) + O(x)$$

$$\log\log \frac{x}{p} = \log\log x + \log\left(1 - \frac{\log p}{\log x}\right) = \log\log x + O\left(\frac{\log p}{\log x}\right)$$
$$p \le \sqrt{x}$$

Theorem 2

(14)

$$\Rightarrow \sum_{n \le x} w^2(n) = x(\log\log x)^2 + (1+2A_1)x \log\log x + O(x)$$

3) By (1), (2) The conclusions obtained are easy to prove:

$$\sum_{n \le x} (w(n) - \log\log x)^2 = x \log\log x + O(x)$$

Note: 1) The derivation process of "$\Omega(n)$" function related conclusion is similar to "$W(n)$".

2) In this paper, "$\sum_{n \le x} (w(n) - \log\log x)^2 = x \log\log x + O(x)$" slightly better than "Hardy-Ramanujan", the results in.

3) With the o "$\sum_{n \le x} w^2(n)$" The similarity method of values will be easy "Selber The formula" expressed in the following form:

$$\theta(x)\log x + \sum_{p \le x} \theta\left(\frac{x}{p}\right) \log p$$

$$= 2x \log x + (2A_1 - 1)x + O\left(x \exp\left(-c \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right)\right)$$

Theorem of seven: if $\varphi(n)$ Is euler function, then:

1) $\liminf_{n\to\infty} \varphi(n) = \frac{n}{e^{Y} \log\log n} + O\left(n \exp\left(-c_5 (\log\log n)^{\frac{3}{5}} (\log\log\log n)^{-\frac{1}{5}}\right)\right)$,

Among them "c_5" For the normal number.

2) If "Riemnn" If the guess is true, then

$$\liminf_{n\to\infty} \varphi(n) = \frac{n}{e^{Y} \log\log n} + O\left(n \log^{-\frac{1}{2}} n\right).$$

Prove: This conclusion mainly uses "theorem 2 (Theorem 2')", the proof process is relatively simple, so skip [10].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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