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# Algebraic Points of Any Degree $l$ with $(l \geq 9)$ over $\mathbb{Q}$ on the Affine Equation Curve $\mathcal{C}_3(11): y^{11} = x^3(x-1)^3$

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## Abstract

In this work, we use the finiteness of the Mordell-weil group and the Riemann Roch spaces to give a geometric parametrization of the set of algebraic points of any given degree over the field of rational numbers $\mathbb{Q}$ on curve $\mathcal{C}_3(11): y^{11} = x^3(x-1)^3$. This result is a special case of quotients of Fermat curves $\mathcal{C}_{r,s}(p): y^p = x^r(x-1)^s$, $1 \leq r, s, r+s \leq p-1$ for $p = 11$ and $r = s = 3$. The results obtained extend the work of Gross and Rohrlich who determined $\bigcup_{[\mathbb{K}:\mathbb{Q}]\leq 2} \mathcal{C}_1(11)(\mathbb{K})$ the set of algebraic points on $\mathcal{C}_1(11)(\mathbb{K})$ of degree at most 2 on $\mathbb{Q}$.

## Keywords

Mordell-Weil Group, Jacobian, Galois Conjugates, Algebraic Extensions, the Abel-Jacobi Theorem, Linear Systems

## 1. Introduction

Let $\mathcal{C}$ be an algebraic curve defined on number field $\mathbb{K}$. We note $\mathcal{C}(\mathbb{K})$ be the set of algebraic points on $\mathcal{C}$ defined on $\mathbb{K}$ and $\bigcup_{[\mathbb{K}:\mathbb{Q}]\leq l} \mathcal{C}(\mathbb{K})$ the set of algebraic points on $\mathcal{C}$ to be coordinated in $\mathbb{K}$ of degree at most $l$ over $\mathbb{Q}$. The degree of an algebraic point $R$ is the degree of its defining field on $\mathbb{Q}$; $\deg(R) = [\mathbb{Q}(R):\mathbb{Q}]$. A famous theorem of Faltings states that if $g \geq 2$ then the set $\mathcal{C}(\mathbb{K})$ of algebraic points on $\mathcal{C}$ defined on $\mathbb{K}$ is finite. A generalization to subvarieties of an abelian variety allows a qualitative study of the set $\bigcup_{[\mathbb{K}:\mathbb{Q}]\le l} \mathcal{C}(\mathbb{K})$ of algebraic points on $\mathcal{C}$ of degree at most $l$ over $\mathbb{Q}$.

We propose to study in detail the set of algebraic points of any degree given on $\mathbb{Q}$ on the curve $\mathcal{C}_3(11)$ of affine equation $y^{11} = x^3(x-1)^3$.

Our affine equation curve $\mathcal{C}_3(11): y^{11} = x^3(x-1)^3$ is a special case of quotients of Fermat curves of equations $\mathcal{C}_{r,s}(p): y^p = x^r(x-1)^s$, $1 \le r, s, r+s \le p-1$ studied in [1].

Let $P_0 = (0:0:0)$, $P_1 = (1:0:1)$ and $P_\infty = (1:0:0)$ denote the point at infinity of $\mathcal{C}_3(11)$. Consider the Jacobian folding defined by

$$\begin{array}{rcl} j: \mathcal{C}_3(11)(\mathbb{Q}) & \to & J(\mathbb{Q}) \\ P & \mapsto & [P - P_\infty] \end{array}$$

We will designate $J$ the Jacobian of $\mathcal{C}_3(11)$ and by $j(P)$ the class denoted $[P - P_\infty]$ of $P - P_\infty$.

Our approach relies on the knowledge of the Mordell-Weil group of the Jacobian $J$-variety of $\mathcal{C}_3(11)$ and the condition that it is finite: it consists in using the Abel-Jacobi theorem to plunge the curve into its Jacobian and to study linear systems on the curve $\mathcal{C}_3(11)$.

The Mordell-Weil group $J(\mathbb{Q})$ of rational points of the Jacobian $J$ of $\mathcal{C}_3(11)$ is finite and given by $J(\mathbb{Q}) \cong (\mathbb{Z}/11\mathbb{Z})$ ([2], p. 219 and [3]).

Our study results from the work of Gross-Rohrlich who determined $\bigcup_{[\mathbb{K}:\mathbb{Q}]\le 2} \mathcal{C}_1(11)(\mathbb{K})$ the set of algebraic points on $\mathcal{C}_1(11)(\mathbb{K})$ of degree at most 2 on $\mathbb{Q}$ and given by the following proposition:

**Proposition 1.**

The set of algebraic points on $\mathcal{C}_1(11)(\mathbb{K})$ of degree at most 2 on $\mathbb{Q}$ is given by

$$\bigcup_{[\mathbb{K}:\mathbb{Q}]\le 2} \mathcal{C}_1(11)(\mathbb{K}) = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}}, y \right) \right\} \cup \{P_\infty\} \quad (1)$$

We extend these results by giving a geometric parametrization of algebraic points of any given degree on $\mathbb{Q}$ on the curve $\mathcal{C}_3(11)$ of affine equation $y^{11} = x^3(x-1)^3$.

Our essential tools are:

1) The Mordell-Weil group $J(\mathbb{Q})$ of the Jacobian of $\mathcal{C}$.

2) The Abel-Jacobi theorem (see in [4] page 156).

3) The study of linear systems on the curve $\mathcal{C}_3(11)$.

4) The theory of intersection.

Our main result is as follows:

**Theorem**

The set of algebraic points of degree $l \ge 9$ on $\mathcal{C}_3(11)$ is:

$$\bigcup_{[\mathbb{K}:\mathbb{Q}]\le l} \mathcal{C}_3(11)(\mathbb{K}) = \mathcal{F}_0 \cup \left( \bigcup_{k=1}^{10} \mathcal{F}_k \right) \quad (2)$$

with

$$\mathcal{F}_0=\left\{\left(-\frac{\sum_{i\le\frac{l}{2}}a_iy^{\frac{i}{3}}}{\sum_{j\le\frac{l-11}{2}}b_jy^{\frac{j}{3}}},y\right)\mid a_0\ne0,a_{\frac{l}{2}}\ne0\text{ if }l\text{ is even,}\right.$$

$$b_{\frac{l-11}{2}}\ne0\text{ if }l\text{ is odd and }y\text{ root of the equation}\tag{3}$$

$$\left.y^{11}\left(\sum_{j\le\frac{l-11}{2}}b_jy^j\right)^2=\left(\sum_{i\le\frac{l}{2}}a_iy^i\right)\left(\sum_{i\le\frac{l}{2}}a_iy^{\frac{i}{3}}+\sum_{j\le\frac{l-11}{2}}b_jy^{\frac{j}{3}}\right)^3\right\}$$

$$\mathcal{F}_k=\left\{\left(-\frac{\sum_{11-k\le i\le\frac{l+11-k}{2}}a_iy^{\frac{i}{3}}}{\sum_{j\le\frac{l-k}{2}}b_jy^{\frac{j}{3}}},y\right)\mid b_0\ne0,a_{\frac{l+11-k}{2}}\ne0\text{ if }l\text{ is even,}\right.$$

$$b_{\frac{l-k}{2}}\ne0\text{ if }l\text{ is odd and }y\text{ root of the equation}$$

$$\left.y^{k}\left(\sum_{j\le\frac{l-k}{2}}b_jy^j\right)^2=\left(\sum_{11-k\le i\le\frac{l+11-k}{2}}a_iy^{i-(11-k)}\right)\left(\sum_{11-k\le i\le\frac{l+11-k}{2}}a_iy^{\frac{i}{3}}+\sum_{j\le\frac{l-k}{2}}b_jy^{\frac{j}{3}}\right)^3\right\}\tag{4}$$

## 2. Auxiliary Results

Let $x$ and $y$ be the rational functions defined on $\mathcal{C}_3(11)$ by: $x(X,Y,Z)=\frac{X}{Z}$ and $y(X,Y,Z)=\frac{Y}{Z}$.

For a divisor $D$ on $\mathcal{C}_3(11)$, let $\mathcal{L}(D)$ be the $\overline{\mathbb{Q}}$-vector space of the rational functions $f$ defined by

$$\mathcal{L}(D)=\left\{f\in\overline{\mathbb{Q}}(\mathcal{C}_3(11))^*\mid div(f)\ge -D\right\}\cup\{0\}\tag{5}$$

The projective equation of the curve $\mathcal{C}_3(11)$ is: $Y^{11}=X^3Z^5(X-Z)^3$.

We have the following Lemma:

**Lemma 1**

$$\mathcal{C}_3(11):y^{11}=x^3(x-1)^3$$

$$div(x)=11P_0-11P_\infty;$$

$$div(y)=3P_0+3P_1-6P_\infty;$$

$$div(x-1)=11P_1-11P_\infty.$$

**Proof 1** It is a calculation of type

$$div(x-i)=((X-iZ)=0).\mathcal{C}_3(11)-(Z=0).\mathcal{C}_3(11) \quad (6)$$

From (6), we have $div(x)=(X=0).\mathcal{C}-(Z=0).\mathcal{C}$.

For $X=0$, the projective equation gives $Y^{11}=0$; and for $Z=1$, we obtain the point $P_0=(0:0:1)$ of multiplicity equal to 11.

For $Z=0$, the projective equation gives $Y^{11}=0$; and for $X=1$, we obtain the point $P_\infty=(1:0:0)$ of multiplicity equal to 11. Thus $div(x)=11P_0-11P_\infty$.

In the same way we show that $div(x-1)=11P_1-11P_\infty$ and $div(y)=3P_0+3P_1-6P_\infty$.

**Consequence 1**

$$11j(P_0)=11j(P_1)=0;$$

$$3j(P_0)+3j(P_1)=0$$

so $j(P_0)$ and $j(P_1)$ generate the same subgroup $J(\mathbb{Q})$.

**Lemma 2** A $\mathbb{Q}$-base of $\mathcal{L}(lP_\infty)$ is given by:

$$\mathcal{B}=\left\{\left(\frac{x^2(x-1)^2}{y^7}\right)^i \mid i\in\mathbb{N}, i\le\frac{l}{2}\right\}\cup\left\{x\left(\frac{x^2(x-1)^2}{y^7}\right)^j \mid j\in\mathbb{N}, j\le\frac{l-11}{2}\right\} \quad (7)$$

**Proof 2.** It is clear that $\mathcal{B}$ is free. It remains to show that

$$dim(\mathcal{B})=dim(\mathcal{L}(lP_\infty)).$$

By the Riemann-Roch theorem, we have $dim(\mathcal{L}(lP_\infty))=l-g+1$ as soon as $l\ge 2g-1$ with $g=\frac{11-1}{2}$

Let us consider the following cases:

**Case 1**: Suppose that $l$ is even, and let $l=2h$. Then we have

$$i\le\frac{l}{2}=h$$

and

$$j\le\frac{l-11}{2}\Leftrightarrow j\le\frac{2h-11}{2}\Leftrightarrow j\le\frac{2h-11-1}{2}=h-6=h-g-1.$$

So we obtain

$$\mathcal{B}=\left\{1,\frac{x^2(x-1)^2}{y^7},\cdots,\left(\frac{x^2(x-1)^2}{y^7}\right)^h\right\}\cup\left\{x,x\frac{x^2(x-1)^2}{y^7},\cdots,x\left(\frac{x^2(x-1)^2}{y^7}\right)^{h-g-1}\right\},$$

and therefore $dim(\mathcal{B})=(h+1)+(h-g)=2h-g+1=l-g+1=dim(\mathcal{L}(lP_\infty))$.

**Case 2**: Suppose that $l$ is odd, and let $l=2h+1$.

$$i\le\frac{l}{2}\Leftrightarrow i\le\frac{2h+1}{2}\Leftrightarrow i\le h$$

and

$$j\le\frac{l-11}{2}\Leftrightarrow j\le\frac{2h-10}{2}=h-g$$

So we obtain

$$\mathcal{B}=\left\{1,\frac{x^2(x-1)^2}{y^7},\cdots,\left(\frac{x^2(x-1)^2}{y^7}\right)^h\right\}\cup\left\{x,x\frac{x^2(x-1)^2}{y^7},\cdots,x\left(\frac{x^2(x-1)^2}{y^7}\right)^{h-g}\right\},$$

and therefore

$$dim(\mathcal{B})=(h+1)+(h-g+1)=2h+1-g+1=l-g+1=dim(\mathcal{L}(lP_\infty)).$$

## 3. Demonstration of the Theorem

Let $R\in\mathcal{C}_3(11)(\overline{\mathbb{Q}})$ with $[\mathbb{Q}(R):\mathbb{Q}]=l$. Let $R_1,\cdots,R_l$ be the Galois conjugates of *R*, and let $t=[R_1+\cdots+R_l-lP_\infty]$ which is a point of $J(\mathbb{Q})=\{mj(P_0),0\le m\le 10\}$; so $t=mj(P_0)$ with $0\le m\le 10$. This gives the relation

$$[R_1+\cdots+R_l-lP_\infty]=mj(P_0). \tag{8}$$

We note that $R\notin\{P_0,P_1,P_\infty\}$.

**Case** $m=0$

Then there exists a rational function *f* such that $div(f)=R_1+\cdots+R_l-lP_\infty$, so $f\in\mathcal{L}(lP_\infty)$. According to Lemma 2, we have

$$f=\sum_{i\le\frac{l}{2}}a_i\left(\frac{x^2(x-1)^2}{y^7}\right)^i+x\sum_{j\le\frac{l-11}{2}}b_j\left(\frac{x^2(x-1)^2}{y^7}\right)^j$$

with $a_{\frac{l}{2}}\ne 0$ if *l* is even (otherwise the $R_i$ would be equal to $P_\infty$) and $b_{\frac{l-11}{2}}\ne 0$ if *l* is odd (otherwise the $R_i$ would be equal to $P_\infty$). At the points $R_i$ we have

$$\sum_{i\le\frac{l}{2}}a_i\left(\frac{x^2(x-1)^2}{y^7}\right)^i+x\sum_{j\le\frac{l-11}{2}}b_j\left(\frac{x^2(x-1)^2}{y^7}\right)^j=0$$

hense

$$x=-\frac{\sum_{i\le\frac{l}{2}}a_i\left(\frac{x^2(x-1)^2}{y^7}\right)^i}{\sum_{j\le\frac{l-11}{2}}b_j\left(\frac{x^2(x-1)^2}{y^7}\right)^j}$$

and therefore

$$y^{11}=x^3(x-1)^3\Leftrightarrow y^{\frac{1}{3}}=\frac{x^2(x-1)^2}{y^7},$$

so

$$x=-\frac{\sum_{i\le\frac{l}{2}}a_iy^{\frac{i}{3}}}{\sum_{j\le\frac{l-11}{2}}b_jy^{\frac{j}{3}}}.$$

So the equation $y^{11} = x^3(x-1)^3$ becomes

$$y^{11}\left(\sum_{j\le\frac{l-11}{2}} b_j y^j\right)^2 = \left(\sum_{i\le\frac{l}{2}} a_i y^i\right)\left(\sum_{i\le\frac{l}{2}} a_i y^{\frac{i}{3}} + \sum_{j\le\frac{l-11}{2}} b_j y^{\frac{j}{3}}\right)^3$$

which is an equation of degree $l$ in $y$. We thus find a family of points of degree $l$

$$\mathcal{F}_0 = \left\{\left(-\frac{\sum_{i\le\frac{l}{2}} a_i y^{\frac{i}{3}}}{\sum_{j\le\frac{l-11}{2}} b_j y^{\frac{j}{3}}}, y\right) \mid a_0 \ne 0, a_{\frac{l}{2}} \ne 0 \text{ if } l \text{ is even}, \right.$$

$$b_{\frac{l-11}{2}} \ne 0 \text{ if } l \text{ is odd and } y \text{ root of the equation}$$

$$\left. y^{11}\left(\sum_{j\le\frac{l-11}{2}} b_j y^j\right)^2 = \left(\sum_{i\le\frac{l}{2}} a_i y^i\right)\left(\sum_{i\le\frac{l}{2}} a_i y^{\frac{i}{3}} + \sum_{j\le\frac{l-11}{2}} b_j y^{\frac{j}{3}}\right)^3\right\}$$

In the same way we show that for $m = k$ with $k \in \{1,\cdots,10\}$, the relation (8) gives $[R_1 + \cdots + R_l - lP_\infty] = kj(P_0) = (k-11)j(P_0)$. Then there exists a rational function $f$ such that $div(f) = R_1 + \cdots + R_l + (11-k)P_0 - (l+11-k)P_\infty$, so $f \in \mathcal{L}(l+11-k)P_\infty$. According to the Lemma 2, we have

$$f = \sum_{i\le\frac{l+11-k}{2}} a_i \left(\frac{x^2(x-1)^2}{y^7}\right)^i + x\sum_{j\le\frac{l-k}{2}} b_j \left(\frac{x^2(x-1)^2}{y^7}\right)^j \text{; and as } ordf_{P_0} = 11-k,$$

therefore

$$f = \sum_{11-k\le i\le\frac{l+11-k}{2}} a_i \left(\frac{x^2(x-1)^2}{y^7}\right)^i + x\sum_{j\le\frac{l-k}{2}} b_j \left(\frac{x^2(x-1)^2}{y^7}\right)^j$$

with $a_{\frac{l+11-k}{2}} \ne 0$ if $l$ is even (otherwise the $R_i$ would be equal to $P_\infty$) and $b_{\frac{l-k}{2}} \ne 0$ if $l$ is odd (otherwise the $R_i$ would be equal to $P_\infty$). At the points $R_i$ we have

$$\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i \left(\frac{x^2(x-1)^2}{y^7}\right)^i + x\sum_{j\le\frac{l-k}{2}} b_j \left(\frac{x^2(x-1)^2}{y^7}\right)^j = 0$$

hense $x = -\dfrac{\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i \left(\frac{x^2(x-1)^2}{y^7}\right)^i}{\sum_{j\le\frac{l-k}{2}} b_j \left(\frac{x^2(x-1)^2}{y^7}\right)^j}$ and therefore $x = -\dfrac{\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i y^{\frac{i}{3}}}{\sum_{j\le\frac{l-k}{2}} b_j y^{\frac{j}{3}}}$.

So the equation $y^{11} = x^3(x-1)^3$ becomes

$$y^k\left(\sum_{j\le\frac{l-k}{2}} b_j y^j\right)^2=\left(\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i y^{i-(11-k)}\right)\left(\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i y^{\frac{i}{3}}+\sum_{j\le\frac{l-k}{2}} b_j y^{\frac{j}{3}}\right)^3$$

which is an equation of degree $l$ in $y$. We thus find a family of points of degree $l$

$$\mathcal{F}_k=\left\{\left(-\frac{\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i y^{\frac{i}{3}}}{\sum_{j\le\frac{l-k}{2}} b_j y^{\frac{j}{3}}},y\right)\middle| b_0\ne 0, a_{\frac{l+11-k}{2}}\ne 0 \text{ if } l \text{ is even,}\right.$$

$b_{\frac{l-k}{2}}\ne 0$ if $l$ is odd and $y$ root of the equation

$$\left.y^k\left(\sum_{j\le\frac{l-k}{2}} b_j y^j\right)^2=\left(\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i y^{i-(11-k)}\right)\left(\sum_{11-k\le i\le\frac{l+11-k}{2}} a_i y^{\frac{i}{3}}+\sum_{j\le\frac{l-k}{2}} b_j y^{\frac{j}{3}}\right)^3\right\}$$

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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