Advances in Pure Mathematics, 2022, 12, 374-391
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Proof of Riemann Conjecture

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How to cite this paper: Chen, C.M. (2022) Proof of Riemann Conjecture. *Advances in Pure Mathematics*, **12**, 374-391.
https://doi.org/10.4236/apm.2022.125028

Received: March 10, 2022
Accepted: May 24, 2022
Published: May 27, 2022

Abstract

Using translation $\beta=\sigma-1/2$ and rotation $s=\sigma+it=1/2+iz$, $z=t-i\beta$, Riemann got two results: (Theorem A) the functional equation $\xi(z)=G(s)\zeta(s)$, where $|G(s)|\approx C(\sigma)t^{(\sigma+3)/2}$ $\mathrm{e}^{-t\pi/4}>0$, $|\zeta(s)|\le Ct$, and (Theorem B) the product expression $\xi(z)=\xi(0)\prod_{j=1}^{\infty}\left(1-z^2/z_j^2\right)$, where z_j are all roots of $\xi(z)$, including complex roots. He proposed Riemann conjecture (RC): All roots of $\xi(z)$ are real. As the product expression can only be used as a tool of contradiction, we prove RC by contradiction. To avoid the zeros of $\zeta(1/2+it)$, define a subset $L(R)=\{t:t\ge 2R>20 \text{ and } |\zeta(1/2+it)|\ge C_0>0\}$. We have basic estimate $\ln C_0\le\ln|\xi(t)/G(s)|=\ln|\zeta(1/2+it)|\le\ln t+O(1)$, on $L(R)$. One can construct $w(t)=\xi(0)\prod_{j=1}^{\infty}\left(1-t^2/t_j^2\right)$ by all real roots t_j of $\xi(t)$. If ξ has no complex roots, then $w(t)=G(s)\zeta(s)$ for $s=1/2+it$. If the product expression has a complex root $z'=t'-i\alpha$, where $0<\alpha\le 1/2$, $R=|z'|>10$, then $\xi(z)$ has four complex roots $\pm(t'\pm i\alpha)$, and should contain fourth order factor $p(z)$, *i.e.* $\xi(z)=w(z)p(z)$. But $p(z)$ can not be contained in $\zeta(s)$, as we have $C_0\le|\zeta(s)|\le Ct$ on $L(R)$ and $p(t)\ge 0.5(t/R)^4$. As a result, we can rewrite $\xi(t)=w(t)p(t)=G(s)\zeta(s)p(t)$ on $L(R)$ and get

$$\ln|\xi(t)/G(s)|=\ln\left|\zeta\left(\frac{1}{2}+it\right)\right|+\ln p(t)\ge 4\ln\frac{t}{R}+O(1),\quad t\gg 2R.$$

This contradicts the basic estimate. Therefore $\xi(z)$ has no complex roots and RC holds.

Keywords

Riemann Conjecture, Distribution of Zeros, Entire Function, Symmetry,

Functional Equation, Product Expression

1. Introduction

B.Riemann (1859) in his famous paper "*On the number of primes less than a given magnitude*" stated an important conjecture: all zeros of ξ-function are real, which is called Riemann conjecture (RC) (see [1], pp-299-305). D. Hilbert (1900) proposed 23 problems in The Second International Conference of Mathematicians [2] (1976). The eighth problem contains Riemann Hypothesis (RH): All the non-trivial zeros of $\zeta(s)$-function have the real part 1/2, which has not been solved in the 20th century. S. Smale [3] (1998, 2000) proposed 18 problems in "*Mathematical problems for the next century*". RH is listed as the first problem. In 2000, Clay Mathematics Institute opened seven Millennium problems, including RH, see E. Bombieri [4] (2000), P. Sarnak [5] (2005) and J. Conrey [6] (2004). Apparently RH has been one of the most difficult problems in mathematics. In recent 20 years, several books are published to give overall introduction to RH, e.g. [1] [7]. So far, there are hundreds of important results based on the assumption that RH holds (see Chapter 5 in [7]). If RH is true, these results will be promoted as "theorems", else, as pointed out by Bombieri [4], "*The failure of the Riemann hypothesis would create havoc in the distribution of prime number*".

In 1970 years, a close connection between the distribution of zeros of ζ-function and the eigenvalues of quantity system with random matrix was found [6]. Later, one shows that RH and quasi-crystals have direct relation, which has added a mystery to RH. Physicist F. Dyson [8] (2008) in "*Birds and Frogs*" (on Einstein lecture of AMS) discussed the relation between RH and quasi-crystals. He said, "*If the Riemann hypothesis is true, then the zeros of the zeta-function form a one-dimensional quasi-crystal according to the definition*". Now RH is not only the aim of mathematicians, but else the interest of physicists.

In the 20th century, extremely large scale computations for ζ ($t = 10^9 \sim 10^{13}$) still did not find a counter example of RH [9] [10] [11] [12]. Indeed, these computations have enhanced our belief to prove RH. We shall look for a clue of the proof by computing, follow Liuhui's thought.

From Riemann's paper, we have found two mysteries. 1) the analytic continuation of Riemann integral $\xi(t)$ is an entire function $\xi(z)$ and satisfies functional equation $\xi(z) = G(s)\zeta(s)$, $s = \sigma + it$, $z = t - i(\sigma - 1/2)$; 2) the product expression has to admit complex roots, thus which can be used only as a tool of contradiction. We have found a flaw: if the product expression $\xi(z)$ has a complex root, then functional equation will be destroyed, so RC holds.. Should point out that A. Hinkkanen [13] and J. Lagarias [14] proved the equivalence between RC and the positivity of ξ, by using the product expression (see Section 5). These works have provided some inspiration for us.

This paper is organized as follows. In Section 2 we analyze Riemann's paper, state two theorems and Riemann conjecture (RC), then propose the method of contradiction; look for the correct research approach by computing ξ and ζ in Section 3; prove RC by contradiction in Section 4; in the last section a rigorous monotone of $|\xi(t-i\beta)|$ in $|\beta|$ is proved.

2. Analysis of Riemann's Paper and New Discovery

2.1. Some Preliminaries

There are two definitions for analytic function $f(s)=u+iv$, $s=\sigma+it$ as follows.

Def. 1. $f(s)$ is continuously differentiable in domain Ω and satisfies Cauchy-Riemann conditions $u_\beta=v_t$, $v_\beta=-u_t$.

Def. 2. $f(s)$ at some point s_0 can be expanded as a power series

$$f(s)=b_0+b_1(s-s_0)+b_2(s-s_0)^2+\cdots+b_n(s-s_0)^n+\cdots,\ |s-s_0|<R,$$

where s is real or complex, as they have the same convergence radius R.

If $f(s)$ is analytic over the whole complex plane, called an entire function, which has product expression (Hadamard theorem). We often meet a broad class of entire function of order 1: for any R, $|f(s)|\le CR^R$ holds for $|s|\le R$. e.g. $e^s,\cos(s),\xi(s)$. If $f(s)$ is analytic over the whole complex plane except several poles, called meromorphic function.

For $s=\sigma+it$, gamma function $\Gamma(s)$ can be defined as

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx,\ \sigma>0,$$

using integration by parts many times we have

$$\Gamma(s)=\frac{1}{s(s+1)\cdots(s+n)}\int_0^\infty x^{s+n}e^{-x}dx,\ n\gg1.$$

Thus $\Gamma(s)$ is a meromorphic function with one order poles $s=0,-1,-2,\cdots$. There is Stirling asymptotic expansion for $|s|\gg1$

$$\Gamma(s)=\sqrt{2\pi}s^{s-1/2}e^{-s}\left(1+\frac{1}{12s}+\frac{1}{288s^2}+\cdots\right),\ |\arg(s)|<\pi, \tag{1}$$

i.e. the negative real axis is excepted by the condition $|\arg(s)|<\pi$, in particular, except all poles $s=0,-1,-2,\cdots$. We describe the growth of $\Gamma(s)$. For $\sigma\gg1$, $\Gamma(\sigma)\approx\sqrt{2\pi}\sigma^{\sigma-1/2}e^{-\sigma}$ increases fast. For
$0\le\sigma\le1, t\gg1, \Gamma(s/2)\approx\sqrt{2\pi}(t/2)^{\sigma/2-1/2}e^{-t\pi/4}$ has exponential decay, see (20).

Euler (1737) proved product formula of the primes

$$\zeta(\sigma)=\sum_{n=1}^\infty\frac{1}{n^\sigma}=\prod_{p\in primes}\left(1-\frac{1}{p^\sigma}\right)^{-1},\ \sigma>1, \tag{2}$$

which diverges for $\sigma\le1$. There are several estimates of $\zeta(s)$ for $t\gg1$, [1] pp. 184, 201,

$$\left|\zeta(\sigma+it)\right| = \begin{cases} O\left(t^{(1-\sigma)/2}\ln t\right), & 0\le\sigma\le 1, \\ O\left(t^{1/6}\right), & \sigma = 1/2. \end{cases} \tag{3}$$

Thus $\zeta(s)$ increases slowly, this is very important. Although $\zeta(s)$ can be continued analytically by Euler-Maclaurin expansion, which is only used as a computational formula in Section 3.

2.2. Introduction of ζ and ξ

In Riemann's eight pages paper, only two pages focused on RC [1], pp. 300-302.

Taking $s=\sigma+it$ and $y=n^2\pi x$ in gamma integral gives

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty y^{s/2-1}\mathrm{e}^{-y}\mathrm{d}y = n^s\pi^{s/2}\int_0^\infty x^{s/2-1}\mathrm{e}^{-n^2\pi x}\mathrm{d}x.$$

Summing over n, Riemann (1895) had

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \pi^{s/2}\Gamma^{-1}\left(\frac{s}{2}\right)\int_0^\infty x^{s/2-1}\psi(x)\mathrm{d}x,\ \ \psi(x) = \sum_{n=1}^\infty \mathrm{e}^{-n^2\pi x},$$

where Jacobi function $\psi(x)$ satisfies $2\psi(x)+1 = x^{-1/2}\left(2\psi\left(\frac{1}{x}\right)+1\right)$. By $z=1/x$, there is

$$\int_0^1 z^{s/2-1}\psi(z)\mathrm{d}z = \frac{1}{s(s-1)} + \int_1^\infty x^{-s/2-1/2}\psi(x)\mathrm{d}x.$$

Riemann got an integral representation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty\left(x^{s/2-1}+x^{-(s+1)/2}\right)\psi(x)\mathrm{d}x, \tag{4}$$

where $\zeta(s)$ has already been continued analytically over the whole complex plane except for a pole point $s=1$. Whereas $\Gamma^{-1}(s/2)$ has zeros $s=-2,-4,\cdots$, called trivial zeros, no interest for us.

Multiplying (4) by $s(s-1)/2$, define functional equation

$$\xi_1(s) = G(s)\zeta(s),\ \ G(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right). \tag{5}$$

As $G(s)\ne 0$ for $\sigma>0$, then $\xi_1(s)$ and $\zeta(s)$ have the same zeros. Inserting ζ into (5) and applying integration by parts twice, one has [1] p. 17,

$$\begin{aligned}\xi_1(s) &= \frac{1}{2} + \frac{s(s-1)}{2}\int_1^\infty\left(x^{s/2-1}+x^{-s/2-1/2}\right)\psi(x)\mathrm{d}x, \\ &= r_1 + \int_1^\infty\left(x^{s/2-1}+x^{-s/2-1/2}\right)f(x)\mathrm{d}x,\ \ f(x) = 2x^2\psi'' + 3x\psi',\end{aligned}$$

where $r_1 = \frac{1}{2}+\psi(1)+4\psi'(1) = 0$ and $f(x) = \sum_{n=1}^\infty\left(2a_n^2x^2 - 3a_nx\right)\mathrm{e}^{-a_nx} > 0$, $a_n = n^2\pi$. However Riemann did not like this. He directly took $s=1/2+it$ and got a real function [1] pp.301-302,

$$\xi(t) := \xi_1(1/2+it) = 2\int_1^\infty \cos\left(\frac{t}{2}\ln x\right)x^{-3/4}f(x)\mathrm{d}x. \tag{6}$$

Riemann said, "*This function is finite for all finite values of t and can be developed as a power series in t^2 which converges very rapidly*" (*i.e.* $\xi(t)$ is an entire function). Actually we can get

$$\xi(t)=A_0+\sum_{j=1}^{\infty}\frac{(-1)^j}{(2j)!}A_{2j}t^{2j},$$

with the coefficients

$$A_{2j}=2\int_1^{\infty}\left(\frac{1}{2}\ln x\right)^{2j}x^{-3/4}f(x)\mathrm{d}x>0,\ j=0,1,2,\cdots.$$

This is an alternative series. As $f(x)$ is of exponential decay $\mathrm{e}^{-\pi x}$, the coefficients A_{2j} tend to 0 fast as $j\to\infty$. Therefore $\xi(t)$ is high-frequency oscillation with exponential decay.

Why Riemann preferred $\xi(t)$ rather than $\xi_1(s)$? This is the first mystery. For the symmetrization, Riemann used translation $\beta=\sigma-1/2$ and rotation,

$$s=\sigma+it=1/2+\beta+it=1/2+iz,\quad z=t-i\beta.$$

Actually we can define an entire function $\xi(z)$ by (6) and $\xi(z)=\xi_1(s)$, as they take the same $\xi(t)$ on the symmetric line $\beta=0$. We state these results as

Theorem A. *The entire function* $\xi(z)$ *satisfies functional equation*

$$\xi(z)=G(s)\zeta(s),\ s=\sigma+it=1/2+iz,\ z=t-i\beta,\ \beta=\sigma-1/2,\tag{7}$$

which has symmetry $\xi(z)=\xi(-z)$, conjugate $\xi(\bar{z})=\overline{\xi(z)}$.

2.3. Riemann Conjecture

Riemann said:

"…, *the function* $\xi(t)$ *can vanish only when the imaginary part of t lies between* $\frac{1}{2}i$ *and* $-\frac{1}{2}i$. *The number of roots of* $\xi(t)=0$ *whose real parts lie between* 0 *and T is about*

$$=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi},\ \{\textit{remark. proved by Mangoldt}, 1905\}$$

because the integral $\int \mathrm{d}\log\xi(t)$ *taken in the positive sense around the domain consisting of all values whose imaginary parts lie between* $\frac{1}{2}i$ *and* $\frac{1}{2}i$ *and whose real parts lie between 0 and T is* (*up to a fraction of the order of magnitude of* $1/T$) *equal to* $[T\log(T/2\pi)-T]i$ *and is, on the other hand, equal to the number of roots of* $\xi(t)=0$ *in the domain multiplied by* $2\pi i$. *One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the research for such a proof after some fleeting vain attempts*, …"

We have seen that in the critical strip $\Omega=\{z=t-i\beta:|\beta|\le 1/2, 0\le t<\infty\}$, Riemann continued to discuss $\xi(z)$ and proposed an exceedingly important statement, *i.e.*

Riemann conjecture (RC). *All the zeros of* $\xi(z)$ *-function are real, i.e., lying on the symmetric line* $\beta = 0$.

D. Hilbert (1900) reported RH and pointed out that [2]

"*… it still remains to prove the correctness of an exceedingly important statement of Riemann, viz., that the zero points of the function* $\zeta(s)$ *defined by the series*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

all have the real part 1/2, except the well-known negative integral real zeros…"

Most researchers use the classical formulation:

Riemann Hypothesis (RH). *All non-trivial zeros of* $\zeta(s)$ *-function lie on critical line* $\sigma = 1/2$.

Although $\xi(z)$ and $\zeta(s)$ have the same roots, but they have quite different properties. $\zeta(s)$ is not an entire function. It is very difficult to study its roots due to the lack of useful analysis tools. As pointed out by J. Conrey [6], "*in my belief, RH is a genuinely arithmetic problem, likely don't succumb to the method of analysis*". Whereas ξ has better properties. We note that some scholars have turned to ξ, for example, Sarnak [5] (2004) pointed out that "*Riemann showed how to continue zeta analytically in s and he established the functional equation*

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s), \tag{8}$$

Γ being the Gamma function. RH is the assertion that all the zeros of $\Lambda(s)$ are on the line of symmetry for the function equation, that is on $R(s) = 1/2$." He has proposed to study $\xi(z)$ and RC. Besides, J. Haglund [15] (2011) directly discussed $\xi(t)$. Hinkkanen [13] and Lagarias [14] have proved the equivalence between RC and the positivity of ξ. Thus studying ξ is a hopeful way.

2.4. Product Expression

Riemann finally discussed the product expression of $\xi(z)$ and said:

"*If one denotes by* α *the roots of the equation* $\xi(\alpha) = 0$, *then one can express* $\log\xi(t)$ *as*

$$\sum \log\left(1 - \frac{t^2}{\alpha^2}\right) + \log\xi(0),$$

because, since the density of roots of size *t* grows only like $\log(t/2\pi)$ as *t* grows, this expression converges and for infinite *t* is only infinite like $t\log t$; thus it differs from $\log\xi(t)$ by a function of t^2 which is continuous and finite for finite *t* and which, when divided by t^2, is infinitely small for infinite *t*. This difference is therefore a constant, the value of which can be determined by setting $t = 0$."

We have seen that Riemann wanted to prove the following

Theorem B. $\xi(z)$ *-function has product expression*

$$\xi(z)=\xi(0)\prod_{j=1}^{\infty}\left(1-\frac{z^2}{z_j^2}\right),\ \xi(0)\neq 0, \tag{9}$$

where $\{z_j\}$ are all zeros of $\xi(z)$ (including complex roots), should take k-ple products for k-ple zeros.

Proof. Riemann's proof is not rigorous, but the conclusion is correct. For this, J. Hadamard [16] (1893) studied the product expression for general entire function, and got for $\xi_1(s)=G(s)\zeta(s)$, [7], p. 16.

$$\xi_1(s)=a^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)\mathrm{e}^{s/\rho},\ s=\sigma+it,$$

where ρ runs over all roots of $\xi_1(s)$ (*i.e.* including ρ and $1-\rho$). Under above translation and rotation, $\xi_1(s)=\xi(z)$ has symmetry $\xi(z)=\xi(-z)$, its roots are $\pm z_j$ and the factors $\mathrm{e}^{\pm z/z_j}$ are canceled, and $\mathrm{e}^{A\pm Bz}$, $B=0$. Therefore (9) is correct (a simplified proof, see [1], pp. 39-47). □

Hadamard's work was called by Von Mangoldt (1895) "*The first real progression in the field in* 34 *years*" since Riemann's paper, [1], p. 39. But they cannot prove these roots to be real. Because, let $W(z)=\prod_{j=1}^{\infty}\left(1-z^2/z_j^2\right)$, then $\xi(z)/W(z)$ is an entire function without zeros, which is a constant proved by Edwards [1] based on new method. Therefore $\xi(z)$ in (9) has to admit complex zeros, which is more extensive than $\xi(z)$ in (7). But it's pity, Riemann had used the same notation, which brought about misunderstanding. This is the second mystery. Whereas the product expression is often used to estimate $\zeta(s)$, this is the second misguiding.

2.5. New Recognition for Riemann's Theorems

For L-functions, including $\zeta(s)$, Bombieri [4] pointed out that "*we do not have algebraic and geometric models to guide our thinking, and entirely new ideas may be needed to study these intriguing objects*". Actually, for $\xi(z)$, Riemann had already provided an (strange) algebraic model: functional equation (7) and product expression (9). But they are different concepts. The functional equation (7) generated by $\zeta(s)$ gives a sharp expression (we can compute all real roots of (7), and no complex roots are found, but no way to prove it, as if some condition is lacked, see Section 3.3). Whereas the product expression (9) must contain all roots, including complex roots (which contradicts RH, how to use it?). We recall that to solve an initial-boundary value problem in PDE's, Fourier method can provide a series solution, whereas its uniqueness is proved by energy method. So we get the following.

Conclusion. *The product expression can only be used as a tool of contradiction.*

Fortunately, we have found a flaw: if product expression has complex roots, then functional equation is destroyed. It derives RC to be true.

Therefore, Riemann's two theorems are successfully combined in the method of contradiction. This is our new idea.

3. Computational Comparison between ζ and ξ

In large scale computations there were two algorithms: Riemann-Siegel formula $Z(t)$ on critical line and Euler-Maclaurin expansion of ζ outside critical line. Many numerical experiments in the 20th century showed the RH is probably true, but cannot provide a clue for the proof. Ancient Chinese greatest mathematician Liuhui (A.D.225-295) [17] had proposed Mathematical methodology: "*computing can distinguish tiny and detect the unknown and method*". "*Analyze the reason by logic, explain the essence by figures*" in preface (A.D.263) of "Nine Chapters Mathematics". We have gradually found the clue of proving RC in the computations.

3.1. Euler Function $\zeta = U + iV$

We consider the Euler-Maclaurin expansion [7]

$$\begin{cases} \zeta(s) = \zeta_N(s) + S_{2m} + R_{2m}, \ \zeta_N(s) = \sum_{n=1}^{N-1} n^{-s}, \ s = \sigma + it, \\ S_{2m} = \left\{ \dfrac{N}{s-1} + \dfrac{1}{2} + \dfrac{B_2}{2}\dfrac{s}{N} + \cdots + \dfrac{B_{2m}}{(2m)!}\dfrac{s(s+1)\cdots(s+2m-2)}{N^{2m-1}} \right\} N^{-s}, \\ R_{2m} = -\dfrac{s(s+1)\cdots(s+2m-1)}{(2m)!} \sum_{j=N}^{\infty} \int_0^1 B_{2m}(x)(j+x)^{-s-2m} \mathrm{d}x, \end{cases} \tag{10}$$

where B_j is j-th Bernoulli number, $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{-1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = \frac{-691}{2730}$, $B_{14} = \frac{7}{6}$, $B_{16} = \frac{-3617}{510}$, $\cdots$, and $B_j(x)$ is j-th Bernoulli polynomial. For $m = 7 \sim 10$ and $N \ge t$, the remainder R_{2m} can be omitted. These curves $\{U, V\}$ are depicted for several σ in **Figure 1**.

We can see that

1) Along the critical line $\sigma = 0.5$, $\{U, V\}$ are not symmetric. Near common zeros $U = V = 0$, $\{U, V\}$ sometimes are tangent each other, one cannot find what structure of ζ.

2) Increasing $\sigma = 0.6, 0.8, 1.0$, the curves U will gradually go away from t-axis, the number of its zeros decreases, whereas V has many zeros. This is why all the theoretical results obtained so far go away from critical line.

3.2. Find Alternative Oscillation of $\xi = u + iv$

For $z = t - i\beta$, we decompose $\cos\left(\frac{z}{2}\ln x\right)$ into real and imaginary parts, and rewrite Riemann integral (6) as

$$\xi(z) = \int_1^{\infty} \left\{ \left(x^{\beta/2} + x^{-\beta/2}\right)\cos\left(\frac{t}{2}\ln x\right) + i\left(x^{\beta/2} - x^{-\beta/2}\right)\sin\left(\frac{t}{2}\ln x\right) \right\} F(x)\mathrm{d}x. \tag{11}$$

So the real part $u(t, \beta)$ is an even function of β and the imaginary part $v(t, \beta)$ is an odd function of β. By Cauchy-Riemann conditions $u_\beta = v_t$, $v_\beta = -u_t$, we have the following

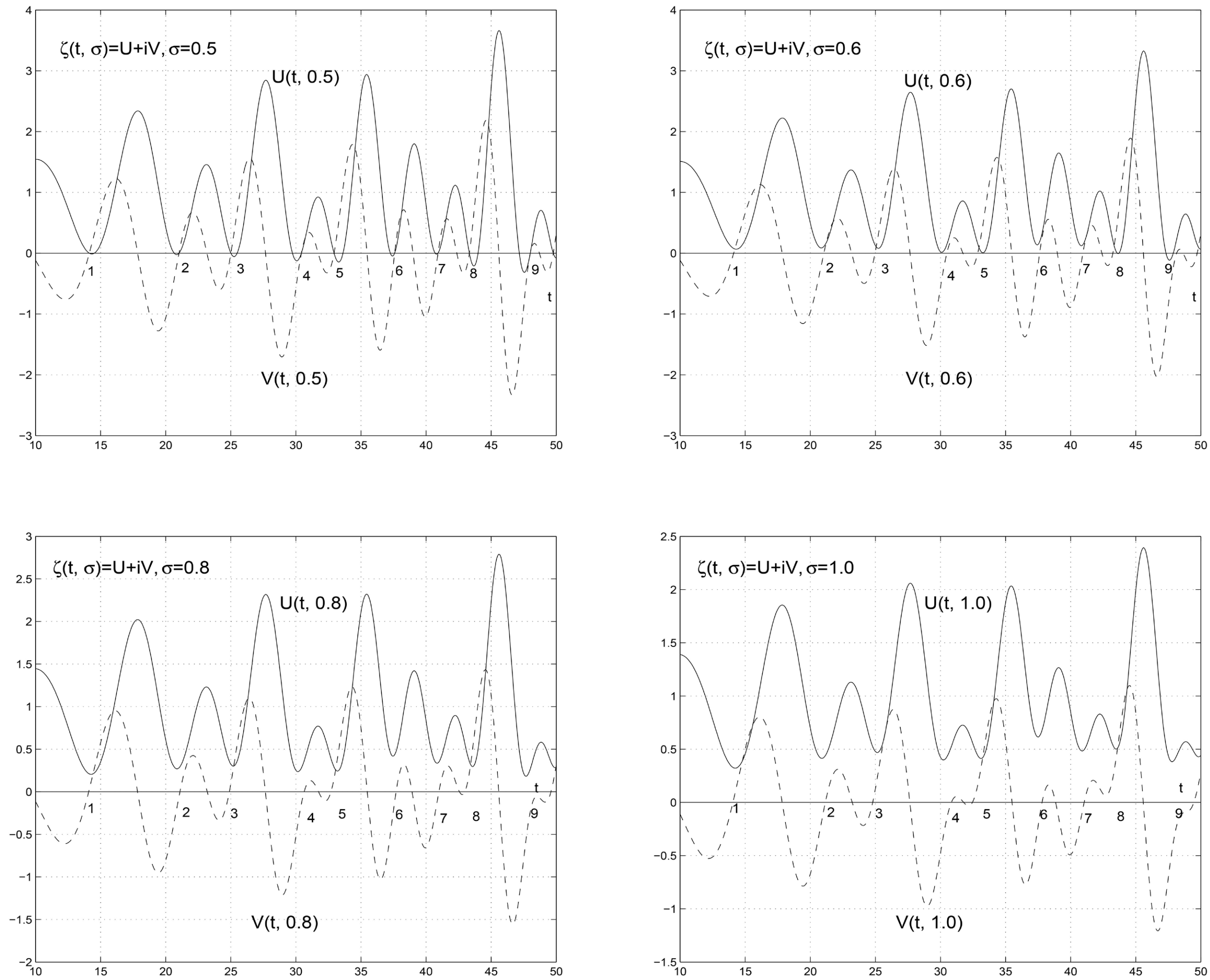

Figure 1. $\{U,V\}$ are not alternative oscillation for any $\sigma \in [0.5,1]$.

Symmetry. If $\beta = 0$, the imaginary part $v(t,0) \equiv 0$. In general,

$$v = 0,\ v_t = u_\beta = 0,\ v_{tt} = -v_{\beta\beta},\ v_{ttt} = u_{\beta\beta\beta} = 0,\ \text{if}\ \beta = 0. \tag{12}$$

Denote the norm $|\xi| = \left(|u|^2 + |v|^2\right)^{1/2}$ and define a strong norm in critical strip

$$\|\xi\| = \begin{cases} |u| + |v|/\beta, & \beta \in (0,1/2], t \in [0,\infty), \\ |u(t,0)| + |u_t(t,0)|, & \beta \to +0,\ t \in [0,\infty). \end{cases} \tag{13}$$

To enlarge curves v, we use $|v|/\beta$. Note that if $\beta \to +0$ and $u(t_j,0) = 0$, $v(t_j,0) = 0$, then $|\xi| = 0$. At the same time, we have $\|\xi\| > 0$, if $|u_t(t_j,0)| \neq 0$. Due to the exponential decay of ξ, we take variable scale $M(t) = (1+t)^{7/4}\,\mathrm{e}^{-t\pi/4}$ (see (20)) and investigate u/M, which increases slowly. In **Figure 2-1**, for $\beta = 0$, $u(t,0)$ is highly oscillating and $v(t,0) \equiv 0$ (symmetry). In **Figure 2-2**, for $\beta = 0.1, 0.2, 0.5$, $\{u, v/\beta\}$ are alternative oscillation without common zeros, which intuitively imply that RC holds. In each root-interval in **Figure 2-3**,

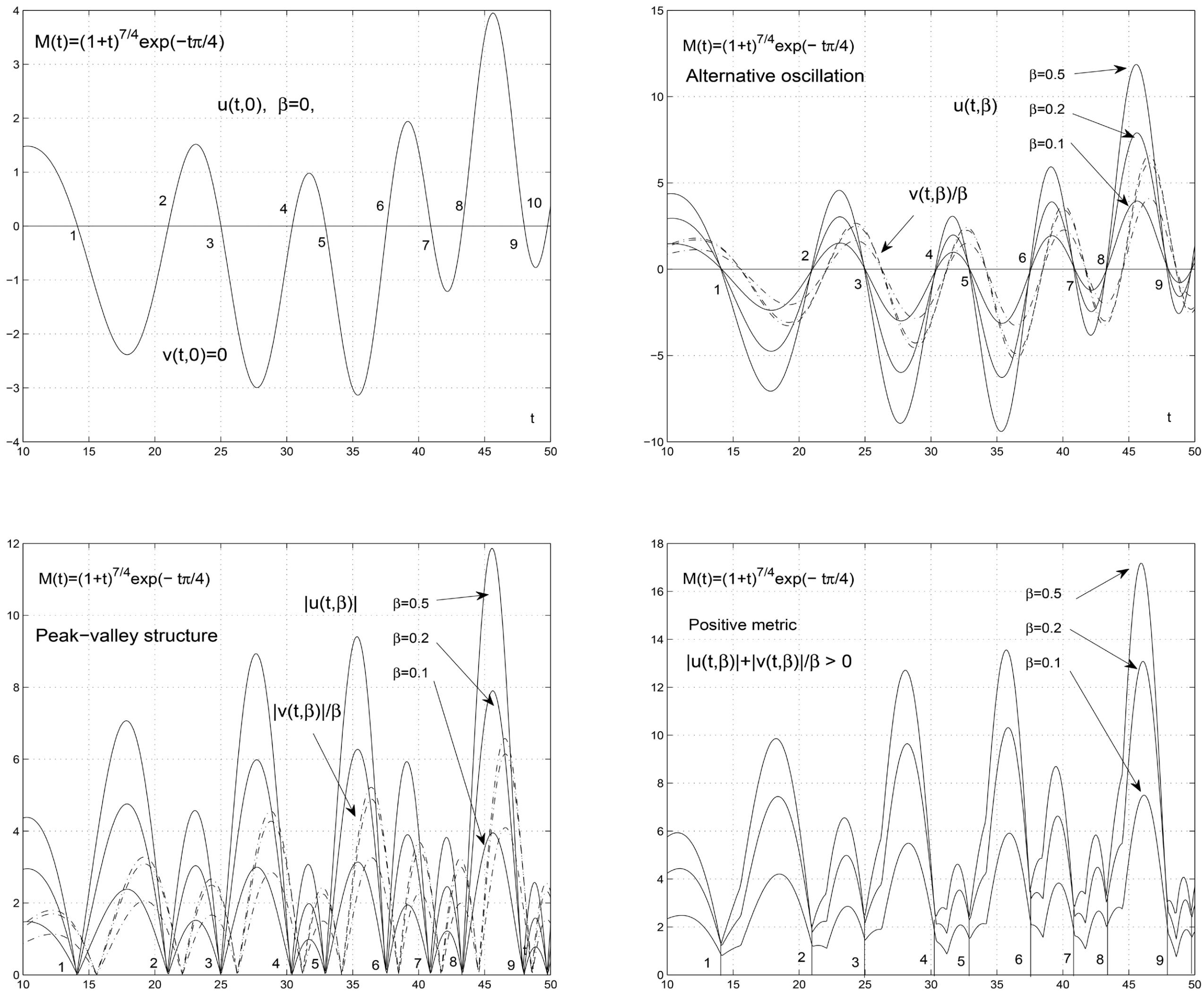

Figure 2. 1-4. Alternative oscillation $\{u, v/\beta\}$ and PVS $\{|u|, |v|/\beta\}$.

$|u| > 0$ is peak, $|v|/\beta$ is valley, and $\{|u|, |v|/\beta\}$ form peak-valley structure. In Figures 2-4, $\|\xi\|/M > 0.59$, so RC holds locally. Note that $\xi(z)$ resembles a complex function $f = \mathrm{e}^{-t}(\cos t + i\beta \sin t)$, $\|f\| = \mathrm{e}^{-t}(|\cos t| + |\sin t|) \ge \mathrm{e}^{-t}$.

3.3. Geometric Model of $\xi = u + iv$

We further consider the peak-valley structure (PVS) of ξ, [18] [19].

Definition 1 (root-interval). $I_n = [t_n, t_{n+1}]$ is called a root-interval, if t_n and t_{n+1} (dependent on β) are two adjacent zeros of $u(t,\beta)$ and $|u(t,\beta)| > 0$ inside I_n.

Definition 2 (single peak). Inside each root-interval $I_n = [t_n, t_{n+1}]$, if $|u(t,\beta)|$ has only one peak, called single peak. Else, called multiple peak.

Using Newton-Leibnitz formula, symmetry $v(t,0) \equiv 0$ and C-R conditions, we get

Lemma 1 (expression of v). The imaginary part v of ξ can be expressed as

$$v(t,\beta)=v(t,0)+\int_0^\beta v_\beta(t,r)\mathrm{d}r$$
$$=-\int_0^\beta u_t(t,r)\mathrm{d}r,\ v(t,0)=0,\ \beta\in(0,1/2]. \tag{14}$$

Corollary 1. $|v(t,\beta)|/\beta$ is uniformly bounded with respect to $\beta\in(0,1/2]$.

Lemma 2 (expression of *u*). The real part u of ξ can be expressed as

$$u(t,\beta)-u(t,0)=\int_0^\beta u_\beta(t,r)\mathrm{d}r$$
$$=u_\beta(t,r)(r-\beta)\Big|_0^\beta-\int_0^\beta u_{\beta\beta}(t,r)(r-\beta)\mathrm{d}r \tag{15}$$
$$=-\int_0^\beta u_{tt}(t,r)(\beta-r)\mathrm{d}r,\ \text{as}\ u_\beta(t,0)=0.$$

Corollary 2. Increasing $\beta>0$, the peak $u(t,\beta)$ develops always toward its convex direction.

Lemma 3. Assume $u(t,\beta)>0$ for $t\in I_n$. If $u_t(t_n,\beta)>0$, then $u_t(t_n,r)>0$ for any $r\in(0,\beta]$. If $u_t(t_{n+1},\beta)<0$, then $u_t(t_{n+1},r)<0$ for any $r\in(0,\beta]$. Similarly discuss $u(t,\beta)<0$.

Using the three Lemmas above, we have proved the following important result.

Lemma 4. *If the initial value* $u(t,0)$ *are single peak and single zero, then RC holds*.

A sketch of proof. Assume that $u(t,0)>0$ is single peak inside root-interval $I^0=\left[t_j^0,t_{j+1}^0\right]$. When increase $\beta\in(0,1/2]$, by corollary 2, $u(t,\beta)>0$ still is single peak inside root-interval $I_j=\left[t_j,t_{j+1}\right]$. As $u(t_j,\beta)=0$ and $u_t(t_j,\beta)>0$ at left end-point t_j, by lemma 3, $u_t(t_j,r)>0$ for all $r\in[0,\beta]$, we have

$$v(t_j,\beta)=-\int_0^\beta u_t(t_j,r)\mathrm{d}r<0,\ \lim_{\beta\to+0}\frac{v(t_j,\beta)}{\beta}=-u_t(t_j^0,0)<0. \tag{16}$$

And $u(t_{j+1},\beta)=0$ and $u_t(t_{j+1},\beta)<0$ at right end-point t_{j+1}, similarly

$$v(t_{j+1},\beta)=-\int_0^\beta u_t(t_{j+1},r)\mathrm{d}r>0,\ \lim_{\beta\to+0}\frac{v(t_{j+1},\beta)}{\beta}=-u_t(t_{j+1}^0,0)>0. \tag{17}$$

So $v(t,\beta)$ has opposite signs at two end-point of I_j, and there exists an inner point t_j' such that $v(t_j',\beta)=0$. Thus $|v(t,\beta)|$ is a valley, and $\{|u|,|v|/\beta\}$ form peak-valley structure, see Figure 3. Continuous function $\|\xi\|$ in the closed interval I_j has a positive lower bound independent of t

$$\min_{t\in I_j}\{|u(t,\beta)|+|v(t,\beta)/\beta|\}=\mu_j(\beta)>0,\ \beta\in(0,1/2], \tag{18}$$

i.e. RC holds in I_j. The peak-valley structure is repeated in each root-interval I_j, we get an irregular infinite series $\{\mu_j(\beta)>0\}$. As the zeros $\{t_j\}$ of analytic function $u(t,\beta)$ do not have finite condensation point (else $u\equiv const.$), Therefore any finite point t surely lies in some root-interval I_j such that $\mu_j(\beta)>0$. That implies RC holds for any t. □

In our geometric analysis, only Theorem A has been used. We have proved RC under some conditions. Here the condition of single zero is not essential, as

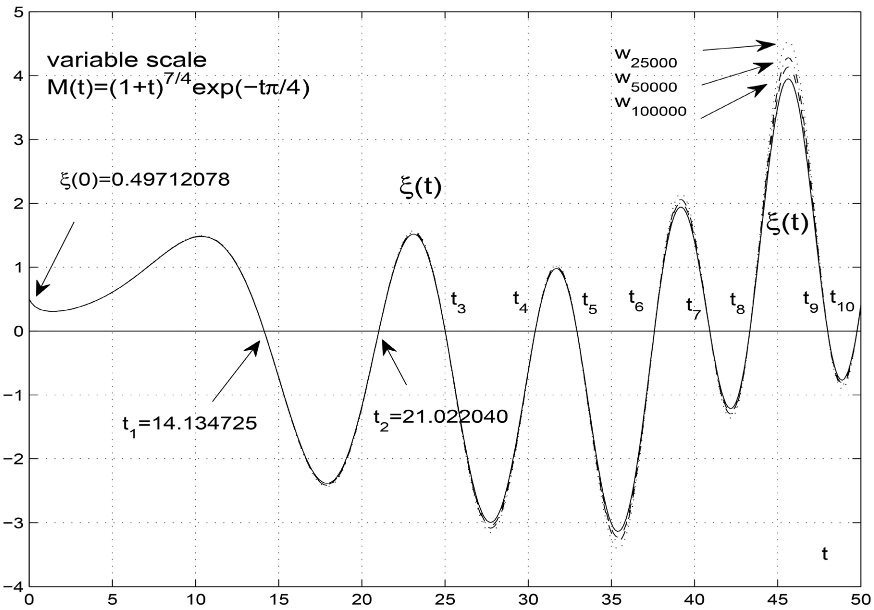

Figure 3. $w_n(t)$ approaches $u(t)$. Scale $M(45)=3.64\times10^{-13}$ very small.

RC can be proved for multiple zero, [19]. But proving the single peak has met essential difficulties, several attempts are unsuccessful. From this we have realized that only Theorem A is not enough, and new tool is needed. Thus we have to consider the product expression (Theorem B). It leads to the proof by contradiction presented in this paper.

3.4. Numerical Experiments of Product Expression

We take the first $n=10^5$ zeros $\{t_j\}$ in Odlyzko [11] and compute $w_n(t)=\xi(0)\prod_{j=1}^{n}\left(1-t^2/t_j^2\right)$, which is a good approximation of $\xi(t)$. By scaling $M(t)=(1+t)^{7/4}\,\mathrm{e}^{-t\pi/4}$, the curves ξ/M are depicted in **Figure 3**, [19]. Therefore we believe that $\xi(t)$ can be uniquely determined by $\xi(0)$ and all real roots $\{t_j\}$ of $\xi(t)$, i.e. $\xi(t)=w(t)$, which inspired us in the proof of RC.

4. Proof of Riemann Conjecture

4.1. Estimates of $\xi(z)$ and $\zeta(s)$

Expanding $\ln\Gamma(s/2)$ by (1) and decomposing the real part and imaginary part, $c=\ln\left(\sqrt{2\pi}\right)$, we have,

$$\ln\Gamma\left(\frac{s}{2}\right)=c+(s/2-1/2)\left\{\ln|s/2|+i\arctan\frac{t}{\sigma}\right\}-\frac{s}{2}+\frac{1}{6s}+O\left(t^{-2}\right),$$

$$=c+\frac{1}{2}(\sigma-1+it)\left\{\ln\frac{t}{2}+\frac{\sigma^2}{2t^2}+i\left(\frac{\pi}{2}-\frac{\sigma}{t}\right)\right\}-\frac{1}{2}(\sigma+it)-\frac{i}{6t}+O\left(t^{-2}\right)$$

$$Re\ln\Gamma=c+\frac{1}{2}(\sigma-1)\ln(t/2)-t\pi/4+O\left(t^{-2}\right),$$

$$Im\ln\Gamma=(t/2)\left(\ln\frac{t}{2}-1\right)+(\sigma-1)\pi/4+\frac{2\sigma-\sigma^2}{4t}-\frac{1}{6t}+O\left(t^{-2}\right).$$

Denote $\sigma = 1/2 + \beta$, then

$$\left|\Gamma(s/2)\right| = \sqrt{2\pi}\left(\frac{t}{2}\right)^{(2\beta-1)/4} \mathrm{e}^{-t\pi/4}\left(1+O\left(t^{-2}\right)\right). \tag{19}$$

Noting that $s(1-s) = t^2\left(1 - i2\beta t^{-1} + \cdots\right)$ and $\pi^{-s/2} = \pi^{-\sigma/2}\mathrm{e}^{-i(t/2)\ln\pi}$, we get

$$\begin{cases} \xi(z) = \left|G(s)\right|\mathrm{e}^{i\phi(t,\beta)}\zeta(\sigma+it), \ s=\sigma+it, z=t-i\beta, \\ \left|G(s)\right| = C(\beta)t^{(2\beta+7)/4}\mathrm{e}^{-t\pi/4}\left(1+O\left(t^{-2}\right)\right) > 0, \\ \phi(t,\beta) = \dfrac{t}{2}\ln\dfrac{t}{2\mathrm{e}\pi} + (2\beta+7)\dfrac{\pi}{8} - \dfrac{2\beta}{t} + \dfrac{1+12\beta(1-\beta)}{48t} + O\left(t^{-2}\right). \end{cases} \tag{20}$$

In the period of Riemann, for $\sigma > 1$, using Euler's method and integration by parts one has

$$\begin{aligned} \zeta(s) = \sum_{n=1}^{\infty} n^{-s} &= \sum_{n=1}^{\infty}\int_0^1\left(n^{-s} - (n+x)^{-s}\right)\mathrm{d}x + \int_1^{\infty} y^{-s}\mathrm{d}y \\ &= \frac{1}{s-1} + s\sum_{n=1}^{\infty}\int_0^1 (n+x)^{-s-1}(1-x)\mathrm{d}x, \end{aligned}$$

which is analytically continued to $Re(s) > 0$. One can get a coarse estimate

$$\left|\zeta(s)\right| \le \left|\frac{1}{s-1}\right| + |s|\left|\int_1^{\infty} x^{-\sigma-1}\mathrm{d}x\right| \le Ct, \ \ \sigma\in[1/2,1], t>10. \tag{21}$$

4.2. Proof of Riemann Conjecture

Main Theorem. *Riemann conjecture holds, i.e.* $\xi(s)$ *has no complex roots.*

Proof. In the functional equation $\xi(z) = G(s)\zeta(s)$, $\left|G(s)\right| > 0$ is exponential decay, so this equation is one-to-one mapping between $\xi(z)$ and $\zeta(s)$.

Follow Riemann's derivation, $\xi(t) = G(s)\zeta(s)$ on $s = 1/2 + it$, where $\zeta(s)$ is unbounded, $\left|\zeta(s)\right| \le Ct$, and has infinitely many zeros (Hardy theorem). To avoid all zeros of $\zeta(s)$, we propose a new technique to define subset of symmetric line (remark 1)

$$L(R) = \left\{t : t \ge 2R, \text{ and} \left|\zeta(1/2+it)\right| \ge C_0 > 0\right\},$$

where $R = \left|t' + i\alpha\right| > 10$ to be defined below and constant $C_0 \in (0,1]$. Thus we have a basic estimate

$$\ln C_0 \le \ln\left|\xi(t)/G(s)\right| = \ln\left|\zeta(s)\right| \le \ln t + O(1), \ \ s = 1/2+it, \ \ t\in L(T). \tag{22}$$

Let $\{t_j\}$ be all real zeros of $\xi(t) = 0$, one can define product expression $w(z) = \xi(0)\prod_j^{\infty}\left(1 - z^2/t_j^2\right)$. If ξ has no complex roots, then $w(z) = G(s)\zeta(s)$.

By contradiction. If the product expression $\xi(z)$ has complex roots, then the functional equation is destroyed.

Actually, if $\xi(z)$ has a complex roots $t' + i\alpha$, where $0 < \alpha \le 1/2$, $R = \sqrt{t'^2 + \alpha^2} > 10$ (because no roots for $t' \le 10$), by Theorem A, $\xi(z)$ has four conjugate complex roots $\pm(t' \pm i\alpha)$. Thus by Theorem B, $\xi(z)$ must contain four factors

$$\begin{aligned} p(z) &= \left(1-\frac{z}{t'+i\alpha}\right)\left(1+\frac{z}{t'+i\alpha}\right)\left(1-\frac{z}{t'-i\alpha}\right)\left(1+\frac{z}{t'-i\alpha}\right) \\ &= \left(1-\frac{z^2}{(t'+i\alpha)^2}\right)\left(1-\frac{z^2}{(t'-i\alpha)^2}\right) \\ &= 1-2z^2\frac{t'^2-\alpha^2}{R^4}+\frac{z^4}{R^4} \\ &= \left(\frac{z^2}{R^2}-1\right)^2+4z^2\frac{\alpha^2}{R^4}. \end{aligned}$$

On symmetric line $\beta = 0$, $p(t) > 0$ and when $t > 2R > 20$,

$$p(t) = g(t/R)\left(\frac{t}{R}\right)^4, \ \frac{1}{2} < g(t/R) = \left(1-\frac{R^2}{t^2}\right)^2 + 4\frac{\alpha^2}{t^2} < 2. \tag{23}$$

We have an estimate of growth

$$\ln p(t) = 4\ln(t/R) + O(1), \ s = 1/2+it, \ t > 2R. \tag{24}$$

We consider product formula $\xi(z) = w(z)p(z)$, this factor $p(z)$ can not be contained in $\zeta(s)$. Because $C_0 \le |\zeta(s)| \le Ct$ on $L(R)$, then $\zeta(s)$ can not contain the fourth order factor $p(t) \ge 0.5(t/R)^4$ and should rewrite $\xi(t) = w(t)p(t) = G(s)\zeta(s)p(t)$. We have

$$\ln|\xi(t)/G(s)| = \ln|\zeta(s)| + \ln|p(t)| \ge 4\ln(t/R) + O(1), \ t \in L(R), \tag{25}$$

which for $t \gg 2R$ contradicts the basic estimate (22). Thus $\xi(z)$ can not have complex roots. If $\xi(t)$ has several factors $p_k(t)$, then (25) contains summation of these $\ln|p_k(t)|$ and still contradicts (22), here should take maximal R in $L(R)$. Therefore the Main theorem is proved. □

By Main Theorem, the Theorem B is promoted as

Theorem 1 (product formula). $\xi(z)$ *is uniquely expressed by product formula*

$$\xi(z) = \xi(0)\prod_{j=1}^{\infty}\left(1-\frac{z^2}{t_j^2}\right), \ z = t - i\beta, \ \xi(0) = 0.497120778188312\cdots, \tag{26}$$

where $\{t_j\}$ are all real roots of $\xi(t)$, should take *k*-ple products for *k*-ple roots.

Remark 1. A hole in [19] is also repaired by introducing the subset $L(R)$.

Remark 2. We point out that Riemann had already approached to a proof of RC, but lacked the last step. Riemann said, "*I have put aside the research for such a proof after some fleeting vain attempts*". It's impossible to know what attempts have been done by Riemann. Siegel (1932) found a computing formula in Riemann's manuscript unpublished, now called R-S formula. Edwards pointed out in [1] (p. 164), "*Siegel states quite positively that the Riemann papers contain no steps toward a proof of the Riemann hypothesis*". Now we find a clue. Recall R-S formula [1] p. 145,

$$\begin{cases} Z(t) = 2\sum_{1\le n\le N} n^{-1/2}\cos(\theta(t) - t\ln n) + R_N, \ N = \left[\sqrt{t/2\pi}\right], \\ R_N(t) = (-1)^{N-1} y^{1/2}\left\{a_0(r) + a_1(r)y^1 + a_2(r)y^2 + a_3(r)y^3 + \cdots\right\}, \end{cases} \tag{27}$$

where $N=[X]$ is the integer part of $X=\sqrt{t/2\pi}$, $r=X-[X]$, $y=1/X$ and the coefficients

$$a_0(r)=\psi(r)=\frac{\cos\left(2\pi\left(r^2-r-1/16\right)\right)}{\cos(2\pi r)},\ a_1(r)=-\frac{1}{2^5\cdot 3\pi^2}\psi'''(r),\cdots.$$

It is easy to get an estimate [1] p.200,

$$\begin{aligned}|Z(t)|&=2|\zeta(1/2+it)|\le 2\sum_{1\le n\le N}n^{-1/2}+|R_N|\\&\le C\int_1^N x^{-1/2}\mathrm{d}x+Ct^{-1/4}\le CN^{1/2}+Ct^{-1/4}\le Ct^{1/4}.\end{aligned}\tag{28}$$

So we guess, Riemann omitted the following fact: if $\xi(z)$ has a complex roots, then $Z(t)$ should contain the fourth order factor $p(t)>0$ and this estimate on $L(R)$ is destroyed (so RC holds). Therefore we know that R-S formula has already provided the first step toward a proof of RC. But this fact has not been found for a long time, and the R-S formula is only used in computing. This is another misunderstanding for Riemann.

4.3. Some Future Research Topics

Riemann's algebraic model contains two theorems:

1) Functional equation $\xi(z)=G(s)\zeta(s)$, where $\zeta(s)$ increases slowly.

2) Entire function $\xi(z)$ has product expression, but admit complex roots (contradicts RC).

From them Riemann conjecture can be proved by contradiction.

In analytic number theory, besides RH, there are several important generalizations, for example, (Grand RH) GRH, (Generalization RH) gRH and (Extension RH) ERH. If they have analytic continuation and corresponding functional equation, therefore our method of contradiction is useful.

Sarnak [5] reviewed RH and GRH. Let χ be a primitive Dirichlet character of modulus q (*i.e.* $\chi(mn)=\chi(m)\chi(n)$, $\chi(1)=1$, $\chi(m+bq)=\chi(m)$), then L-series is defined by

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}=\prod_p\left(1-\chi(p)p^{-s}\right)^{-1}.$$

As with zeta, $L(s,\chi)$ extends to an entire function and satisfies the functional equation

$$\Lambda(s,\chi)=\pi^{(s+a_\chi)/2}\Gamma\left(\frac{s+a_\chi}{2}\right)L(s,\chi),$$

where $a_\chi=(1+\chi(-1))/2$. There is

Grand Riemann Hypothesis: *The zeros of* $\Lambda(s,\chi)$ *all lie on* $Re(s)=1/2$.

Sarnak added a footnote:

"*Hardy* (*Collected Papers*, *Vol.1*, *p.*560) *assures us that latter will be proven within a week of a proof of the former*".

Besides, to compute $\xi(z)$ by (20), reducing the factor $\mathrm{e}^{-t\pi/4}$ and symmetric function $s(1-s)$ etc., we get a simplified formula

$$\begin{cases} Z(t,\beta) = (t/2\pi)^{\beta/2} \mathrm{e}^{i\theta(t,\beta)} \zeta(s), \ s = \sigma + it, \ \beta = \sigma - 1/2, \\ \theta(t,\beta) = \frac{t}{2} \ln \frac{t}{2\mathrm{e}\pi} + (2\beta+7)\frac{\pi}{8} + \frac{1+12\beta(1-\beta)}{48t} + \cdots \end{cases} \tag{29}$$

where $\zeta(s)$ is computed by E-M formula (10). This formula is of product-type, the algorithm simple. We have computed $Z(t,\beta)$ to $t = 10^6$ and peak-valley structure is preserved. However E-M formula requires $N \ge t$, its efficiency is lower. We expect that follow R-S formula, consider a contour integral [1] (p. 137)

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + \frac{\Gamma(-s+1)}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-Nx}(-x)^s}{\mathrm{e}^x - 1} \cdot \frac{\mathrm{d}x}{x},$$

and get a computational formula to take $N = O(\sqrt{t})$ enough. This is an open problem.

5. A Stronger Conclusion

Hinkkanen [13] (1997) and Lagarias [14] (1999) proved the positivity of $\xi_1(s) = G(s)\zeta(s)$ to get

Lemma 3 (Equivalence) [13] [14]. *For any* $\sigma > 1/2$, $t \ge 0$, *the positivity* $Re\left(\frac{\xi_1'(s)}{\xi_1(s)}\right) > 0$ *is equivalent to RC.*

This is a new property and shape conclusion.

Extension to $|\beta| \le 1/2$. Let $s = 1/2 + \beta + it = iz$, $z = t - i\beta$. By Theorem A, we have $\xi_1(s) = \xi(z)$ and derivatives $D_s\xi_1(s) = D_\beta\xi(z) = -i\xi'(z)$. Take the logarithm in (26) and derivation, we have

$$\frac{D_\beta \xi(z)}{\xi(z)} = -i\frac{\xi'(z)}{\xi(z)} = -i\sum_{j=1}^{\infty}\left\{\frac{1}{z-t_j} + \frac{1}{z+t_j}\right\}$$

$$= -i\sum_{j=1}^{\infty}\left\{\frac{t-t_j+i\beta}{(t-t_j)^2+\beta^2} + \frac{t+t_j+i\beta}{(t_j+t)^2+\beta^2}\right\}, \ |\beta| \le 1/2.$$

Its real part is

$$J = Re\left(\frac{D_\beta\xi(z)}{\xi(z)}\right) = \beta\sum_{j=1}^{\infty}\left\{\frac{1}{(t_j-t)^2+\beta^2} + \frac{1}{(t_j+t)^2+\beta^2}\right\} > 0, \ \beta > 0. \tag{30}$$

If $\beta < 0$ then $J < 0$. This series converges for finite *t*. □

Theorem 2 (monotone) [18] [19]. *The strict monotone* $|\xi(t-i\beta)| > |\xi(t-i\beta_0)|$ *holds for* $|\beta| > |\beta_0|$.

Proof. Denoting $\xi(z) = u + iv$ and $D_\beta\xi = u_\beta + iv_\beta$, we rewrite

$$J = Re\frac{D_\beta\xi(z)\bar{\xi}(z)}{|\xi(z)|^2} = \frac{\psi(t,\beta)}{|\xi(z)|^2}, \psi(t,\beta) = uu_\beta + vv_\beta.$$

Using $|\xi|^2 = u^2 + v^2$ and $D_\beta|\xi|^2 = 2\psi$, for $|\beta| > |\beta_0|$, we have

$$|\xi(t-i\beta)|^2 - |\xi(t-i\beta_0)|^2 = 2\int_{\beta_0}^{\beta}\psi(t,\beta)\mathrm{d}\beta > 0, \tag{31}$$

and strict monotone (*i.e.* ordering, **Figure 4**) $|\xi(t-i\beta)| > |\xi(t-i\beta_0)|$ for any *t*. □

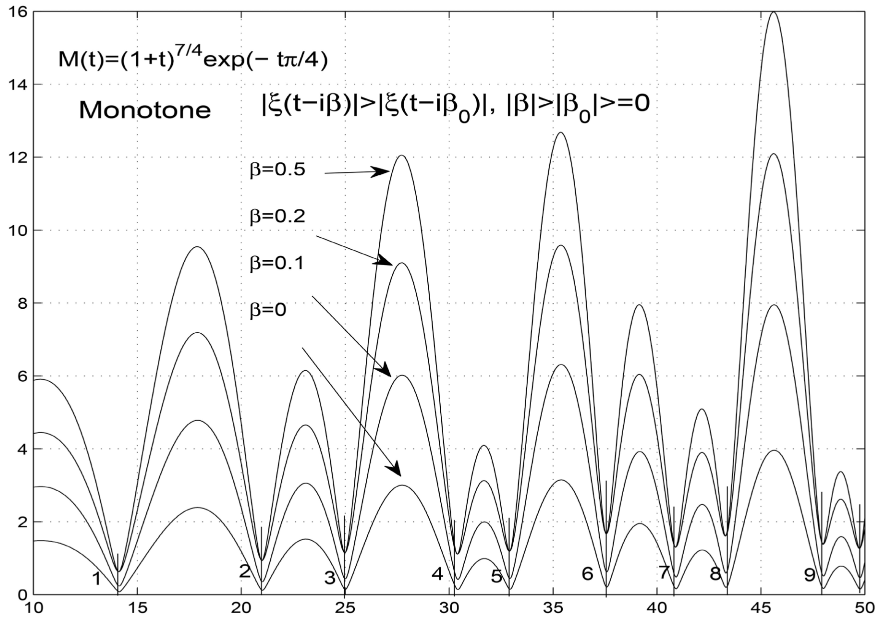

Figure 4. Monotone $\left|\xi(t-i\beta)\right|>\left|\xi(t-i\beta_0)\right|$, $\beta>\beta_0\geq 0$.

Ancient Greek Aristotle thought, "order and symmetry are important elements of beauty". Therefore we say, the symmetry and ordering of ξ are mathematical beauty of Riemann conjecture.

Acknowledgements

The author expresses sincere gratitude to the reviewer for his valuable and constructive comments. I should thank Prof. Zhengtin Hou and Prof. Xinwen Jiang for their precious opinion in discussions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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