Inequalities for Higher Order Riesz-Laguerre Transforms

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Abstract

The weak-type (1, 1) boundedness of the higher order Riesz-Laguerre transforms associated with the Laguerre polynomials and the boundedness for the Riesz-Laguerre transforms of order 2 are considered. We discuss a polynomial weight w that makes the Riesz-Laguerre transforms of order greater than or equal to 2 continuous from $L^1(wd\mu_\alpha)$ into $L^{1,\infty}(d\mu_\alpha)$, under specific value α, where μ_α is the Laguerre measure.

Keywords

Riesz-Laguerre Transform, Polynomial Expansion, Weak-Type, Stein Complex Interpolation Theorem, Calderon-Zygmund-Type, Riesz-Gauss Transform

1. Introduction

The aim of this paper is to discuss the weak-type (1, 1) boundedness of $\mathcal{R}_\alpha^{m+1}$ and the polynomial weight w that makes the Riesz-Laguerre transforms of order greater than or equal to 2 continuous from $L^1(wd\mu_\alpha)$ into $L^{1,\infty}(d\mu_\alpha)$, under specific value α. Following the same notions appear in [1] [2]. The $(m+1)$th Riesz-Laguerre transform with $m \in Z_{\geq 0}^d$ associated with the multidimensional Laguerre operator $\mathcal{L}_\alpha$, where $\alpha = (\alpha_1, \cdots, \alpha_d)$ is a multi-index with $\alpha_i \geq 0, i = 1, \cdots, d$.

The Laguerre operator $\mathcal{L}_\alpha$, is a self-adjoint "Laplacian" on $L^2(d\mu_\alpha)$, where μ_α is the Laguerre measure of type $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i > -1, i = 1, \cdots, d$; defined on $R_+^d = \{x \in R^d : x_i > 0, \text{ for each } i = 1, \cdots, d\}$, by

$$d\mu_\alpha(x) = \prod_{i=1}^d \frac{x_i^{\alpha_i} e^{-x_i}}{\Gamma(\alpha_i + 1)} dx.$$

It is well known that the spectral resolution of $\mathcal{L}_\infty$ is

$$\mathcal{L}_\infty = \sum_{n=0}^\infty n \mathcal{P}_n^\infty,$$

where $\mathcal{P}_n^\infty$ is the orthogonal projection on the space spanned by Laguerre polynomials of total degree n and type ∞ in d variables [3] [4]. The operator $\mathcal{L}_\infty$ is the infinitesimal generator of a "heat" semigroup, called the Laguerre semigroup, $\left\{ e^{-t\mathcal{L}_\infty} : t \geq 0 \right\}$, defined in the spectral sense as

$$e^{-t\mathcal{L}_\infty} = \sum_{n=0}^\infty e^{-nt} \mathcal{P}_n^\infty.$$

For any multi-index $m+1 = (a_1, \cdots, a_d) \in Z_{\geq 0}^d$, the Riesz-Laguerre transforms $\mathcal{R}_\infty^{m+1}$ of order $|m+1| = a_1 + \cdots + a_d$ are defined by

$$\mathcal{R}_\infty^{m+1} = \nabla_\alpha^{m+1} \left(\mathcal{L}_\alpha \right)^{-|m+1|/2} \mathcal{P}_0^{\infty\perp},$$

where ∇_∞ is associated to $\mathcal{L}_\infty$ defined as $\nabla_\infty = \left(\sqrt{x_1} \partial_{x_1}, \cdots, \sqrt{x_d} \partial_{x_d} \right)$, and $\mathcal{P}_0^{\infty\perp}$, denotes the orthogonal projection onto the orthogonal complement of the eigenspace corresponding to the eigenvalue 0 of $\mathcal{L}_\infty$.

In order to use the well-known relationship with the Ornstein-Uhlenbeck context, but not too much exploited in the weak-type inequalities, we are going to perform a change of coordinates in R_+^d. If $x = (x_1, \cdots, x_d)$ is a vector R_+^d, then x^2 will denote the vector $(x_1^2, \cdots, x_d^2)$. Let $\Psi : R_+^d \to R_+^d$ be defined as $\Psi(x) = x^2$ and let $d\tilde{\mu}_\infty = d\mu_\infty o \Psi^{-1}$ be the pull-back measure from $d\mu_\infty$. Then the modified Laguerre measure $d\tilde{\mu}_\infty$ is the probability measure

$$d\tilde{\mu}_\infty(x) = 2^d \prod_{i=1}^d \frac{x_i^{2\infty_i+1} e^{-x_i^2}}{\Gamma(\infty_i+1)} dx = 2^d \prod_{i=1}^d \frac{x_i^{2\infty_i+1}}{\Gamma(\infty_i+1)} e^{-|x|^2} dx, \tag{1}$$

on R_+^d.

The map $f \to \mathcal{U}_\Psi f = f o \Psi$ is an isometry from $L^q(d\mu_\alpha)$ onto $L^q(d\tilde{\mu}_\alpha)$ and from $L^{q,\infty}(d\mu_\alpha)$ onto $L^{q,\infty}(d\tilde{\mu}_\alpha)$, for every q in $[1, \infty]$. So we may reduce the problem of studying the weak-type boundedness of $\mathcal{R}_\infty^{m+1}$ to the study of the same boundedness for the modified Riesz-Laguerre transforms $\tilde{\mathcal{R}}_\infty^{m+1} = \mathcal{U}_\Psi \mathcal{R}_\infty^{m+1} \mathcal{U}_\Psi^{-1}$ with respect to the measure $d\tilde{\mu}_\infty$.

Observe that $\tilde{\mathcal{R}}_\infty^{m+1}$ coincides, up to a multiplicative constant, with $\nabla^{m+1} \left(\tilde{\mathcal{L}}_\infty \right)^{-|m+1|} \tilde{\mathcal{P}}_0^{\infty\perp}$, being $\tilde{\mathcal{L}}_\infty = \mathcal{U}_\Psi \mathcal{L}_\infty \mathcal{U}_\Psi^{-1}$, $\tilde{\mathcal{P}}_0^{\infty\perp} = \mathcal{U}_\Psi \mathcal{P}_0^{\infty\perp} \mathcal{U}_\Psi^{-1}$ andes ∇ the gradient of R^d associated to the Laplacian operator [5].

For the sequel, it is convenient to express the kernel of $\tilde{\mathcal{R}}_\infty^{m+1}$ with respect to the Polynomial measure $(m+1)_\infty$ defined on R_+^d as

$$d(m+1)_\infty(x) = e^{|x|^2} d\tilde{\mu}_\infty(x). \tag{2}$$

According to [6] [7], for $\infty_i > -1/2, i = 1, \cdots, d$, the kernel of the modified Riesz-Laguerre transforms of order $|m+1|$ with respect to the polynomial measure $(m+1)_\infty$ is defined, off the diagonal, as

$$\mathcal{K}^{m+1}(x,s) = \int_{[-1,1]^d} \mathcal{K}^{m+1}(x,s) \Pi_\alpha(s)(s) ds$$

with

$$\mathcal{K}^{m+1}(x,s) = \int_0^1 \left(\sqrt{r}\right)^{|m+1|-2} \left(-\frac{\log r}{1-r}\right)^{\frac{|m+1|-2}{2}} \prod_{i=1}^d H_{a_i}\left(\frac{x_i\left(\sqrt{r}-s_i\right)}{\sqrt{1-r}}\right) \frac{e^{-\frac{q-\left(rx^2,x^2,s\right)}{1-r}}}{(1-r)^{|\alpha|+d+1}} dr \quad (3)$$

where H_{a_i} is the Hermite polynomial of degree a_i and

$$q_{\pm}(x,s) = \sum_{i=1}^d 2x_i\left(1 \pm s_i\right),$$

$$\Pi_\alpha(s) = \prod_{i=1}^d \frac{\Gamma\left(\alpha_i+1\right)}{\Gamma\left(\alpha_i+\frac{1}{2}\right)\sqrt{\pi}}\left(1-s_i^2\right)^{\alpha_i-1/2},$$

$$\cos\theta = \cos\theta(x,s) = \frac{\sum_{i=1}^d x_i s_i}{|x|^2},$$

$$\sin\theta = \sin\theta(x,s) = \left(1-\cos^2\theta\right)^{1/2} = \left(1-\left(\frac{\sum_{i=1}^d x_i s_i}{|x|^2}\right)^2\right)^{1/2}.$$

The symbol $a \lesssim b$ means $a \le Cb$ where C is a constant that may be different on each occurrence. And we write $a \sim b$ whenever $a \lesssim b$ and $b \lesssim a$.

2. Main Results

For every multi-index $\propto$ we have the following result see [3].

Theorem 1: The second order Riesz-Laguerre transforms map $L^1\left(d\mu_\alpha\right)$ continuously into $L^{1,\infty}\left(d\mu_\alpha\right)$.

Proof: The result follows by splitting the modified Riesz-Laguerre transforms of second order into a local operator and a global one. Let us observe that for a simple covering Lemma, we may pass from estimates with respect to the measure $(m+1)_\infty$ on the local part R_0 to estimates with respect to the modified Laguerre measure $\tilde{\mu}_\infty$. Therefore the local operator is equivalent to T_0^{m+1} for $|m+1| = 2$. The global operators bounded weak type (1, 1) and therefore so are the second order modified Riesz-Laguerre transforms.

From [8] and [9], it is known that an upper bound for $\left|\mathcal{K}^{m+1}(x,s)\right|$ on G is

$$\tilde{\mathcal{K}}^{m+1}(x,s)$$
$$= \begin{cases} \left(|2x|^2\right)^{\frac{|m+1|-2}{2}} e^{-|x|^2}, & \cos\theta < 0 \\ \frac{1}{2}\left(4|x|^4 \sin^2\theta\right)^{\frac{|m+1|-2}{4}}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{|\alpha|-d}{2}}\left(1+(4|x|^4\sin^2\theta)^{\frac{1}{4}}e^{-u_0}\right), & \cos\theta \ge 0 \end{cases} \quad (4)$$

with

$$u_0 = \frac{\left(q_+\left(x^2,s\right)q_-\left(x^2,s\right)\right)^{1/2}}{2}.$$

Proposition 2: For $|m| = 2$,

$$\left|\mathcal{K}^m\left(x,y,s\right)\right| \lesssim e^{|x|^2} \mathcal{K}^*\left(x,y,s\right)e^{-|y|^2}$$

on the global region

$$G = R_+^d \times [-1,1]^d \setminus R_0 = R_1 \cup R_2 \cup R_3 \cup R_4,$$

with

$$R_0 = \left\{(y,s) \in R_+^d \times [-1,1]^d : q - \left(x^2, y^2, s\right)^{1/2} \le \frac{C}{1+|X|}\right\},$$

$$R_1 = \left\{(y,s) \notin R_0 : \cos\theta < 0\right\},$$

$$R_2 = \left\{(y,s) \notin R_0 : \cos\theta \ge 0, |y| \le |x|\right\},$$

$$R_3 = \left\{(y,s) \notin R_0 : \cos\theta \ge 0, |x| \le |y| \le 2|x|\right\},$$

$$R_4 = \left\{(y,s) \notin R_0 : \cos\theta \ge 0, |y| \le 2|x|\right\},$$

and

$$\mathcal{K}^*\left(x,y,s\right) = \begin{cases} e^{-|x|^2} & (y,s) \in R_1 \quad (5) \\[2mm] |x|^{2|\alpha|+2d} e^{-c|x|q_+^{1/2}\left(x^2,y^2,s\right)} & (y,s) \in R_2 \quad (6) \\[2mm] |x|^{2|\alpha|+2d}\left(1 \wedge \dfrac{e^{-\frac{c|x|^4\sin^2\theta}{|y|^2-|x|^2+\sin\theta|x|^2}}}{\left(|y|^2-|x|^2+\sin\theta|x|^2\right)^{\frac{2|\alpha|+2d-1}{2}}}\right) & (y,s) \in R_3 \quad (7) \\[2mm] \left(1+|x|\right)e^{-\sin^2\theta|x|^2} & (y,s) \in R_4 \quad (8) \end{cases}$$

Proof. In this proposition, $|m| = 2$. If $\cos\theta < 0$, it is immediate that

$$\left|\mathcal{K}^m\left(x,y,s\right)\right| \lesssim e^{|x|^2} \mathcal{K}^*\left(x,y,s\right)e^{-|y|^2}.$$

Let us then assume that $\cos\theta \ge 0$.

1) First let us consider $|x| > |y|$.

Since $\cos\theta \ge 0, q_+^{1/2} \ge |x|$ and since $|x| > |y|$, then $q_+^{1/2} \le 2|x|$. Therefore $q_+^{1/2} \sim |x|$. On the other hand, since $q_-^{1/2} \ge \dfrac{C}{1+|x|}$ then $|x| \ge c$. Thus

$$\left|\mathcal{K}^m\left(x,y,s\right)\right| \lesssim \left[\left(\frac{q_+}{q_-}\right)^{\frac{|\alpha|+d}{2}} + \left(\frac{q_+}{q_-}\right)^{\frac{|\alpha|+d}{2}}\left(q_+q_-\right)^{1/4}\right]e^{-u_0}$$

$$\lesssim \left[|x|^{|\alpha|+d}\left(1+|x|\right)^{|\alpha|+d} + \frac{\left(q_+^{1/2}\right)^{|\alpha|+d+1/2}}{\left(q_-^{1/2}\right)^{|\alpha|+d-1/2}}\right]e^{-u_0}$$

$$\lesssim |x|^{2|\alpha|+2d} e^{-u_0} = e^{|x|^2}|x|^{2|\alpha|+2d} e^{-\frac{\left(q_+q_-\right)^{1/2}}{2}\frac{|y|^2-|x|^2}{2}}e^{-|y|^2}$$

$$\lesssim e^{|x|^2}|x|^{2|\alpha|+2d} e^{-\frac{|x|q_-^{1/2}\left(x^2,y^2,s\right)}{2}}e^{-|y|^2} = e^{|x|^2}\mathcal{K}^*\left(x,y,s\right)e^{-|y|^2}.$$

2) Now let us assume $|y| \ge |x|$ and rewrite u_0 in the following way:

$$u_0 = \frac{|y|^2 - |x|^2}{2} + \frac{\left(q_+\left(x^2, y^2, s\right)q_-\left(x^2, y^2, s\right)\right)^{1/2}}{2}$$

$$= |y|^2 - |x|^2 + \frac{\left(q_+\left(x^2, y^2, s\right)q_-\left(x^2, y^2, s\right)\right)^{1/2} - \left(|y|^2 - |x|^2\right)}{2}$$

$$= |y|^2 - |x|^2 \frac{q_+ q_- - \left(|y|^2 - |x|^2\right)^2}{2\left(|y|^2 - |x|^2 + \left(q_+ q_-\right)^{1/2}\right)} \tag{9}$$

$$= |y|^2 - |x|^2 + \frac{2\sin^2\theta|x|^2 - |y|^2}{|y|^2 - |x|^2 + \left(q_+ q_-\right)^{1/2}}.$$

Since

$$q_+ q_- = \left(|x|^2 + |y|^2\right)^2 - 4|x|^2|y|^2\cos^2\theta = \left(|y|^2 - |x|^2\right)^2 + 4|x|^2|y|^2\sin^2\theta,$$

and taking into account that $\sin\theta$ is non-negative, we obtain that

$$\left(q_+ q_-\right)^{1/2} \sim |y|^2 - |x|^2 + |x||y|\sin\theta \geq |y|^2 - |x|^2 + |x|^2\sin\theta. \tag{10}$$

Thus, from (9) together with (10) we get

$$u_0 \geq |y|^2 - |x|^2 + \frac{c|x|^4\sin^2\theta}{|y|^2 - |x|^2 + |x||y|\sin\theta}. \tag{11}$$

Claim 3: $\max\left(|y|^2 - |x|^2, |x|^2\sin\theta\right) \geq 1$.

Proof. If $\sin\theta \geq \dfrac{1}{|x|^2}$, the inequality is immediate.

If $\sin\theta \leq \dfrac{1}{|x|^2}$, then $|y|^2 \geq |x|^2 + 1$. This inequality is immediate when $|x| \leq 1$ by adjusting conveniently the constant C in the definition of the global zone and it is also immediate for $d = 1$ and $|x| > 1$. Now let us assume that $d \geq 2$ and $|x| > 1$.

$$\frac{\left(C/2\right)^2}{|x|^2} \leq \frac{C^2}{\left(1 + |x|\right)^2} \leq q_-\left(x^2, y^2, s\right) = |x|^2 + |y|^2 - 2|x||y|\sqrt{1 - \sin^2\theta}$$

$$\leq |x|^2 + |y|^2 - 2|x||y|\sqrt{1 - \frac{1}{|x|^4}}.$$

Hence

$$|x|^2 - 2|x|\sqrt{1 - \frac{1}{|x|^4}}|y| - |x|^2 - \frac{\left(C/2\right)^2}{|x|^2} \geq 0$$

for all $|y| \geq |x|$, then

$$|y| \geq |x|\sqrt{1 - \frac{1}{|x|^4}} + \frac{\sqrt{\left(C/2\right)^2 - 1}}{|x|}$$

which implies that

$$|y|^2 \geq |x|^2 + 2\sqrt{(C/2)^2 - 1}\sqrt{1 - \frac{1}{|x|^4}} + \frac{(C/2)^2 - 2}{|x|^2} \geq |x|^2 + 1.$$

Therefore by applying this claim to inequality (10) we obtain that $q_+ q_- \geq c$ in this context. If $|x| \leq |y| \leq 2|x|$, by taking into account (11), we get

$$u_0 \geq |y|^2 - |x|^2 + \frac{c|x|^4 \sin^2 \theta}{|y|^2 - |x|^2 + |x|^2 \sin \theta}, \tag{12}$$

then

$$\left|\mathcal{K}^m (x,y,s)\right| \lesssim \left(\frac{q_+}{q_-}\right)^{\frac{|\alpha|+d}{2}} (q_+ q_-)^{1/4} \, \mathrm{e}^{-u_0} \lesssim \frac{q_+^{\frac{|\alpha|+d}{2}}}{(q_+ q_-)^{\frac{2|\alpha|+2d}{4}}} (q_+ q_-)^{1/4} \, \mathrm{e}^{-u_0}$$

$$\lesssim \frac{|x|^{2|\alpha|+2d}}{\left[(q_+ q_-)^{1/2}\right]^{\frac{2|\alpha|+2d-1}{2}}} \mathrm{e}^{-u_0}$$

$$\lesssim \mathrm{e}^{|x|^2} \frac{|x|^{2|\alpha|+2d}}{\left(|y|^2 - |x|^2 + |x|^2 \sin \theta\right)^{\frac{2|\alpha|+2d-1}{2}}} \mathrm{e}^{-\frac{c|x|^4 \sin^2 \theta}{|y|^2 - |x|^2 + |x|^2 \sin \theta} \mathrm{e}^{-|y|^2}}.$$

To get the last inequality we have used (10) and (12). On the other hand, since $q_+ q_- \geq c$ it is immediate the following inequality

$$\left|\mathcal{K}^m (x,y,s)\right| \lesssim |x|^{2|\alpha|+2d} \, \mathrm{e}^{|x|^2 - |y|^2}.$$

Thus

$$\left|\mathcal{K}^m (x,y,s)\right| \lesssim \mathrm{e}^{|x|^2} K^* (x,y,s) \mathrm{e}^{-|y|^2}.$$

Now if $|y| \geq 2|x|$ then $(q_+ q_-)^{1/4} \leq \left(|x|^2 + |y|^2\right)^{1/2} \lesssim |y|^2$, and thus

$$\left(1 + (q_+ q_-)^{1/4} \frac{|x||y|}{|x|^2 + |y|^2}\right) \lesssim (1 + |x|).$$

Besides $q_- \geq \left(|y| - |x|\right)^2 \geq c|y|^2$ and $q_+ \leq C|y|^2$ therefore $\frac{q_+}{q_-} \leq C$. On the other hand, from (9) together with $(q_+ q_-)^{1/2} \leq |y|^2 - |x|^2 + 2|x||y|\sin \theta$ and $|y|^2 - |x|^2 + |x||y|\sin \theta \leq 2|y|^2$ we get

$$u_0 \geq |y|^2 - |x|^2 + \frac{\sin^2 \theta |x|^2 |y|^2}{|y|^2 - |x|^2 + |x||y|\sin \theta} \geq |y|^2 - |x|^2 + \frac{\sin^2 \theta}{2}|x|^2.$$

Therefore

$$\left|\mathcal{K}^m (x,y,s)\right| \lesssim \mathrm{e}^{|x|^2} K^* (x,y,s) \mathrm{e}^{-|y|^2}.$$

Proposition 4: The operator $\mathcal{K}^*$ defined as

$$\mathcal{K}^* f(x) = \mathrm{e}^{|x|^2} \int_{R_+^d} \int_{[-1,1]^d} \chi_G (x,s) \mathcal{K}^* (x,s) \Pi_\alpha (s) \left|f(x)\right| d\tilde{\mu}_\alpha (x),$$

is of weak type (1, 1) with respect to the measure $\tilde{\mu}_\infty$.

Proof. The method of proof used in [1] is an adaptation to our context of the techniques developed in [10] [11] [12] which allows us to get rid of the classical one called "forbidden regions technique".

The kernels (5) and (6) define strong type (1, 1) operators. Indeed,

$$e^{|x|^2} \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_1}(x,s) e^{-|x|^2} \Pi_\infty(s) |f(x)| d\tilde{\mu}_\infty(x) \le C \|f\|_1.$$

Moreover, for semi-integer values of the parameter α, by [6]

$$|x|^{2(|\infty|+d)} e^{-C|x|^2 (2(1-\cos\theta))^{1/2}} \tag{13}$$

is in $L^1(d\mu_\alpha)$ uniformly in x and s and so the operator is of strong type with respect to $\tilde{\mu}_\infty$ on R_2. Finally the result for the other values of ∞ is obtained via the multidimensional Stein's complex interpolation Theorem. So to get the weak-type (1, 1) inequality for the operator $\mathcal{K}^*$ it suffices to prove that the operators

$$S_i f(x) = e^{|x|^2} \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_{i+3}}(x,s) \mathcal{K}^*(x,s) \Pi_\infty(s) ds |f(x)| d\tilde{\mu}_\infty(x), i = 0,1$$

map $L^1(d\tilde{\mu}_\infty)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\infty)$.

Without loss of generality, we may assume that $f \ge 0$. Fix $\lambda > 0$ and let

$$E_i = \left\{ x \in R_+^d : S_i f(x) > \lambda \right\},$$

for $i = 0,1$. We must prove that $\tilde{\mu}_\infty(E_i) \le C \dfrac{\|f\|_1}{\lambda}$. Let r_0 and r_1 be the positive roots of the equations

$$r_0^{2(|\infty|+d)} e^{r_0^2} \|f\|_1 = \lambda \quad \text{and} \quad r_1 e^{r_1^2} \|f\|_1 = \lambda.$$

We may observe that indeed, if $E_i \cap \left\{ x \in R_+^d : |x| < r_i \right\} = \varnothing$: indeed, if $|x| < r_i$, we have

$$S_0 f(x) \le |x|^{2(|\infty|+d)} e^{|x|^2} \|f\|_1 < \lambda,$$

$$S_1 f(x) \le |x| e^{|x|^2} \|f\|_1 < \lambda.$$

On the other hand, we may take $\lambda > K \|f\|$ in [1], and by choosing K large enough we may assume that both r_0 and r_1 are larger that one. Hence

$$\tilde{\mu}_\infty \left\{ x \in R_+^d : |x| < 2r_i \right\} \le \int_{|x| < 2r_i} \prod_{j=1}^d x_j^{2\infty_j + 1} e^{-|x|^2} dx \lesssim r_i^{2|\alpha|} e^{-4r_i^2} \le C \dfrac{\|f\|_1}{\lambda}.$$

Thus we only need to estimate $\tilde{\mu}_\infty \left\{ x \in R_+^d : r_i \le |x| \le 2r_i \right\}$.

We let E_i' denote the set of $x' \in S^{d-1}$ for which there exists a $\rho \in [r_i, 2r_i]$ with $\rho x' \in E$. For each $x' \in E_i'$ we let $\rho(x')$ be the smallest such ρ. Observe that

$$\sin\theta(x,s) = \sin\theta(x',s) = \sin\theta.$$

Then $S_i f(\rho(x') x') = \lambda$, by continuity. This implies for $i = 0$ and $x' \in E_0'$,

$$\lambda = S_0 f\left(\rho_0(x')x'\right)$$

$$= \int_{R_+^d} \int_{[-1,1]^d} x_{R_3} e^{|x|^2} x^{2(|\alpha|+d)} \left(1 \wedge \frac{e^{-\frac{c|x|^2 \sin^2\theta}{\sin\theta|x|^2}}}{\left(\sin\theta|x|^2\right)^{\frac{2(|\alpha|+d)-1}{2}}}\right) \Pi_\alpha(s) ds\, f(x) d_{\tilde{\mu}_\alpha}(x)$$

$$\lesssim e^{p_0^2(x')^2} r_0^{2(|\alpha|+d)} \Pi_\alpha(s) ds\, f(x) d_{\tilde{\mu}_\alpha}(x) \tag{14}$$

$$\times \int_{R_+^d} \int_{[-1,1]^d\{|x|\geq r_0\}} x(x) \left(1 \wedge \frac{e^{-\frac{cr_0^4 \sin^2\theta}{|x|^2-r_0^2+\sin\theta r_0^2}}}{\left(|x|^2 - r_0^2 + \sin\theta r_0^2\right)^{\frac{2(|\alpha|+d)-1}{2}}}\right),$$

and for $i=1$ and $x' \in E_1'$,

$$\lambda = S_1 f\left(p_i(x')x'\right)$$

$$= \int_{R_+^d} \int_{[-1,1]^d} x_{R_5}(x) e^{|x|^2} \left(1+|x|\right) e^{-\sin^2\theta|x|^2} \Pi_\alpha(s) ds\, f(x) d_{\tilde{\mu}_\alpha}(x) \tag{15}$$

$$\lesssim e^{p_0^2(x')^2} r_1 \int_{R_+^d} \int_{[-1,1]^d\{|x|>r_1\}} x(x) e^{-c\sin^2\theta r_1^2} \Pi_\alpha(s) ds\, f(x) d_{\tilde{\mu}_\alpha}(x)$$

Clearly, since r_0 and r_1 are greater than one, we have

$$\tilde{\mu}_\alpha \left\{x \in E_i : r_i \leq |x| \leq 2r_i\right\} \leq \int_{E_i'} d\sigma(x') \int_{\rho_i(x')}^{2r_i} e^{-\rho^2} \rho^{2(|\alpha|+d)-1} d\rho$$

$$\lesssim \int_{E_i'} e^{-\rho_i(x')^2} r_i^{2(|\alpha|+d-1)} d\sigma(x')$$

combining this estimate for $i=0$ with (14), we get

$$\tilde{\mu}_\alpha \left\{x \in E_i : r_i \leq |x| \leq 2r_i\right\} \leq \frac{C}{\lambda} \int_{E_0'} r_0^{2((2|\alpha|+2d)-1)} d\sigma(x')\left(I_0 + II_0\right), \tag{16}$$

with

$$I_0 = \int_{[-1,1]^d} \int_{\{\sin\theta r_0^2 \leq c\}} \frac{e^{-\frac{cr_0^4 \sin^2\theta}{|x|^2-r_0^2+\sin\theta r_0^2}}}{\left(|x|^2 - r_0^2 + \sin\theta r_0^2\right)^{\frac{2(|\alpha|+d)-1}{2}}} f(x) d_{\tilde{\mu}_\alpha}(x) \Pi_\alpha(s) ds,$$

and

$$II_0 = \int_{[-1,1]^d} \int_{\{|x|\geq r_1 \sin\theta r_0^2 \leq c\}} f(x) d_{\tilde{\mu}_\alpha}(x) \Pi_\alpha(s) ds.$$

Similarly for $i=1$ with (15), we obtain

$$\tilde{\mu}_\alpha \left\{x \in E_1 : r_1 \leq |x| \leq 2r_1\right\} \leq \frac{C}{\lambda} \int_{E_1'} r_1^{2(|\alpha|+d)-1} d\sigma(\bar{x})\left(I_1 + II_1\right), \tag{17}$$

with

$$I_1 = \int_{[-1,1]^d} \int_{\{|x|\geq r_1 \sin\theta r_1^2 \geq c\}} e^{-c\sin^2\theta r_1^2} f(x) d_{\tilde{\mu}_\alpha}(x) \Pi_\alpha(s) ds,$$

and

$$II_1 = \int\limits_{[-1,1]^d} \int\limits_{\{\sin\theta r_1^2 \le c\}} f(x) d_{\tilde{\mu}_\alpha}(x) \Pi_\alpha(s) ds .$$

It is immediate to verify that

$$r_0^{2(2(|\alpha|+d)-1)} \int\limits_{[-1,1]^d} \int\limits_{\{x':\sin\theta r_0^2 \le c\}} \sigma(x') \Pi_\alpha(s) ds \le C$$

and

$$r_1^{2(|\alpha|+d)-1} \int\limits_{[-1,1]^d} \int\limits_{\{x':\sin\theta r_0^2 \le c\}} d\sigma(x') \Pi_\alpha(s) ds \le C.$$

Which give, after changing the order of integration in (16) and (17), the desired estimate for the terms involving II_0 and II_1, respectively as in [5]. Now let us prove that for $|x| \ge r_0$

$$r_0^{2(2(|\alpha|+d)-1)} \int\limits_{[-1,1]^d} \int\limits_{\{x':\sin\theta r_0^2 \le c\}} \frac{e^{-\frac{c r_0^4 \sin^2\theta}{|x|^2 - r_0^2 + \sin\theta r_0^2}}}{\left(|x|^2 - r_0^2 + \sin\theta r_0^2\right)^{\frac{2(|\alpha|+d)-1}{2}}} d\sigma(x') \Pi_\alpha(s) ds \le C$$

and for $|x| > r_1$

$$r_1^{2(|\alpha|+d)-1} \int\limits_{[-1,1]^d} \int\limits_{\{x':\sin\theta r_1^2 \ge c\}} e^{-c\sin^2\theta r_1^2} d\sigma(x') \Pi_\alpha(s) ds \le C.$$

Firstly, one considers the case where $\alpha = \left(\dfrac{n_1}{2}, -1, \cdots, \dfrac{n_d}{2} - 1\right)$ with $n_i \in N$ and $n_i > 1$ for each $i = 1, \cdots, d$. In this case the inner integrals can be interpreted as integrals over $S^{|n|-1}$ with respect to the Lebesgue measure, expressed in polyradial coordinates in [11]. The same estimates are obtained also for $\alpha \in \dfrac{N^d}{2} - 1 + iR^d$.

Finally the result for the other values of α are obtained via the multidimensional Stein's complex interpolation Theorem. Indeed, let $F : C^d \to C$ the function defined by

$$F(\xi) = r_0^{2(2\xi + 2d - 1)} \int\limits_{\{\sin\theta r_0^2 \le c\}} \frac{e^{-\frac{c r_0^4 \sin^2\theta}{|x|^2 - r_0^2 + \sin\theta r_0^2}}}{\left(|x|^2 - r_0^2 + \sin\theta r_0^2\right)^{\frac{2(|\xi|+d)-1}{2}}} \Pi_\xi(s) ds.$$

We have seen that $\left|F\left(\dfrac{n}{2} - 1\right)\right| \le C$ and it is easy to prove that

$\left|F\left(\dfrac{n}{2} - 1 + i\zeta\right)\right| \le \left|F\left(\dfrac{n}{2} - 1\right)\right|$, whenever n is a integer vector and $\zeta \in R^d$.

Now we introduce the possible roots of the equations mentioned in the following Remark see [13]

Remark 5: 1) a) if $r_0 = r_1$ then we have

$$r_0^{2|\alpha| + 2d - 1} = 1,$$

and

$$2|\alpha| + 2d - 1 = 0,$$

which implies that

$$|\alpha| = \frac{1}{2}(1 - 2d),$$

b) if $r_0 \neq r_1$ we have the quadratic equation

$$(2|\alpha| + 2d)\ln r_0 = \ln r_1 + r_1^2 - r_0^2$$

we assume, for simplicity, that $r_0 = e^n$ and $r_1 = e^{2n}$ we can find

$$(2|\alpha| + 2d)\ln e^n = \ln e^{2n} + e^{4n} - e^{2n}$$

$$\left(e^{2n}\right)^2 - e^{2n} + 2n(1 - |\alpha| - d) = 0$$

so that

$$e^{2n} = \frac{1 \pm \sqrt{1 - 8n(1 - |\alpha| - d)}}{2},$$

where $n \geq 1$, we can easily find r_0.

2) S_0 and S_1 aremonotone.

3) Since $\dfrac{|x|^{2(|\alpha|+d)}}{|x|} < 1$, then $|x|^{2(|\alpha|+d)} \leq |x| < C$.

Proposition 6: For all *m*, the operator

$$T_0^{m+1} f(x) = p.v. \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_0}(x,s) \mathcal{K}^{m+1}(x,s) \Pi_\alpha(s) ds f(x) d(m+1)_\alpha(x),$$

which is the modified Riesz-Laguerre transform restricted to the local region R_0, is of weak type (1, 1) with respect to the measure $\tilde{\mu}_\alpha$

Proof. The proof of this result follows the same steps like the proof of the weak-type boundedness on the local zone of the first order Riesz-Laguerre transforms done in [4] [9]. For the former we have the Calderon–Zygmund-type estimates for the kernel K^{m+1}.

Lemma 7: There exists a constant *C* such that

$$\left|\mathcal{K}^{m+1}(x,s)\right|\varphi(x,s) \leq C\left(2|x|^2(1-\cos\theta)\right)^{-(|\alpha|+d)},$$

$$\left|\nabla_{(x,x)}\left(\mathcal{K}^{m+1}(x,s)\varphi(x,s)\right)\right| \leq C\left(2|x|^2(1-\cos\theta)\right)^{-(|\alpha|+d+1/2)}$$

being $\varphi(x,s)$ acut-off function defined in [3] and $(x,s) \in R_0$.

Proof. Since $\left|x_i\left(\sqrt{r} - s_i\right)\right| \leq q_-^{1/2}\left(rx^2, x^2, s\right)$, then

$$\left|\prod_{i=1}^d H_{a_i}\left(\frac{x_i\left(\sqrt{r} - s_i\right)}{\sqrt{1-r}}\right)\right| e^{-\frac{q_-\left(rx^2, x^2, s\right)}{1-r}}$$

$$\lesssim \sum_{k=0}^{|m+1|}\left(\frac{q_-^{1/2}\left(rx^2, x^2, s\right)}{1-r}\right)^{k/2} e^{-\frac{q_-\left(rx^2, x^2, s\right)}{1-r}}$$

$$\lesssim e^{-\frac{q_-\left(rx^2,x^2,s\right)}{2(1-r)}} \lesssim e^{-c\frac{q_-\left(rx^2,x^2,s\right)}{1-r}},$$

where last inequality follows from this one:

$$q_-\left(rx^2,x^2,s\right) \geq \left(2|x|^2\left(1-\cos\theta\right)\right)^{1/2} - 2C\left(1-r^{1/2}\right)$$

when $(x,s) \in R_0$ in [6]. Thus on R_0

$$\left|\mathcal{K}^{m+1}(x,s)\right|\varphi(x,s) \lesssim \int_{1/2}^1 \left(\sqrt{r}\right)^{|m+1|-1}\left(-\frac{\log r}{1-r}\right)^{\frac{|m+1|-2}{2}} \frac{e^{-c\frac{q_-\left(rx^2,x^2,s\right)}{1-r}}}{\left(1-r\right)^{|\alpha|+d+1}} dr$$

$$\lesssim \int_0^{1/2}\left(\sqrt{r}\right)^{|m+1|-2}\left(-\log r\right)^{\frac{|m+1|-2}{2}} dr + \int_{1/2}^1 \frac{e^{-c\frac{q_-\left(rx^2,x^2,s\right)}{1-r}}}{\left(1-r\right)^{|\alpha|+d+1}} dr$$

$$\lesssim 1 + \left(2|x|^2\left(1-\cos\theta\right)\right)^{-(|\alpha|+d)}.$$

In computing the gradient to f the kernel with respect to x we are going to have integrals such as $\mathcal{K}^{m+1}(x,s)\partial_{x_j}(x,s)$,

$$\int_0^1 \left(\sqrt{r}\right)^{|m+1|-1}\left(-\frac{\log r}{1-r}\right)^{\frac{|m+1|-2}{2}}\prod_{i\neq j}H_{a_i}\left(\frac{x_i\left(\sqrt{r}-s_i\right)}{\sqrt{1-r}}\right)$$

$$\times H_{a_i-1}\left(\frac{x_i\left(\sqrt{r}-s_i\right)}{\sqrt{1-r}}\right)\frac{e^{-\frac{q_-\left(rx^2,x^2,s\right)}{1-r}}}{\left(1-r\right)^{|\alpha|+d+3/2}} dr,$$

with $H_{-1} \equiv 0$ and

$$\int_0^1 \left(\sqrt{r}\right)^{|m+1|-2}\left(-\frac{\log r}{1-r}\right)^{\frac{|m+1|-2}{2}}H_{a_i}\left(\frac{x_i\left(\sqrt{r}-s_i\right)}{\sqrt{1-r}}\right)\times\frac{x_j\left(\sqrt{r}-s_j\right)}{\sqrt{1-r}}\frac{e^{-\frac{q_-\left(rx^2,x^2,s\right)}{1-r}}}{\left(1-r\right)^{|\alpha|+d+3/2}} dr.$$

In order to estimate these three integrals we use the same estimates described at the beginning of this proof. For the first one we have to use

$$2\left|\nabla_x\varphi(x,s)\right| \lesssim \frac{1}{\left(2|x|^2\left(1-\cos\theta\right)\right)^{1/2}}$$

The gradient with respect to x is treated similarly.

For the latter we have the following Theorem regarding the $L^p d\tilde{\mu}_\infty$-boundedness for $1 < p < \infty$ of the modified Riesz-Laguerre transform of any order on G.

Theorem 8: The operator

$$\mathcal{R}_g^{m+1}f(x) = \int_{R_+^d}\int_{[-1,1]^d}\chi_G(x,s)\mathcal{K}^{m+1}(x,s)\Pi_\infty(s)dsf(x)d\mu_\infty(x)$$

is strong-type (p,p) for $1 < p < \infty$ with respect to the measure $\tilde{\mu}_\infty$.

Proof. The proof of this result in [1] is an adaptation to our context of the same result for the higher order Riesz–Gauss transform s done in [6]. Taking into account that on G, $q_+\left(x^2,s\right)q_-\left(x^2,s\right) \geq c$ when $\cos\theta \geq 0$, an upperbound

for $\left|\mathcal{K}^{m+1}(x,s)\right|$ is

$$\tilde{\tilde{\mathcal{K}}}^{m+1}(x,s) = \begin{cases} \left(2|x|^2\right)^{\frac{|m+1|-2}{2}} e^{-|x|^2} & \text{if } \cos\theta < 0, \\ \left[2|x|^2(1+\cos\theta)\right]^{|\alpha|+d} \left(2|x|^2\sin\theta\right)^{\frac{|m+1|-1}{2}} e^{-|x|^2\sin\theta} & \text{if } \cos\theta \geq 0. \end{cases}$$

Thus

$$\int_{R_+^d}\left|\int_{G\cap\{\cos\theta<0\}}\mathcal{K}^{m+1}(x,s)\Pi_\alpha(s)\,ds\,f(x)\,d\mu_\alpha(x)\right|^p d\tilde{\mu}_\alpha$$

$$\lesssim \int_{R_+^d}\left(\int_{G\cap\{\cos\theta<0\}}\tilde{\tilde{\mathcal{K}}}^{m+1}(x,s)\Pi_\alpha(s)\,ds\,|f(x)|\,d\mu_\alpha(x)\right)^p d\tilde{\mu}_\alpha$$

$$\lesssim \int_{R_+^d}\left(\int_{R_+^d}\left(2|x|^2\right)^{\frac{p'(|m+1|-2)}{2}}d\tilde{\mu}_\alpha\right)^{p-1}d\tilde{\mu}_\alpha(x)\|f\|^p_{L^p(d\tilde{\mu}_\alpha)}.$$

For the region $G\cap\{\cos\theta\geq 0\}$ we are going to use the following estimates:

$$2x^2 \leq q_+ \leq |2x|^2, q_- \geq 0, (q_+q_-)^{\frac{1}{2}} \geq 0,$$

$$0 \leq \left|\frac{1}{p}-\frac{1}{2}\right||x|^2\sin\theta < |x|^2\sin\theta, \text{ since } p > 1,$$

$$\int_{R_+^d}\left|\int_{G\cap\{\cos\theta<0\}}\mathcal{K}^{m+1}(x,s)\Pi_\alpha(s)\,ds\,f(x)\,d\mu_\alpha(x)\right|^p d\tilde{\mu}_\alpha$$

$$\lesssim \int_{R_+^d}\left(\int_{G\cap\{\cos\theta<0\}}\tilde{\tilde{\mathcal{K}}}^{m+1}(x,s)\Pi_\alpha(s)\,ds\,|f(x)|\,d\mu_\alpha(x)\right)^p d\tilde{\mu}_\alpha$$

$$\lesssim \int_{R_+^d}\left(\int_{G\cap\{\cos\theta<0\}}|2x|^{2(|\alpha|+d)}\left(2|x|^2\sin\theta\right)^{\frac{(m+1)-1}{2}}e^{-\frac{2|x|^2\sin\theta}{2}}\Pi_\alpha(s)\,ds \times |f(x)|e^{-\frac{|x|^2}{p}}d\mu_\alpha(x)\right)^p d\mu_\alpha(x)$$

$$\lesssim \int_{R_+^d}\left(\int_{G\cap\{\cos\theta<0\}}|2x|^{2(|\alpha|+d)}\left(2|x|^2\sin\theta\right)^{\frac{(m+1)-1}{2}}e^{-\left(\frac{1}{2}-\left|\frac{1}{p}-\frac{1}{2}\right|\right)|x|^2\sin\theta}\Pi_\alpha(s)\,ds \times |f(x)|e^{-\frac{|x|^2}{p}}d\mu_\alpha(x)\right)^p d\mu_\alpha(x)$$

$$\lesssim \int_{R_+^d}\left(\int_{G\cap\{\cos\theta<0\}}|2x|^{2(|\alpha|+d)}\left(2|x|^2\sin\theta\right)^{\frac{(m+1)-1}{2}}e^{-c|x|^2\sin\theta}\Pi_\alpha(s)\,ds \times |f(x)|e^{-\frac{|x|^2}{p}}d\mu_\alpha(x)\right)^p d\mu_\alpha(x).$$

To finish the proof we just need to check that the kernel

$$H(x,s) := |2x|^{2(|\alpha|+d)}e^{-c_p 2|x|^2\sin\theta}\chi_{\mathfrak{G}\cap\{\cos\theta\geq 0\}},$$

for $\mathfrak{G} = \left\{(x,s): q_-^{\frac{1}{2}}(x^2,s) \geq \dfrac{c}{1+2|x|}\right\}$ is in $L^1\left(d(m+1)_\alpha(x)\right)$

and independently of the remaining variables. Due to the symmetry of the kernel we are going to check only the first Claim given in [1].

$$\int_{R_+^d} H(x,s)\,d(m+1)_\alpha(x)$$

$$\lesssim \int_{0\leq 1}|x|^{2(|\alpha|+d)}e^{-c_p|x|\left(2|x|^2(1-\cos\theta)\right)^{1/2}}d(m+1)_\alpha(x)$$

$$+ \int_{0>1}2|x|^{2(|\alpha|+d)}e^{-\tilde{c}_p(2|x|)}d(m+1)_\alpha(x).$$

It is clear that the second integral is bounded independentl *y* of *x* and *s*, for the first one see (13) for any *x*.

It is known that the first order Riesz-Laguerre transforms are weak-type (1, 1). Furthermore, we also know from that the the Riesz-Laguerre transforms of order higher than 2 need not be weak-type (1, 1) with respect to μ_α. However, we can prove the following result that has to do with certain kind of weights we can add on the domain of these transforms to make them satisfy a weak-type inequality.

Let us mention that in the Gaussian context something quite similar occur with the higher order Riesz-Gauss transforms. Perez proved that for $|m+1| > 2$, the Riesz-Gauss transforms of order $|m+1|$ associated to the Ornstein-Uhlenbeck semigroup, map $L^1\left(\left(1+|x|^{|m+1|-2}\right)d\gamma\right)$ continuously into $L^{1,\infty}(d\gamma)$, with $d\gamma(x) = e^{-|x|^2}dx$. Regarding the weights for the Riesz-Laguerre transforms of order higher than 2, then [1] proved the following

Theorem 9: The Riesz-Laguerre transforms order $|m+1|$ with $|m+1| > 2$, map $L^1(wd\mu_\alpha)$ continuously into $L^{1,\infty}(d\mu_\alpha)$. Where

$$w(x) = \left(1+\sqrt{|x|}\right)^{|m+1|-2}$$

Proof. As we mention in the preliminaries to prove this theorem is equivalent to prove that the modified Riesz-Laguerre transforms of order higher than 2 map $L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\alpha)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\alpha)$, with $\tilde{w}(x) = \left(1+|x|\right)^{|m+1|-2}$. For each $x \in R_+^d$. Let us write

$$R_+^d \times [-1,1]^d = \bigcup_{i=0}^4 R_i.$$

Therefore, in order to get the result, it will be enough to prove that each of the following operators

$$T_i^{m+1} f(x) = \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_i}(x,s) \mathcal{K}^{m+1}(x,s) \Pi_\alpha(s) ds f(x) d\mu_\alpha(x),$$

for $i = 0, \cdots, 4$ maps $L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\alpha)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\alpha)$.

Observe that for all $m+1$ the operator T_0^{m+1} is weak-type (1, 1) with respect to $\tilde{\mu}_\alpha$. On the other hand, for the "global parts": $R_1, R_2, R_3,$ and R_4, we have the following estimate for the kernel $\mathcal{K}^{m+1}$

$$\left|\mathcal{K}^{m+1}(x,s)\right| \lesssim \begin{cases} \left(2|x|^2\right)^{\frac{|m+1|-2}{2}} \mathcal{K}^*(x,s), & \text{if } \cos\theta < 0, \\ \left(2|x|^2 \sin\theta\right)^{\frac{|m+1|-2}{2}} \mathcal{K}^*(x,s), & \cos\theta \geq 0 \end{cases}$$

If $(x,s) \in R_i$, $\left|k^{m+1}(x,s)\right|$ is controlled by $C\left(1+\{|x|\}\right)^{|m+1|-2} e^{-|x|^2}$ and there for it is immediate to prove that T_1^{m+1} maps $L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\alpha)$ into $L^1(d\tilde{\mu}_\alpha)$.

Now if $(x,s) \in R_i$, with $i = 2,3,4$, we claim that

$$\left|\mathcal{K}^{m+1}(x,s)\right| \lesssim \tilde{w}(x)\mathcal{K}^*(x,s)$$

If $(x,s) \in R_2$ since

$$q_+ \leq \left(2|x|\right)^2 \lesssim |x|^2,$$

then

$$\left| \mathcal{K}^{m+1}(x,s) \right| \lesssim \left(2|x|^2 \sin\theta \right)^{\frac{|m+1|-2}{2}} \mathrm{e}^{-C\left(|x|^4(1-\cos\theta)\right)^{1/2}} \lesssim \tilde{w}(x) \mathrm{e}^{-C\left(|x|^4(1-\cos\theta)\right)^{1/2}}.$$

Also

$$q_+ q_- = 4|x|^4 \sin^2\theta.$$

Thus

$$\left| \mathcal{K}^{m+1}(x,s) \right| \lesssim \left(2|x|^2 \sin\theta \right)^{\frac{|m+1|-2}{2}} \mathcal{K}^*(x,s) \lesssim \tilde{w}(x) \mathcal{K}^*(x,s).$$

And this concludes the proof of the Theorem.

It should be noted that there is another proof of Theorem 9 for multi-indices of half-integer type by taking f_w as the function f in [3] [7].

Now we introduce a sharp estimate for w.

Corollary 10: The Riesz-Laguerre transforms of order $|m+1|$ with $|m+1| > 2$, map $L^1(w d\mu_\alpha)$ continuously into $L^{1,\infty}(d\mu_\alpha)$. Where

$$|\infty| = \frac{8n(1-d) - 1 - \left(2\mathrm{e}^{2n} - 1\right)^2}{8n}$$

and

$$w(x) \leq \left(1 + \sqrt{|C|} \right)^{|m+1|-2} = K^{|m+1|-2}$$

Proof. From Theorem 9 and Remark 5: We can directly see that

$$w(x) = \left(1 + \sqrt{|x|} \right)^{|m+1|-2} \leq \left(1 + \sqrt{|C|} \right)^{|m+1|-2} \leq K^{|m+1|-2},$$

where $m \geq 2$.

Theorem 11: The weight w is the optimal polynomial weight needed to get the weak type (1, 1) inequality for the Riesz-Laguerre transforms of order $|m+1|$.

Proof. This proof follows essentially in [9]. With the notation of that Theorem 1 One takes $\eta \in \mathbb{R}_+^d$ with $|\eta|$ sufficiently large, away from the axis and obtains the following lower bound for $\mathcal{K}^{m+1}(x,\eta)$

$$\mathcal{K}^{m+1}(x,\eta) = C \int_{[-1,1]^d} \mathcal{K}^{m+1}(x,\eta,s) \Pi_\infty(s) ds \geq C|\eta|^{|m+1|-2|\infty|-d-1} \mathrm{e}^{\xi^2 - |\eta|^2} \quad (18)$$

for $x \in J = \left\{ \xi \dfrac{\eta}{|\eta|} + v : v \perp \eta, |v| < 1, \dfrac{1}{2}|\eta| < \xi < \dfrac{3}{2}|\eta| \right\}$.

Now if we assume that the Riesz-Laguerre transforms of order $|m+1| > 2$ map $L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\infty)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\infty)$ with $\tilde{w}_\varepsilon = (1+|x|)^\varepsilon$ and $0 < \varepsilon < |m+1|-2$ then by taking $f \geq 0$ in $L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\infty)$ close to an approximation of a point mass at η, with $\|f\|_{L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\infty)} = 1$ we have that $\mathcal{R}_\infty^{m+1} f(x)$ is close to $\mathrm{e}^{|\eta|^2} \mathcal{K}^{m+1}(x,\eta)|\eta|^{-\varepsilon}$ and by applying in equality (18) we get that

$$e|n|k^{m+1}(x,\eta)|\eta|^{-\epsilon} \geq |\eta|^{|m+1|-2|\alpha|-d-1-\epsilon} \mathrm{e}^{-\left(\frac{|n|}{2}\right)^2}. \text{ Therefore setting}$$

$$\lambda = \left| \eta \right|^{|m+1|-2|\alpha|-d-1-\epsilon} \mathrm{e}^{-\left(\frac{|\eta|}{2} \right)^2}$$

we obtain

$$\mathrm{e}^{-\left(\frac{|\eta|}{2} \right)^2} \left| \eta \right|^{2|\alpha|+d-1} \lesssim \tilde{\mu}_\alpha \left(J \right) \le \tilde{\mu}_\alpha \left\{ x \in R_+^d : R_\alpha^{m+1} f \left(x \right) > \lambda \right\}$$

$$\lesssim \frac{1}{\lambda} = C \left| \eta \right|^{2|\alpha|+d-|m+1|+1+\epsilon} \mathrm{e}^{-\left(\frac{|\eta|}{2} \right)^2}.$$

Hence $\left| \eta \right|^{|m+1|-2-\epsilon}$ must be bounded which is a contradiction. Therefore the conclusion of Theorem 11 holds.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Forzani, L., Sasso, E. and Scotto, R. (2009) Weak-Type Inequalities for Higher Order Riesz-Laguerre Transforms. *Journal of Functional Analysis*, **256**, 258-274. https://doi.org/10.1016/j.jfa.2008.09.004

[2] Betancor, J.J. and León-Contreras, M.D. (2021) Variation Inequalities for Riesz Transforms and Poisson Semigroups Associated with Laguerre Polynomial Expansions. https://arxiv.org/abs/2110.03493

[3] Graczyk, P., Loeb, J., Lopez, I., Nowak, A. and Urbina, W. (2005) Higher Order Riesz Transforms, Fractional Derivatives and Sobolev Spaces for Laguerre Expansions. *Journal de Mathématiques Pures et Appliquées*, **84**, 375-405. https://doi.org/10.1016/j.matpur.2004.09.003

[4] Szego, G. (1959) Orthogonal Polynomials. American Mathematical Society, New York, USA.

[5] Sasso, E. (2005) Maximal Operators for the Holomorphic LAGUERRE Semigroup. *Mathematica Scandinavica*, **97**, 235-265. https://doi.org/10.7146/math.scand.a-14974

[6] Sasso, E. (2004) Spectral Multipliers of Laplace Transform Type for the Laguerre Operator. *Bulletin of the Australian Mathematical Society*, **69**, 255-266. https://doi.org/10.1017/S0004972700035991

[7] Gutierrez, C., Incognito, A. and Torrea, J.L. (2001) Riesz Transforms, G-Functions and Multipliers for the Laguerre Semigroup. *Houston Journal of Mathematics*, **27**, 579-592.

[8] Perez, S. and Soria, F. (2000) Operators Associated with the Ornstein-Uhlenbeck Semigroup. *Journal of the London Mathematical Society*, **61**, 857-871. https://doi.org/10.1112/S0024610700008917

[9] Sasso, E. (2007) Weak Type Estimates for the Riesz-Laguerre Transforms. *Bulletin of the Australian Mathematical Society*, **75**, 397-408. https://doi.org/10.1017/S0004972700039320

[10] Forzani, L., Harboure, E. and Scotto, R. (2012) Weak Type Inequality for a Family of Singular Integral Operators Related with the Gaussian Measure. *Potential Analysis*, **31**,103-116. https://doi.org/10.1007/s11118-009-9124-x

[11] Forzani, L., Harboure, E. and Scotto, R. (2009) Harmonic Analysis Related to Hermite Expansions. *Applied and Numerical Harmonic Analysis*, **978**, 73-96.
https://doi.org/10.1007/978-0-8176-4588-5_5

[12] Garcia-Cuerva, J., Mauceri, G., Meda, S., Sjogren, P. and Torrea, J.L. (2003) Maximal Operators for the Holomorphic Ornstein-Uhlenbeck Semigroup. *Journal of the London Mathematical Society*, **67**, 219-234.
https://doi.org/10.1112/S0024610702003733

[13] Hussein, S. and Elobied, A. (2013) Verification of the Sharp Inequalities for Higher Order Riesz-Laguerre Transforms. *Journal of Natural and Medical Sciences*, **14**, 25-41.