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On the Domains of General Ordinary Differential Operators in the Direct Sum Spaces

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Abstract

Given general quasi-differential expressions $\tau_1, \tau_2, \cdots, \tau_n$, each of order n with complex coefficients and their formal adjoint are $\tau_1^+, \tau_2^+, \cdots, \tau_n^+$ on the interval $[a,b)$ respectively, we give a characterization of all regularly solvable operators and their adjoints generated by a general ordinary quasi-differential expression τ_{jp} in the direct sum Hilbert spaces $L_w^2(a_p, b_p), p = 1, \cdots, N$. The domains of these operators are described in terms of boundary conditions involving $L_w^2(a_p, b_p)$-solutions of the equations $\tau_{jp}[y] = \lambda w y$ and their adjoint $\tau_{jp}^+[z] = \bar{\lambda} w z$ $(\lambda \in \mathbb{C})$ on the intervals $[a_p, b_p)$. This characterization is an extension of those obtained in the case of one interval with one and two singular end-points of the interval (a,b), and is a generalization of those proved in the case of self-adjoint and *J*-self-adjoint differential operators as a special case, where *J* denotes complex conjugation.

Keywords

Quasi-Differential Expressions, Regular and Singular Equations, Minimal and Maximal Operators, Regularly Solvable Operators, *J*-Self-Adjoint Extension, Boundary Conditions

1. Introduction

Jiangang, Zheng and Jiong Sun [1] considered the problem of Sturm-Liouville differential equation:

$$-(py')' + qy = \lambda w y \text{ on } (a,b), \; -\infty \le a < b \le +\infty, \tag{1.1}$$

where p, q are complex functions, $p(x) \ne 0$ and $w(x) > 0$ a.e. on (a,b), p^{-1}, q, w are all locally integrable functions on (a,b), λ is the so-called spectral parameter. They studied the classification of Equation (1.1) according to the number of square-integrable solutions of Equation (1.1) in suitable weighted integrable spaces. This type of classification of differential equations plays an important role in the spectral theory of differential operators as it can tell us how to obtain the operator realizations associated with the differential equations.

Amos [2] considered the problem that all solutions of the second-order ordinary differential equation $\tau[y]=\lambda wy\ (\lambda\in\mathbb{C})$ are in $L_w^2(a,\infty)$ when τ is a second-order symmetric ordinary differential expression of the form $\tau[y]=-(py')'+qy$ on $[a,\infty)$ under sufficient conditions on the coefficients *p* and *q*. The case that not all solutions are in $L_w^2(a,\infty)$ was considered by Atkinson and Evans in ([3], Theorem 1). Sobhy El-Sayed and others [4] extend their results for a second-order non-symmetric ordinary differential expression $\tau[y]=-(p(y'-ry))'+\overline{r}p(y'-ry)+qy$ with complex coefficients.

Everitt and Zettl [5] considered the problem of characterizing all self-adjoint differential operators which can be generated by a formally symmetric Sturm-Liouville differential expression τ_p defined on two intervals $I_p,(p=1,2)$ with boundary conditions at the endpoints. Their work was motivated by Sturm-Liouville problems which occur in the literature in which the coefficients have a singularity in the interior of the underlying interval. An interesting feature of their work is the possibility of generating self-adjoint operators in this way which are not expressible as the direct sum of self-adjoint operators defined in the separate intervals.

Jiong Sun [6] gives a characterization of the self-adjoint extensions of the minimal operator T_0 generated in $L_w^2(0,b)$ by a formally-symmetric differential expression τ of arbitrary order n. If the minimal operator T_0 has deficiency indices (ℓ,ℓ), the domain of any self-adjoint extension of T_0 is described in terms of ℓ boundary conditions involving the square-integrable solutions of the differential equation $\tau[u]=\lambda u$ for $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Thus Sun Jiong has completely solved a problem of central importance which has evaded the efforts of mathematicians for the last two decades.

J. Knowles [7] and Zai-Jiu-Shang (1988) (see [8]) gave a characterization of the boundary conditions which determine the domain of any *J*-self-adjoint extension of the minimal operator T_0 with maximal deficiency index in the case when the field of regularity, $\Pi(T_0)$, of T_0 was non-empty. This is achieved by using Sun Jiong's results (1986) (see [6]) with only one singular endpoint.

Evans and Sobhy El-Sayed [9] gave a characterization of all regularly solvable operators and their adjoints generated by a general differential expression in Hilbert space $L_w^2(a,b)$ in the case of one interval with one singular endpoint. Also, in [4] [10]-[20], Sobhy El-Sayed gives a characterization of all regularly solvable operators in the case of one interval with two singular endpoints *a* and *b*, and a characterization of Sturm-Liouville differential operators in direct sum spaces. The domains of these operators are described in terms of boundary conditions featuring $L_w^2(a,b)$-solutions of $\tau[u]=\lambda wu$ and $\tau^+[v]=\overline{\lambda}wv$ at both singular end points *a* and *b*. Their results include those of Sun Jiong [6] concerning self-adjoint realizations of symmetric expression τ when the minimal operator has equal deficiency indices, and Zai-Jiu Shang in [8] concerning the *J*-self-adjoint operators as a special case.

Our objective in this research is to generalize the results of Evans and Sobhy El-Sayed, Jiong Sun, Naimark, Zettl and Zai-Jiu-Shang's results in [5]-[12] [21] [22] [23] for the general ordinary quasi-differential expressions $\tau_1, \tau_2, \cdots, \tau_n$ each of order n with complex coefficients generated by general Shin-Zettl matrices (see [9] [24] [25]) in the direct sum spaces such that the operators defined on each of the separate intervals $I_p = (a_p, b_p)$, $p = 1, 2, \cdots, N$. The left-hand endpoint of I_p is assumed to be regular but the right-hand end-point may be regular or singular.

2. Notation and Preliminaries

We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [4] [9] [10]-[20] [23] [24] [25] ([26], Chapter III) and [27].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written $nul(T)$, is the dimension of $N(T)$ and the deficiency of T, written $def(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed, then $def(T) = nul(T^*)$. The Fredholm domain of T is (in the notation of [9] [24] [26]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \in \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $ind(T - \lambda I) = nul(T - \lambda I) - def(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators A and B acting in H are said to form an adjoint pair if $A \subset B^*$ and consequently, $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(.,.)$ denotes the inner-product on H.

The field of regularity $\Pi(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that:

$$\|(A - \lambda I)x\| \geq K(\lambda)\|x\| \quad \text{for all} \quad x \in D(A) \tag{2.1}$$

or, equivalently, on using the Closed Graph Theorem, and $nul(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

The joint field of regularity $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, $\bar{\lambda} \in \Pi(B)$ and both $def(A - \lambda I)$ and $def(B - \bar{\lambda} I)$ may be finite. An adjoint pair of A and B is said to be compatible if $\Pi(A, B) \neq \varnothing$.

Now, we define a second order quasi-differential equations and quasi-derivatives.

Definition 2.1: Let the set $Z_2(I)$ denotes the collection of all square matrices $A = \{a_{rs}\}$ of order 2 × 2 and satisfy the conditions:

1) $a_{rs} : I \to \mathbb{C}\,(r, s = 1, 2)$,

2) $a_{rs} \in L_{loc}(r,s=1,2)$, (2.2)

3) $a_{12}(x) \neq 0$ (almost all $x \in I$),

where a_{12}^{-1} denote, in view of condition 3) in (2.2), the reciprocal function $a_{12}^{-1}(x) := (a_{12}(x))^{-1}$ (almost all $x \in I$).

Given $A \in Z_2(I)$, we define the quasi-derivatives $\{f_A^{[r]} : r = 0,1,2\}$ on I of a function $f : I \to \mathbb{C}$ by:

$$\left.\begin{aligned} &f_A^{[0]} := f,\ f_A^{[1]} := a_{12}^{-1}(f' - a_{11} f), \\ &y_A^{[2]} := \left(a_{12}^{-1}(f' - a_{11} f)\right)' - a_{22}a_{12}^{-1}(f' - a_{11} f) - a_{21} f \end{aligned}\right\} \quad (2.3)$$

where the prime ′ denotes classical differentiation on *I*. Also, we define the linear manifold $D_A \subset AC_{loc}(I) \subset L_{loc}(I)$ by:

$$D_A := \left\{ f : I \to \mathbb{C} \mid f_A^{[r-1]} \in AC_{loc}(I)(r = 1,2) \right\}, \quad (2.4)$$

where the notations $AC_{loc}(I)$ and $L_{loc}(I)$, denote the linear space of functions with values in the complex field $\mathbb{C}$, which are absolutely continuous and Lebesgue integrable, respectively over all compact sub-intervals of the interval $I = (a,b)$ of the real line $\mathbb{R}$.

The general linear ordinary quasi-differential equation of second-order given by:

$$y_A^{[2]} := 0 \quad \text{on } I. \quad (2.5)$$

The quasi-differential Equation (2.5) is said to be Lagrange symmetric when the matrix $A \in Z_2(I)$ satisfies the additional conditions:

1) a_{12} and $a_{21} : I \to \mathbb{R}$,

2) $a_{22} = -a_{11}$ on *I*. (2.6)

As examples of the homogeneous quasi-differential Equation (2.5), *i.e.*, $y_A^{[2]} := 0$ on *I*, we have:

1) Let:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{on } I. \quad (2.7)$$

Then $A \in Z_2(I), y_A^{[0]} = y, y_A^{[1]} = y^{(1)}$ and (2.5) takes the form:

$$y^{(2)} = 0 \quad \text{on } I;$$

here $y^{(1)}, y^{(2)}$ denote the classical derivatives y', y''.

2) If $a_0, a_1 : I \to \mathbb{C}$ and are continuous on *I*, and if:

$$A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \quad \text{on } I. \quad (2.8)$$

Then $A \in Z_2(I)$ with $y_A^{[0]} = y, y_A^{[1]} = y^{(1)}$ and the Equation (2.5) takes the form:

$$y_A^{[2]} = y^{(2)} + a_1 y^{(1)} + a_0 y = 0 \quad \text{on } I, \quad (2.9)$$

which is the classical equation of the second-order with continuous coefficients.

3) The most general Lagrange symmetric equation of the second-order, see (2.3) and (2.6) is given by:

$$A=\begin{pmatrix} r & p^{-1} \\ q & \overline{r} \end{pmatrix} \text{ on } I, \tag{2.10}$$

where $p,q:I\to\mathbb{R}, p\neq 0$ almost everywhere on *I*, $r:I\to\mathbb{C}$ and $p^{-1},q,r\in L_{loc}(I)$. This yields the symmetric quasi-differential equation in the standard form:

$$\left(p\left(y'-ry\right)\right)'-\overline{r}p\left(y'-ry\right)-qy=0 \text{ on } I, \tag{2.11}$$

with:

$$y_A^{[0]}=y, y_A^{[1]}=p\left(y'-ry\right)\in AC_{loc}(I).$$

If $r=0$ on I then this equation reduces to the generalized Sturm-Liouville equation:

$$p\left(y'\right)'-qy=0 \text{ on } I, \tag{2.12}$$

for which $y_A^{[0]}=y, y_A^{[1]}=py'\in AC_{loc}(I)$.

4) If $p:I\to\mathbb{R}$, $p\neq 0$ almost everywhere on *I*, $q,r:I\to\mathbb{C}$ and $p^{-1},q,rp^{-1}\in L_{loc}(I)$. Let,

$$A=\begin{pmatrix} 0 & p^{-1} \\ q & rp^{-1} \end{pmatrix} \text{ on } I, \tag{2.13}$$

then the quasi-derivatives and the quasi-differential equation associated with A are defined as follows:

$$y_A^{[0]}=y, y_A^{[1]}=py'\in AC_{loc}(I),$$

$$-\left(py'\right)'+ry'+qy=0 \text{ on } I, \tag{2.14}$$

(Evans's differential expression, see [9] and [26]). If $r=0$ on *I*, Equation (2.14) reduces to (2.12).

We now turn to the quasi-differential expressions defined in terms of a Shin-Zettl matrix A on an interval *I*.

Definition 2.2: The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $n\times n$-matrices $A=\{a_{rs}\},1\le r,s\le n$, whose entries are complex-valued functions on I which satisfy the following conditions:

$$\left.\begin{array}{lll} a_{rs}\in L_{loc}^{1}(I) & & \\ a_{r,r+1}\neq 0 & a.e. \text{ on } I & (1\le r\le n-1) \\ a_{rs}=0 & a.e. \text{ on } I & (2\le r+1<s\le n) \end{array}\right\} \tag{2.15}$$

For $A\in Z_n(I)$, the quasi-derivatives associated with A are defined by:

$$\left.\begin{array}{l} y^{[0]}:=y, \\ y^{[r]}:=\left(a_{r,r+1}\right)^{-1}\left\{\left(y^{[r-1]}\right)'-\sum_{s=1}^{r}a_{rs}y^{[s-1]}\right\},(1\le r\le n-1) \\ y^{[n]}:=\left(y^{[n-1]}\right)'-\sum_{s=1}^{n}a_{ns}y^{[s-1]} \end{array}\right\} \tag{2.16}$$

The quasi-differential expression τ associated with the matrix A is given by:

$$\tau[y] := i^n y^{[n]}, (n \geq 2) \tag{2.17}$$

this being defined on the set:

$$V(\tau) := \left\{ y : y^{[r-1]} \in AC_{loc}(I), r = 1, 2, \cdots, n \right\}. \tag{2.18}$$

The formal adjoint τ^+ of τ defined by the matrix $A^+ \in Z_n(I)$ is given by:

$$\tau^+[z] := i^n z_+^{[n]}, \text{ for all } z \in V(\tau^+); \tag{2.19}$$

this being defined on the set:

$$V(\tau^+) := \left\{ z : z_+^{[r-1]} \in AC_{loc}(I), r = 1, 2, \cdots, n \right\}, \tag{2.20}$$

where $z_+^{[r-1]}$, the quasi-derivatives associated with the matrix $A^+ \in Z_n(I)$,

$$A^+ = \left\{ a_{rs}^+ \right\} = (-1)^{r+s+1} \bar{a}_{n-s+1, n-r+1} \text{ for each } r, s, \ 1 \leq r, s \leq n \tag{2.21}$$

are therefore:

$$\left.\begin{aligned} & z_+^{[0]} := z, \\ & z_+^{[r]} := \left(\bar{a}_{n-r, n-r+1}\right)^{-1} \left(\left(z_+^{[r-1]}\right)' - \sum_{s=1}^{r} (-1)^{r+s+1} \bar{a}_{n-s+1, n-r+1} z_+^{[s-1]} \right) \\ & z_+^{[n]} := \left(z_+^{[n-1]}\right)' - \sum_{s=1}^{n} (-1)^{n+s+1} \bar{a}_{n-s+1, 1} z_+^{[s-1]}, (1 \leq r \leq n-1) \end{aligned}\right\} \tag{2.22}$$

Note that: $(A^+)^+ = A$ and so $(\tau^+)^+ = \tau$. We refer to [1] [4] [5] [7] [9]-[13] [20]-[27] for a full account of the above and subsequent results on quasi-differential expressions.

Definition 2.3: For $u \in V(\tau), v \in V(\tau^+)$ and $\alpha, \beta \in I$, we have the Green's formula:

$$\int_\alpha^\beta \left\{ \bar{v}\tau[u] - u\overline{\tau^+[v]} \right\} \mathrm{d}x = [u, v](\beta) - [u, v](\alpha), \tag{2.23}$$

where,

$$\begin{aligned} [u, v](x) &= i^n \left(\sum_{r=0}^{n-1} (-1)^{n+r+1} u^{[r]}(x) \bar{v}_+^{[n-r-1]}(x) \right) \\ &= (-i)^n \left(u, \cdots, u^{n-1}\right) J_{n \times n} \begin{pmatrix} \bar{v} \\ \vdots \\ \bar{v}_+^{[n-1]} \end{pmatrix} (x); \end{aligned} \tag{2.24}$$

see [4] [9]-[20] ([23], Corollary 1) and [24] [26].

Let the interval I have end-points a, b ($-\infty \leq a < b \leq \infty$) and let $w: I \to \mathbb{R}$ be a non-negative weight function with $w \in L_{loc}^1(I)$ and $w(x) > 0$ (for almost all $x \in I$). Then $H = L_w^2(I)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int w\|f\|^2 < \infty$; the inner-product is defined by:

$$(f, g) := \int w(x) f(x) \overline{g(x)} \mathrm{d}x \ \left(f, g \in L_w^2(I)\right). \tag{2.25}$$

The equation,

$$\tau[y]-\lambda wy=0\ (\lambda\in\mathbb{C})\ \text{ on } I, \tag{2.26}$$

is said to be **regular** at the left end-point $a\in R$, if for all $X\in(a,b)$,

$$a\in\mathbb{R};\ w,a_{rs}\in L^1(a,X),(r,s=1,2,\cdots,n). \tag{2.27}$$

Otherwise (2.26) is said to be **singular** at *a*. If (2.26) is regular at both end-points, then it is said to be regular; in this case we have,

$$a,b\in\mathbb{R};\ w,a_{rs}\in L^1(a,b),(r,s=1,2,\cdots,n). \tag{2.28}$$

We shall be concerned with the case when *a* is a regular end-point of (2.26), the end-point *b* being allowed to be either regular or singular. Note that, in view of (2.22) an end-point of *I* is regular (see [4] [9]-[20] [22] [23]) for the Equation (2.26), if and only if it is regular for the equation,

$$\tau^+[z]-\bar{\lambda}wz=0\ (\lambda\in\mathbb{C})\ \text{ on } I. \tag{2.29}$$

Note that, at a regular end-point *a*, say, $y^{[r-1]}(a)\left(z_+^{[r-1]}(a)\right),r=1,\cdots,n$ is defined for all $y\in V(\tau)\left(z\in V(\tau^+)\right)$. Set,

$$\left.\begin{aligned}D(\tau)&:=\left\{y:y\in V(\tau),y\text{ and }w^{-1}\tau[y]\in L_w^2(a,b)\right\}\\ D(\tau^+)&:=\left\{z:z\in V(\tau^+),z\text{ and }w^{-1}\tau^+[z]\in L_w^2(a,b)\right\}\end{aligned}\right\} \tag{2.30}$$

The subspaces $D(\tau)$ and $D(\tau^+)$ of $L_w^2(a,b)$ are the domains of the so-called maximal operators $T(\tau)$ and $T(\tau^+)$ respectively, defined by:

$$T(\tau)y:=w^{-1}\tau[y]\ \left(y\in D(\tau)\right)\ \text{ and }\ T(\tau^+)z:=w^{-1}\tau[z]\ \left(z\in D(\tau^+)\right).$$

For the regular problem the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$, are the restrictions of $w^{-1}\tau[y]$ and $w^{-1}\tau[z]$ to subspaces:

$$\left.\begin{aligned}D_0(\tau)&:=\left\{y:y\in D(\tau),y^{[r-1]}(a)=y^{[r-1]}(b)=0,r=1,\cdots,n\right\}\\ D_0(\tau^+)&:=\left\{z:z\in D(\tau^+),z_+^{[r-1]}(a)=z_+^{[r-1]}(b)=0,r=1,\cdots,n\right\}\end{aligned}\right\} \tag{2.31}$$

respectively. The subspaces $D_0(\tau)$ and $D_0(\tau^+)$ are dense in $L_w^2(a,b)$, $T_0(\tau)$ and $T_0(\tau^+)$ are closed operators (see [1] [4] [9]-[20] [22] ([23], Section 3) and [24] [26] [27] [28]).

In the singular problem we first introduce the operators $T_0'(\tau)$ and $T_0'(\tau^+)$; $T_0'(\tau)$ being the restriction of $w^{-1}\tau[.]$ to the subspace:

$$D_0'(\tau):=\left\{y:y\in D(\tau),supp\ y\subset(a,b)\right\} \tag{2.32}$$

and with $T_0'(\tau^+)$ defined similarly. These operators are densely-defined and closable in $L_w^2(a,b)$; and we defined the minimal operators $T_0'(\tau)$ and $T_0'(\tau^+)$ to be their respective closures (see [1] [5] [9] [23] [24] [26] [28] [29]). We denote the domains of $T_0(\tau)$ and $T_0(\tau^+)$ by $D_0(\tau)$ and $D_0(\tau^+)$ respectively. It can be shown that:

$$\left.\begin{aligned}y\in D_0(\tau)&\Rightarrow y^{[r-1]}(a)=0,(r=1,\cdots,n)\\ z\in D_0(\tau^+)&\Rightarrow z_+^{[r-1]}(a)=0,(r=1,\epsilon,n)\end{aligned}\right\} \tag{2.33}$$

because we are assuming that a is a regular end-point. Moreover, in both regular and singular problems, we have

$$T_0^*(\tau) = T(\tau^+) \text{ and } T^*(\tau^+) = T_0(\tau), \tag{2.34}$$

see ([14] Section 5) in the case when $\tau = \tau^+$ and compare with treatment in [4] [9]-[20] [22] and ([26], Section III.10.3) in general case. Note that $T_0(\tau)$ and $T(\tau)$ are closed and densely-defined operators on *H*.

3. The Operators in Direct Sum Spaces

The operators here are no longer symmetric but direct sums:

$$T_0(\tau) = \oplus_{p=1}^N T_0(\tau_p) \text{ and } T_0(\tau^+) = \oplus_{p=1}^N T_0(\tau_p^+), \tag{3.1}$$

on any finite number of intervals $I_p = (a_p, b_p), p = 1, 2, \cdots, N$, where $T_0(\tau_p)$ is the minimal operator generated by τ_p in I_p and τ_p^+ denotes the formal adjoint of τ_p, which form an adjoint pair of closed operators in $\oplus_{p=1}^N L_{w_p}^2(I_p)$. Let H be the direct sum,

$$H = \oplus_{p=1}^N H_p = \oplus_{p=1}^N L_{w_p}^2(I_p). \tag{3.2}$$

The elements of H will be denoted by $\tilde{f} = \{f_1, \cdots, f_N\}$ with $f_1 \in H_1, \cdots, f_N \in H_N$. When $I_i \cap I_j = \varnothing, i \neq j; i, j = 1, \cdots, N$, the direct sum space $\oplus_{p=1}^N L_{w_p}^2(I_p)$ can be naturally identified with the space $L_{w_p}^2\left(\bigcup_{p=1}^N I_p\right)$ where $w = w_p$ on $I_p, p = 1, \cdots, N$. This is of particular significance when $\bigcup_{p=1}^N I_p$ may be taken as a single interval; see [13] [19] [20] [22] [28].

We now establish by [5] [10] [12] some further notation.

$$\left.\begin{array}{l} D_0(\tau) = \oplus_{p=1}^N D_0(\tau_p), \; D(\tau) = \oplus_{p=1}^N D(\tau_p) \\ D_0(\tau^+) = \oplus_{p=1}^N D_0(\tau_p^+), \; D(\tau^+) = \oplus_{p=1}^N D(\tau_p^+) \end{array}\right\} \tag{3.3}$$

$$\left.\begin{array}{l} T_0(\tau)\tilde{f} = \{T_0(\tau_1) f_1, \cdots, T_0(\tau_N) f_N\}, \; f_1 \in D_0(\tau_1), \cdots, f_N \in D_0(\tau_N) \\ T_0(\tau^+)\tilde{g} = \{T_0(\tau_1^+) g_1, \cdots, T_0(\tau_N^+) g_N\}, \; g_1 \in D_0(\tau_1^+), \cdots, g_N \in D_0(\tau_N^+) \end{array}\right\} \tag{3.4}$$

Also,

$$\left.\begin{array}{l} T(\tau)\tilde{f} = \{T(\tau_1) f_1, \cdots, T(\tau_N) f_N\}, \; f_1 \in D(\tau_1), \cdots, f_N \in D(\tau_N) \\ T(\tau^+)\tilde{g} = \{T(\tau_1^+) g_1, \cdots, T(\tau_N^+) g_N\}, \; g_1 \in D(\tau_1^+), \cdots, g_N \in D(\tau_N^+) \end{array}\right\} \tag{3.5}$$

$$[\tilde{f}, \tilde{g}] = \sum_{p=1}^N \{[f_p, g_p](b_p) - [f_p, g_p](a_p)\}, \; \tilde{f} \in D(\tau), \tilde{g} \in D(\tau^+) \tag{3.6}$$

$$(\tilde{f}, \tilde{g}) = \sum_{p=1}^N (f_p, g_p)_p, \tag{3.7}$$

where $\tilde{f} = \{f_1, \cdots, f_N\}, \tilde{g} = \{g_1, \cdots, g_N\}$ and $(.,.)_p$ the inner-product defined in (2.13). Note that $T_0(\tau)$ is a closed densely-defined operator in *H*.

We summarize a few additional properties of $T_0(\tau)$ in the form of a Lemma.

Lemma 3.1: We have:

1) $[T_0(\tau)]^* = \oplus_{p=1}^N [T_0(\tau_p)]^* = \oplus_{p=1}^N T(\tau_p^+)$,

$$\left[T_0\left(\tau^+\right)\right]^* = \oplus_{p=1}^N \left[T_0\left(\tau_p^+\right)\right]^* = \oplus_{p=1}^N T\left(\tau_p\right).$$

In particular,

$$D\left[T_0\left(\tau\right)\right]^* = D\left[T\left(\tau^+\right)\right] = \oplus_{p=1}^N D\left[T\left(\tau_p^+\right)\right],$$

$$D\left[T_0\left(\tau^+\right)\right]^* = D\left[T\left(\tau\right)\right] = \oplus_{p=1}^N D\left[T\left(\tau_p\right)\right],$$

2) $nul\left[T_0\left(\tau\right)-\lambda I\right] = \sum_{p=1}^N nul\left[T_0\left(\tau_p\right)-\lambda I\right]$,

$$nul\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right] = \sum_{p=1}^N nul\left[T_0\left(\tau_p^+\right)-\bar{\lambda} I\right].$$

3) The deficiency indices of $T_0\left(\tau\right)$ are given by

$$def\left[T_0\left(\tau\right)-\lambda I\right] = \sum_{p=1}^N def\left[T_0\left(\tau_p\right)-\lambda I\right] \text{ for all } \lambda \in \Pi\left[T_0\left(\tau\right)\right],$$

$$def\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right] = \sum_{p=1}^N def\left[T_0\left(\tau_p^+\right)-\bar{\lambda} I\right] \text{ for all } \lambda \in \Pi\left[T_0\left(\tau^+\right)\right]$$

Proof: Part 1) follows immediately from the definition of $T_0\left(\tau\right)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part 1) or follows immediately from the definitions.

Lemma 3.3: If $S_p, p = 1,\cdots,N$ are regularly solvable with respect to $T_0\left(\tau_p\right)$ and $T_0\left(\tau_p^+\right)$, then $S = \oplus_{p=1}^N S_p$ is regularly solvable with respect to:

$$T_0\left(\tau\right) = \oplus_{p=1}^N T_0\left(\tau_p\right) \text{ and } T_0\left(\tau^+\right) = \oplus_{p=1}^N T_0\left(\tau_p^+\right).$$

Proof: The proof follows from Lemmas 3.1 and 3.2.

Lemma 3.4: For $\lambda \in \Pi\left[T_0\left(\tau\right), T_0\left(\tau^+\right)\right]$,

$def\left[T_0\left(\tau\right)-\lambda I\right] + def\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right]$ is constant and:

$$0 \le def\left[T_0\left(\tau\right)-\lambda I\right] + def\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right] \le 2nN.$$

In the case with one singular end-point:

$$nN \le def\left[T_0\left(\tau\right)-\lambda I\right] + def\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right] \le 2nN.$$

In the regular problem:

$$def\left[T_0\left(\tau\right)-\lambda I\right] + def\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right] = 2nN, \text{ for all } \lambda \in \Pi\left[T_0\left(\tau\right), T_0\left(\tau^+\right)\right].$$

Proof: The proof is similar to that in ([3], lemma 3.1), [10] [11] [12] and therefore omitted.

For $\lambda \in \Pi\left[T_0\left(\tau\right), T_0\left(\tau^+\right)\right]$, we define r, s and m as follows:

$$\left.\begin{aligned}
r = r(\lambda) &:= def\left[T_0\left(\tau\right) ?\ \lambda I\right] = \sum_{p=1}^N def\left[T_0\left(\tau\right)-\lambda I\right] \\
&= \sum_{p=1}^N nul\left[T\left(\tau_p^+\right)-\bar{\lambda} I\right] = \sum_{p=1}^N r_p \\
s = s(\lambda) &:= def\left[T_0\left(\tau^+\right)-\bar{\lambda} I\right] = \sum_{p=1}^N def\left[T_0\left(\tau_p^+\right)-\bar{\lambda} I\right] \\
&= \sum_{p=1}^N nul\left[T\left(\tau_p\right)-\lambda I\right] = \sum_{p=1}^N s_p \\
m &:= r + s = \sum_{p=1}^N r_p + \sum_{p=1}^N s_p = \sum_{p=1}^N \left(r_p + s_p\right) = \sum_{p=1}^N m_p
\end{aligned}\right\} \tag{3.8}$$

Then $0 \le r, s \le nN$ and by Lemma 3.4, m is constant on $\Pi\left[T_0\left(\tau\right), T_0\left(\tau^+\right)\right]$, and:

$$nN \le m \le 2nN. \tag{3.9}$$

For $\Pi\left[T_0(\tau), T_0(\tau^+)\right] \ne \varnothing$, the operators which are regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ are characterized by the following theorem:

Theorem 3.5: For $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$, let r and m be defined by (3.8), and let $\tilde{\Psi}_j\,(j=1,2,\cdots,r), \tilde{\Phi}_k\,(k=r+1,\cdots,m)$ be arbitrary functions satisfying:

1) $\left\{\tilde{\Psi}_j\,(j=1,2,\cdots,r)\right\} \subset D(\tau)$ is linearly independent modulo $D_0(\tau)$ and $\left\{\tilde{\Phi}_k\,(k=r+1,\cdots,m)\right\} \subset D(\tau^+)$ is linearly independent modulo $D_0(\tau^+)$;

2) $\left[\tilde{\Psi}_j, \tilde{\Phi}_k\right] = \sum_{p=1}^{N}\left(\left[\Psi_{jp}, \Phi_{kp}\right](b_p) - \left[\Psi_{jp}, \Phi_{kp}\right](a_p)\right) = 0$, $(j=1,\cdots,r; k=r+1,\cdots,m)$.

Then the set:

$$\left\{\tilde{u} : \tilde{u} \in D(\tau), \left[\tilde{u}, \tilde{\Phi}_k\right] = \sum_{p=1}^{N}\left(\left[u_p, \Phi_{kp}\right](b_p) - \left[u_p, \Phi_{kp}\right](a_p)\right) = 0, (k=r+1,\cdots,m)\right\} \tag{3.10}$$

is the domain of an operator $S = \oplus_{p=1}^{N} S_p$ which is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ and:

$$\left\{\tilde{v} : \tilde{v} \in D(\tau^+), \left[\tilde{\Psi}_j, \tilde{v}\right] = \sum_{p=1}^{N}\left(\left[\Psi_{jp}, v_p\right](b_p) - \left[\Psi_{jp}, v_p\right](a_p)\right) = 0, (j=1,2,\cdots,r)\right\} \tag{3.11}$$

is the domain of an operator $S^* = \left[\oplus_{p=1}^{N} S_p\right]^*$, moreover $\lambda \in \Delta_4(S)$.

Conversely, if S is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ and $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right] \cap \Delta_4(S)$, then with r and m defined by (3.8) there exist functions $\tilde{\Psi}_j\,(j=1,2,\cdots,r)$ and $\tilde{\Phi}_k\,(k=r+1,\cdots,m)$ which satisfied 1) and 2) and are such that (3.10) and (3.11) are the domains of the operators S and S^* respectively.

S is self-adjoint (J-self-adjoint) if, and only if, $\tau^+ = \tau$, $r = s$ and $\tilde{\Phi}_k = \tilde{\Psi}_{k-r}\,(k=r+1,\cdots,m)$; S is J-self-adjoint if $\tau^+ = \bar{\tau}$, (J complex conjugate), $r = s$ and $\tilde{\Phi}_k = \overline{\tilde{\Psi}_{k-r}}\quad (k=r+1,\cdots,m)$.

Proof: The proof is similar to that in [6] [8] [9] [10] [11] ([26], Theorem III.3.6) and [30].

For $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$, define r_p, s_p and m_p be defined by (3.8). Let $\left\{\Psi_{jp}\left(j=1,2,\cdots,s_p\right)\right\}$, $\left\{\Phi_{kp}\left(k=s_p+1,\cdots,m_p\right)\right\}$ be bases for $N\left[T_0(\tau_p) - \lambda I\right]$ and $N\left[T_0(\tau_p^+) - \bar{\lambda} I\right]$ respectively; thus $\Psi_{jp}, \Phi_{kp} \in L_w^2(a_p, b_p)$ $\left(j=1,2,\cdots,s_p; k=s_p+1,\cdots,m_p; p=1,2,\cdots,N\right)$ and

$$\tau_p\left[\Psi_{jp}\right] = \lambda w \Psi_{jp},\ \tau_p^+\left[\Phi_{kp}\right] = \bar{\lambda} w \Phi_{kp} \text{ on } \left[a_p, b_p\right),\ (P=1,2,\cdots,N). \tag{3.12}$$

Since $\left[T_0(\tau_p^+) - \bar{\lambda} I\right]$ has closed range, so does its adjoint $\left[T(\tau) - \lambda I\right]$ and moreover:

$$R\left[T(\tau_p) - \lambda I\right]^\perp = N\left[T_0(\tau_p^+) - \bar{\lambda} I\right] = \{0\},\ p=1,2,\cdots,N.$$

Hence:

$$R\left[T_0\left(\tau_p\right)-\lambda I\right]=L_w^2\left(a_p,b_p\right) \text{ and } R\left[T_0\left(\tau_p^+\right)-\bar{\lambda}I\right]=L_w^2\left(a_p,b_p\right).$$

We can therefore define the following:

$$\left.\begin{array}{l} x_{jp}:=\Psi_{jp}\left(j=1,2,\cdots,s_p\right) \\ \left[T\left(\tau_p\right)-\lambda I\right]x_{jp}:=\Phi_{kp}\left(j=s_p+1,\cdots,m_p\right) \end{array}\right\} \tag{3.13}$$

$$\left.\begin{array}{l} \left[T\left(\tau_p^+\right)-\bar{\lambda}I\right]y_{jp}:=\Psi_{jp}\left(j=1,\cdots,s_p\right) \\ y_{jp}:=\Psi_{jp}\left(j=s_p+1,\cdots,m_p\right) \end{array}\right\} \tag{3.14}$$

Next, we state the following results, the proofs are similar to those in [4] [10] [11] [12] [13] [19]-[23] and ([26], Section 4).

Lemma 3.6: ([23], Lemma 3.3). The sets $\left\{x_{jp}: j=1,2,\cdots,m_p\right\}$ and $\left\{y_{jp}: j=1,2,\cdots,m_p\right\}$ are bases of $N\left(\left[T\left(\tau_p^+\right)-\bar{\lambda}I\right]\left[T\left(\tau_p\right)-\lambda I\right]\right)$ and $N\left(\left[T\left(\tau_p\right)-\lambda I\right]\left[T\left(\tau_p^+\right)-\bar{\lambda}I\right]\right)$ respectively, $p=1,2,\cdots,N$.

On applying ([26], Theorem III.3.1), [10] [11] [12] [13] [19] [20] we obtain:

Corollary 3.7: Any $z_p\in D\left(\tau_p\right)$ and $z_p^+\in D\left(\tau_p^+\right)$ have the unique representations

$$z_p=z_{0p}+\sum_{j=1}^{m_p}a_{jp}x_{jp}\quad\left(z_{0p}\in D_0\left(\tau\right),a_{jp}\in\mathbb{C},\,p=1,2,\cdots,N\right), \tag{3.15}$$

$$z_p^+=z_{0p}^++\sum_{j=1}^{m_p}b_{jp}y_{jp}\quad\left(z_{0p}^+\in D_0\left(\tau_p^+\right),b_{jp}\in\mathbb{C},\,p=1,2,\cdots,N\right). \tag{3.16}$$

A central role in the argument is played by the matrices.

Lemma 3.8: Let,

$$E_{m_p\times m_p}:=\left(\left[x_{jp},y_{kp}\right]\left(b_p\right)\right)_{1\le j,k\le m_p}, \tag{3.17}$$

and:

$$E_{s_p\times r_p}^{1,2}:=\left(\left[x_{jp}y_{kp}\right]\left(b_p\right)\right)_{1\le j\le s_p,s_p+1\le k\le m_p}, \tag{3.18}$$

Then,

$$\text{Rank } E_{s_p\times r_p}^{1,2}=\text{rank } E_{m_p\times m_p}=m_p-n,\ p=1,2,\cdots,N. \tag{3.19}$$

In view of Lemma 3.6 and since $r_p,s_p\ge m_p-n$, $p=1,2,\cdots,N$, we may suppose, without loss of generality, that the matrices,

$$E_{(m_p-n)\times(m_p-n)}^{1,2}:=\left(\left[x_{jp},y_{kp}\right]\left(b_p\right)\right)_{1\le j\le m_p-n,n+1\le k\le m_p}, \tag{3.20}$$

satisfy:

$$\text{Rank } E_{(m_p-n)\times(m_p-n)}^{1,2}=m_p-n,\ p=1,2,\cdots,N. \tag{3.21}$$

If we partition $E_{m_p\times m_p}$ as:

$$E_{m_p\times m_p}=\begin{pmatrix} E_{(m_p-n)\times n}^{1,1} & \cdots & E_{(m_p-n)\times(m_p-n)}^{1,2} \\ \vdots & \ddots & \vdots \\ E_{n\times n}^{2,1} & \cdots & E_{n\times(m_p-n)}^{2,2} \end{pmatrix} \tag{3.22}$$

and let:

$$\left\{\begin{array}{l} E^{1}_{(m_p-n)\times n} = E^{1,1}_{(m_p-n)\times n} \oplus E^{1,2}_{(m_p-n)\times(m_p-n)} \\ E^{2}_{n\times m_p} = E^{2,1}_{n\times n} \oplus E^{2,2}_{n\times(m_p-n)} \end{array}\right\} \tag{3.23}$$

$$\left\{\begin{array}{l} F^{1}_{m_p\times n} = E^{1,1}_{(m_p-n)\times n} \oplus^{\top} E^{2,1}_{n\times n} \\ F^{2}_{m_p\times(m_p-n)} = E^{1,2}_{(m_p-n)\times(m_p-n)} \oplus^{\top} E^{2,2}_{n\times(m_p-n)} \end{array}\right\} \tag{3.24}$$

Then (3.21) yields the result:

$$\operatorname{Rank} E^{1}_{(m_p-n)\times m_p} = \operatorname{rank} F^{2}_{m_p\times(m_p-n)} = m_p - n,\ p = 1,2,\cdots,N. \tag{3.25}$$

Lemma 3.9: Let $D_1(\tau_p)$ be the linear span $\{z_{ip} : i = 1,2,\cdots,n\}$, where $z_{ip} \in D(\tau_p)$ satisfy the following conditions for $k = 1,2,\cdots,n$ and some $c_p \in (a_p, b_p)$, $p = 1,2,\cdots,N$;

$$z^{[k-1]}_{ip}(a_p) = \delta_{ik},\ z^{[k-1]}_{ip}(c_p) = 0,\ z_{ip}(t) = 0 \ \text{ for } t \ge c_p, \tag{3.26}$$

and let $D_2(\tau_p)$ be the linear span of $\{x_{ip} : i = 1,2,\cdots,m_p - n\}$ with (3.21) satisfied. Then,

$$D(\tau_p) = D_0(\tau_p) \dot{+} D_1(\tau_p) \dot{+} D_2(\tau_p),\ p = 1,2,\cdots,N. \tag{3.27}$$

If $D_1(\tau^+_p)$ and $D_2(\tau^+_p)$ be the linear spans of $\{z^+_{ip} : i = 1,2,\cdots,n\}$ and $\{y_{ip} : i = n+1,\cdots,m_p\}$ respectively, then:

$$D(\tau^+_p) = D_0(\tau^+_p) \dot{+} D_1(\tau^+_p) \dot{+} D_2(\tau^+_p),\ p = 1,2,\cdots,N \tag{3.28}$$

4. The Boundary Conditions Featuring L^2_w-Solutions

We shall now characterize all the operators which are regularly solvable with respect to $T_0(\tau) = \oplus^N_{p=1} T_0(\tau_p)$ and $T_0(\tau^+) = \oplus^N_{p=1} T_0(\tau^+_p)$ in terms of boundary conditions featuring $L^2_w(I) = \oplus^N_{p=1} L^2_{w_p}(I_p)$-solutions of the equations $[\tau - \lambda I]y = 0$ and $[\tau^+ - \bar{\lambda} I]z = 0, (\lambda \in \mathbb{C})$ on any finite number of the intervals with one regular end-point and the other may be regular or singular. The results in this section are extension of those in [1] [4] [5] [7] [9]-[22] [27] [28] [29].

Theorem 4.1: Let $\lambda \in \Pi[T_0(\tau), T_0(\tau^+)]$, let r, s and m be defined by (3.8), and let $x_i(i = 1,2,\cdots,m_p), y_j(j = 1,2,\cdots,m_p)$ be defined in (3.13) and (3.14) respectively, and arranged to satisfy (3.21). Let $K^p_{r_p\times n}, L^p_{r_p\times(m_p-n)}, M^p_{s_p\times n}$ and $N^p_{s_p\times(m_p-n)}$, $p = 1,2,\cdots,N$ be numerical matrices which satisfy the following conditions:

1) $\operatorname{Rank}\left(\sum^N_{p=1}\left\{K^p_{r_p\times n} \oplus L^p_{r_p\times(m_p-n)}\right\}\right) = \sum^N_{p=1} r_p = r$ and

$\operatorname{Rank}\left(\sum^N_{p=1}\left\{M^p_{s_p\times n} \oplus N^p_{s_p\times(m_p-n)}\right\}\right) = \sum^N_{p=1} s_p = s$.

2) $$\sum_{p=1}^{N}\left(L^{p}_{r_p\times(m_p-n)}E^{1,2}_{(m_p-n)\times(m_p-n)}\left(N^{p}_{s_p\times(m_p-n)}\right)^{\top}\right)_{r_p\times s_p} +(-i)^n\sum_{p=1}^{N}\left(K^{p}_{r_p\times n}J_{n\times n}\left(M^{p}_{s_p\times n}\right)^{\top}\right)_{r_p\times s_p} = \sum_{p=1}^{N}0_{r_p\times s_p}=0_{r\times s}$$

$J_{n\times n}=(-1)^r\delta_{r,n+1-s}$, $1\le r,s\le n$, δ being the Kronecker delta.

The set of all $u\in D[T(\tau)]$ such that,

$$\sum_{p=1}^{N}\left(M^{p}_{s_p\times n}\begin{pmatrix}u(a_p)\\ \vdots\\ u^{[n-1]}(a_p)\end{pmatrix}-N^{p}_{s_p\times(m_p-n)}\begin{pmatrix}[u,y_{(n+1),p}](b_p)\\ \vdots\\ [u,y_{(m_p),p}](b_p)\end{pmatrix}\right)=0_{s\times1}, \tag{4.1}$$

is the domain of an operator $S=\oplus_{p=1}^{N}S_p$ which is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ and $D(S^*)$ is the set of all $v\in D[T(\tau^+)]$ which are such that:

$$\sum_{p=1}^{N}\left(K^{p}_{r_p\times n}\begin{pmatrix}\bar v(a_p)\\ \vdots\\ \bar v_+^{[n-1]}(a_p)\end{pmatrix}-L^{p}_{r_p\times(m_p-n)}\begin{pmatrix}[x_{1p},v](b_p)\\ \vdots\\ [x_{(m_p-n)p},v](b_p)\end{pmatrix}\right)=0_{r\times1}. \tag{4.2}$$

Proof: Let,

$$M^{p}_{s_p\times n}J^{-1}_{n\times n}=-i^n\left(\alpha^{p}_{jk}\right)_{\substack{r_p+1\le i\le m_p\\ n+1\le k\le m_p}},\quad N^{p}_{s_p\times(m_p-n)}=\left(\beta^{p}_{jk}\right)_{\substack{r_p+1\le i\le m_p\\ n+1\le k\le m_p}} \tag{4.3}$$

and set,

$$g_{jp}:=\sum_{k=n+1}^{m_p}\overline{\beta^{p}_{jk}}y_{kp},\ \left(j=r_p+1,\cdots,m_p;\ p=1,2,\cdots,N\right). \tag{4.4}$$

Then $g_{jp}\in D[T(\tau_p^+)]$, by [5] [10] and [12] we may choose $\Phi_{jp}\left(j=r_p+1,\cdots,m_p\right)\in D[T(\tau_p^+)]$ such that for $k=1,2,\cdots,n$ and some $c_p\in(a_p,b_p)$

$$\left\{\begin{matrix}(\Phi_{jp})_+^{[k-1]}(a_p)=\overline{\alpha_{jk}},\ (\Phi_{jp})_+^{[k-1]}(c_p)=(g_{jp})_+^{[k-1]}(c_p)\\ \Phi_{jp}=g_{jp}\ \text{on}\ [c_p,b_p),\ \left(j=r_p+1,\cdots,m_p;\ p=1,2,\cdots,N\right)\end{matrix}\right\} \tag{4.5}$$

This gives:

$$\sum_{p=1}^{N}\left(M^{p}_{s_p\times n}\begin{pmatrix}u(a_p)\\ \vdots\\ u^{[n-1]}(a_p)\end{pmatrix}\right)$$

$$=-i^n\sum_{p=1}^{N}\left(\left[(\bar\Phi_{jp})_+^{[k-1]}(a_p)\right]_{\substack{r_p+1\le i\le m_p\\ 1\le k\le n}}J_{n\times n}\begin{pmatrix}u(a_p)\\ \vdots\\ u^{[n-1]}(a_p)\end{pmatrix}\right)$$

$$=\sum_{p=1}^{N}\left(\left[[u,\Phi_{(r_p+1)p}](a),[u,\Phi_{(r_p+2)p}](a),\cdots,[u,\Phi_{(m_p)p}](a_p)\right]^{\top}\right),$$

by (2.11). Also, since $\Phi_{jp} = g_{jp}$ on $[c_p, b_p)$, $(j = r_p + 1, \cdots, m_p)$. Then,

$$\sum_{p=1}^{N}\left(N^{p}_{s_p\times(m_p-n)} \begin{pmatrix} [u, y_{(n+1)p}](b) \\ \vdots \\ [u, y_{(m_p)p}](b) \end{pmatrix} \right)$$

$$= \sum_{p=1}^{N}\left(\left[\left[u, \sum_{k=n+1}^{m_p} \overline{\beta^{p}_{r_p+1,k}} y_{kp}\right](b_p), \cdots, \left[u, \sum_{k=n+1}^{m_p} \overline{\beta^{p}_{m_p,k}} y_{kp}\right](b_p) \right]^{\top} \right)$$

$$= \sum_{p=1}^{N}\left(\left[\left[u, \Phi_{(r_p+1)p}\right](b_p), \left[u, \Phi_{(r_p+2)p}\right](b_p), \cdots, \left[u, \Phi_{(m_p)p}\right](b_p) \right]^{\top} \right).$$

The boundary condition (4.1) therefore coincides with that in (3.10). Similarly (4.2) coincides with (3.11) on making the following choices:

$$K^{p}_{r_p\times n} J^{-1}_{n\times n} = (-i)^n \left(\tau^{p}_{jk}\right)_{\substack{1\le j\le r_p \\ 1\le k\le n}}, \quad L^{p}_{r_p\times(m_p-n)} = \left(\varepsilon^{p}_{jk}\right)_{\substack{1\le j\le r_p \\ 1\le k\le m_p-n}}, \tag{4.6}$$

$$h_{jp} := \sum_{k=1}^{m_p-n} \varepsilon^{p}_{jk} x_{kp}, \left(j = 1, \cdots, r_p; p = 1, 2, \cdots, N \right) \tag{4.7}$$

and by [5] [10] and [12] we may choose $\Psi_{jp}\left(j = 1, \cdots, r_p\right) \in D\left[T\left(\tau_p\right)\right]$ such that for $k = 1, 2, \cdots, n$ and some $c_p \in [a_p, b_p)$,

$$\left.\begin{cases} \Psi^{[k-1]}_{jp}\left(a_p\right) = \tau^{p}_{jk}, \Psi^{[k-1]}_{jp}\left(c_p\right) = h^{[k-1]}_{jp}\left(c_p\right), \\ \Psi_{jp} = h_{jp} \text{ on } \left[c_p, b_p\right), \left(j = r_p + 1, \cdots, m_p, p = 1, 2, \cdots, N \right) \end{cases}\right\} \tag{4.8}$$

It remains to show that the above functions $\left\{\Phi_{kp} : k = r_p + 1, \cdots, m_p\right\} \subset D\left[T\left(\tau^{+}_p\right)\right]$ and $\left\{\Psi_{jp}\left(j = 1, \cdots, r_p\right)\right\} \subset D\left[T\left(\tau_p\right)\right]$ are linearly independent modulo $D_0\left[T\left(\tau^{+}_p\right)\right]$ and $D_0\left[T\left(\tau_p\right)\right]$ respectively and satisfy conditions 1) and 2) in Theorem 3.5. First, suppose that $\left\{\tilde{\Psi}_j\right\} = \left\{\Psi_{jp} : j = 1, \cdots, r_p\right\}$ is not linearly modulo $D_0\left(\tau_p\right)$ that is, there exist constants $c_1, \cdots, c_{r_p}$ not all zero, such that $u = \sum_{j=1}^{r_p} c_j \Psi_{jp} \in D_0\left(r_p\right)$. Then, from (2.24), (4.6) and (4.8),

$$0_{n\times 1} = \begin{pmatrix} u\left(a_p\right) \\ \vdots \\ u^{[n-1]}\left(a_p\right) \end{pmatrix} = \left(\tau^{p}_{jk}\right)_{\substack{1\le k\le n \\ 1\le j\le r_p}} \begin{pmatrix} c_1 \\ \vdots \\ c_{r_p} \end{pmatrix} = i^n J_{n\times n} \left(K^{p}_{r_p\times n}\right)^{\top} \begin{pmatrix} c_1 \\ \vdots \\ c_{r_p} \end{pmatrix}.$$

On noting that:

$$J^{-1}_{n\times n} = (-1)^{n+1} J_{n\times n} = \left(J_{n\times n}\right)^{\top}.$$

But $J_{n\times n}$ has rank n and so we infer that:

$$\left(c_1, \cdots, c_{r_p}\right) K^{p}_{r_p\times n} = 0_{1\times n}. \tag{4.9}$$

Since $u = \left\{u_p\right\} \in D_0\left(r_p\right)$, we have that $[u, v]\left(b_p\right) = 0$ for all $v = \left\{v_p\right\} \in D_0\left(\tau^{+}_p\right)$.

Hence,

$$0_{1\times m}=\sum_{p=1}^{N}\left(\left(\left[u,y_{1p}\right]\left(b_p\right),\cdots,\left[u,y_{(m_p)p}\right]\left(b_p\right)\right)\right)$$
$$=\sum_{p=1}^{N}\left(\left[\sum_{j=1}^{r_p}c_j\sum_{k=1}^{m_p-n}\varepsilon_{jk}^{p}x_{kp},y_{1p}\right]\left(b_p\right),\cdots,\left[\sum_{j=1}^{r_p}c_j\sum_{k=1}^{m_p-n}\varepsilon_{jk}^{p}x_{kp},y_{(m_p)p}\right]\left(b_p\right)\right)$$
$$=\sum_{p=1}^{N}\left(\left(c_1,\cdots,c_{r_p}\right)L_{r_p\times(m_p-n)}^{p}E_{(m_p-n)\times m_p}^{1}\right)$$

on using the notation in (3.23). In view of (3.25), we conclude that:

$$\sum_{p=1}^{N}\left(\left(c_1,\cdots,c_{r_p}\right)L_{r_p\times(m_p-n)}^{p}\right)=\sum_{p=1}^{N}0_{1\times(m_p-n)}=0_{1\times(m-n)}. \tag{4.10}$$

We obtain from (4.9) and (4.10) that:

$$\sum_{p=1}^{N}\left(\left(c_1,\cdots,c_{r_p}\right)\left(K_{r_p\times n}^{p}\oplus L_{r_p\times(m_p-n)}^{p}\right)\right)=\sum_{p=1}^{N}\left(0_{1\times m_p}\right)=0_{1\times m},$$

which contradicts the assumption that $\sum_{p=1}^{N}\left(K_{r_p\times n}^{p}\oplus L_{r_p\times(m_p-n)}^{p}\right)$ has rank *r*.

It follows similarly that, $\{\Phi_k\}=\{\Phi_{kp}:k=r_p+1,\cdots,m_p\}\subset D\left[T\left(\tau_p^{+}\right)\right]$ is linearly independent modulo $D_0\left(\tau_p^{+}\right)$.

Finally, we prove 2) in Theorem 3.5,

$$\left(\left[\Psi_j,\Phi_k\right](a)\right)_{\substack{1\le j\le r,\\ r+1\le k\le m}}$$
$$=(-i)^n\sum_{p=1}^{N}\left(\left[\Psi_{jp}^{[k-1]}\left(a_p\right)\right]_{\substack{1\le j\le r_p\\1\le k\le n}}J_{n\times n}\left[\left(\bar{\Phi}_{kp}\right)_{+}^{[j-1]}\left(a_p\right)\right]_{\substack{1\le j\le n\\ r_p+1\le k\le m_p}}\right)$$
$$=(-i)^n\sum_{p=1}^{N}\left(\left(\tau_{jk}^{p}\right)_{\substack{1\le j\le r_p\\1\le k\le n}}J_{n\times n}\left(\alpha_{jk}^{p}\right)_{\substack{r_p+1\le j\le m_p\\ n+1\le k\le m_p}}\right)$$

By (4.5) and (4.8),

$$=-(-i)^n\sum_{p=1}^{N}\left(\left\{K_{r_p\times n}^{p}J_{n\times n}\left(M_{s_p\times n}^{p}\right)^{\top}\right\}_{r_p\times s_p}\right). \tag{4.11}$$

Next, we see that:

$$\left(\left[\Psi_i,\Phi_j\right](b)\right)=\sum_{p=1}^{N}\left(\left(\left[\sum_{l=1}^{m_p-n}\varepsilon_{il}^{p}x_{lp},\sum_{i=n+1}^{m_p}\overline{\beta_{jk}^{p}}y_{ki}\right]\left(b_p\right)\right)_{\substack{1\le i\le r_p\\ r_p+1\le j\le m_p}}\right),$$

Hence,

$$\begin{aligned}\left(\left[\Psi_j,\Phi_k\right](b)\right)_{\substack{1\le i\le r\\ r+1\le k\le m}}&=\sum_{p=1}^{N}\left(\left(\varepsilon_{jl}^{p}\right)_{\substack{1\le j\le r_p\\1\le l\le m_p-n}}\left(\left[x_{lp},y_{ip}\right]\left(b_p\right)\right)_{\substack{1\le l\le m_p-n\\ n+1\le i\le m_p}}\left(\beta_{ki}^{p}\right)_{\substack{r_p+1\le k\le m_p\\ n+1\le i\le m_p}}^{\top}\right)\\&=\sum_{p=1}^{N}\left(L_{r_p\times(m_p-n)}^{p}E_{(m_p-n)\times(m_p-n)}^{1,2}\left(N_{s_p\times(m_p-n)}^{p}\right)^{\top}\right)\end{aligned} \tag{4.12}$$

From 2), (4.11) and (4.12) it follows that condition 2) in Theorem 3.5 is satisfied. The proof is therefore complete.

The converse of Theorem 4.1 is

Theorem 4.2: Let $S=\oplus_{p=1}^{N}S_p$ be regularly solvable with respect to $T_0(\tau)$

and $T_0(\tau^+)$, let $\lambda \in \Pi[T_0(\tau), T_0(\tau^+)] \cap \Delta_4(S)$, let r, s and m be defined by (3.8), and suppose that (3.21) is satisfied. Then there exist numerical matrices $K^p_{r_p \times n}$, $L^p_{r_p \times (m_p - n)}$, $M^p_{s_p \times n}$ and $N^p_{s_p \times (m_p - n)}$ such that conditions 1) and 2) in Theorem 4.1 are satisfied and $D(\hat{S})$ is the set of $u \in D[T(r)]$ satisfying (4.1) while $D(S)^*$ is the set of $v \in D[T(\tau^+)]$ satisfying (4.2).

Proof: Let $\{\Psi_{jp}(j = 1, \cdots, r_p)\} \subset D[T(\tau_p)]$ and $\{\Phi_{kp} : k = r_p + 1, \cdots, m_p\} \subset D[T(\tau_p^+)]$ satisfy the second part of Theorem 3.5. From (3.27) and (3.28), we have:

$$\Phi_{jp} = y_{j0} + \sum_{k=1}^{n} \eta^p_{jk} z^+_{kp} + \sum_{k=n+1}^{m_p} \beta^p_{jk} y_{kp}, \left(j = r_p + 1, \cdots, m_p; p = 1, 2, \cdots, N\right), \quad (4.13)$$

for some $y_{j0} \in D[T_0(\tau_p^+)]$ and complex constants η^p_{jk} and β^p_{jk}. Let:

$$M^p_{s_p \times n} = -i^n \left[\left(\overline{\Phi}_{jp}\right)_+^{[k-1]}(a_p)\right]_{\substack{r_p+1 \le j \le m_p \\ 1 \le k \le n}} J_{n \times n}, \quad (4.14)$$

$$N^p_{s_p \times (m_p - n)} = \left(\overline{\beta^p_{jk}}\right)_{\substack{r_p+1 \le j \le m_p \\ n+1 \le k \le m_p}}, \; p = 1, 2, \cdots, N \quad (4.15)$$

Then,

$$\begin{pmatrix} [u, \Phi_{(r_p+1),p}](a_p) \\ \vdots \\ [u, \Phi_{(m_p),p}](a_p) \end{pmatrix} = \left[(-i)^n \left[u(a_p), \cdots, u^{[n-1]}(a_p)\right] J_{n \times n} \left[\left(\overline{\Phi}_{kp}\right)_+^{[j-1]}(a_p)\right]_{\substack{1 \le j \le n \\ r_p+1 \le k \le m_p}}\right]^\top = M^p_{s_p \times n} \begin{pmatrix} u(a_p) \\ \vdots \\ u^{[n-1]}(a_p) \end{pmatrix}.$$

Moreover, for all $u = \{u_p\} \in D[T(r_p)]$,
$[u, y_{j0}](b_p) = [u, z^+_k](b_p) = 0$, $\left(j = r_p + 1, \cdots, m_p; k = 1, 2, \cdots, n; p = 1, 2, \cdots, N\right)$, and hence, from (4.13),

$$\begin{pmatrix} [u, \Phi_{(r_p+1),p}](b_p) \\ \vdots \\ [u, \Phi_{(m_p),p}](b_p) \end{pmatrix} = \begin{pmatrix} \left[u, \sum_{k=n+1}^{m} \beta_{r_p+1,k} y_{kp}\right](b_p) \\ \vdots \\ \left[u, \sum_{k=n+1}^{m_p} \beta_{m_p,k} y_{kp}\right](b_p) \end{pmatrix} = N^p_{s_p \times (m_p - n)} \begin{pmatrix} [u, y_{(n+1)p}](b_p) \\ \vdots \\ [u, y_{(m_p)p}](b_p) \end{pmatrix}.$$

Therefore, we have shown that the boundary conditions (4.1) coincide with those in (3.10). Similarly (4.2) and the conditions in (3.11) can be shown to coincide if we choose,

$$K^p_{r_p\times n} := (-i)^n \left[\Psi^{[k-1[}_{jp}\left(a_p\right)\right]_{\substack{1\le j\le r_p\\1\le k\le n}} J_{n\times n}, \tag{4.16}$$

and:

$$L^p_{r_p\times(m_p-n)} := \left(\varepsilon^p_{jk}\right)_{\substack{1\le j\le r_p\\1\le k\le m_p-n}}, \tag{4.17}$$

where the ε^p_{jk} are the constants uniquely determined by the decomposition:

$$\Psi_{jp} = x_{j0} + \sum_{k=1}^{n}\xi^p_{jk} z_{kp} + \sum_{k=1}^{m_p-n}\varepsilon^p_{jk} x_{kp}, \left(j=1,\cdots,r_p; p=1,2,\cdots,N\right), \tag{4.18}$$

derived from Lemma 3.9. Next, we prove that 1) and 2) in Theorem 4.7 are consequences of the fact that $\left\{\Psi_{jp}\left(j=1,\cdots,r_p\right)\right\}\subset D\left[T\left(r_p\right)\right]$ and $\left\{\Phi_{kp} : k = r_p+1,\cdots,m_p\right\}\subset D\left[T\left(\tau^+_p\right)\right]$ are linearly independent modulo $D_0\left[T\left(r_p\right)\right]$ and $D_0\left[T\left(\tau^+_p\right)\right]$ respectively. Suppose that:

$$\text{Rank}\left(\sum_{p=1}^{N}\left(K^p_{r_p\times n}\oplus L^p_{r_p\times(m_p-n)}\right)\right) < r.$$

Then there exist constants $c_1, c_2, \cdots, c_{r_p}$ not all zero, such that:

$$\sum_{p=1}^{N}\left(\left(c_1, c_2, \cdots, c_{r_p}\right)\left(K^p_{r_p\times n}\oplus L^p_{r_p\times(m_p-n)}\right)\right) = 0_{1\times m}. \tag{4.19}$$

This implies that,

$$\begin{aligned}0_{1\times n} &= \sum_{p=1}^{N}\left(\left(c_1, c_2, \cdots, c_{r_p}\right)\left(K^p_{r_p\times n}\right)\right)\\ &= (-i)^n \sum_{p=1}^{N}\left(\left(c_1, c_2, \cdots, c_{r_p}\right)\left[\Psi^{[k-1[}_{jp}\left(a_p\right)\right]_{\substack{r\le j\le r_p\\1\le k\le n}} J_{n\times n}\right)\end{aligned}$$

and as $J_{n\times n}$ non-singular, it follows that $u = \left\{u_p\right\} = \sum_{j=1}^{r_p} c_j \Psi_{jp}$, satisfies

$$\left(u\left(a_p\right), \cdots, u^{[n-1]}\left(a_p\right)\right) = 0_{1\times n}. \tag{4.20}$$

We also, infer from (4.19) that:

$$\begin{aligned}0_{1\times(m-n)} &= \sum_{p=1}^{N}\left(\left(c_1, c_2, \cdots, c_{r_p}\right)\left(L^p_{r_p\times(m_p-n)}\right)\right)\\ &= \sum_{p=1}^{N}\left(\sum_{j=1}^{r_p} c_j\varepsilon^p_{j1}, \cdots, \sum_{j=1}^{r_p} c_j\varepsilon^p_{j(m_p-n)}\right).\end{aligned}$$

Consequently, on substituting (4.18), we obtain:

$$u = \sum_{j=1}^{r_p} c_j x_{j0} + \sum_{j=1}^{r_p}\sum_{k=1}^{n} c_j \xi^p_{jk} z_{kp}. \tag{4.21}$$

For arbitrary $v\in D\left(\tau^+\right)$ it follows that $[u,v](b) = 0$. This fact and (4.20) together imply that $u\in D_0(\tau)$ and hence that $\left\{\Psi_{jp} : j=1,\cdots,r_p\right\}$ is linearly independent modulo $D_0\left(\tau_p\right)$ contrary to assumption. We have therefore proved that $\sum_{p=1}^{N}\left(\left\{K^p_{r_p\times n}\oplus L^p_{r_p\times(m_p-n)}\right\}\right)$ has Rank *r*. The proof of $\text{Rank}\left(\sum_{p=1}^{N}\left\{M^p_{s_p\times n}\oplus N^p_{s_p\times(m_p-n)}\right\}\right) = s$ is similar. From (4.14) and (4.16),

$$\begin{aligned}
&\left(\left[\tilde{\Psi}_j,\tilde{\Phi}_k\right](a)\right)_{\substack{1\le j\le r,\\ r+1\le k\le m}}\\
&=(-i)^n\sum_{p=1}^N\left(\left[\Psi_j^{[k-1]}(a_p)\right]_{\substack{1\le j\le r_p\\1\le k\le n}}J_{n\times n}\left[\left(\bar{\Phi}_k\right)_+^{[j-1]}(a_p)\right]_{\substack{1\le j\le n\\ r_p+1\le k\le m_p}}\right)\\
&=(-i)^n\sum_{p=1}^N\left(\left(i^nK_{r_p\times n}^pJ_{n\times n}^{-1}\right)J_{n\times n}\left(-(-i)^nM_{s_p\times n}^pJ_{n\times n}^{-1}\right)^{\top}\right)\\
&=-(-i)^n\sum_{p=1}^N\left(\left\{K_{r_p\times n}^pJ_{n\times n}\left(M_{s_p\times n}^p\right)^{\top}\right\}_{r_p\times s_p}\right).
\end{aligned}$$

On using (4.13), (4.18) and the fact that $z_{jp}=z_{jp}^+=0$ on $\left[c_p,b_p\right)$, $(j=1,\cdots,n;p=1,2,\cdots,N)$ and $[u,v](b)=0$ if either $u\in D(\tau)$ and $v\in D_0(\tau^+)$ or $u\in D_0(\tau)$ and $v\in D(\tau^+)$, we obtain:

$$\begin{aligned}
\left(\left[\tilde{\Psi}_i,\tilde{\Phi}_j\right](b)\right)_{\substack{1\le i\le r\\ r+1\le j\le m}}&=\sum_{p=1}^N\left(\left(\left[\sum_{l=1}^{m_p-n}\varepsilon_{il}^px_l,\sum_{k=n+1}^{m_p}\beta_{jk}^py_p\right](b_p)\right)_{\substack{1\le i\le r_p\\ r_p+1\le j\le m_p}}\right)\\
&=\sum_{p=1}^N\left(L_{r_p\times(m_p-n)}^pE_{(m_p-n)\times(m_p-n)}^{1,2}\left(N_{s_p\times(m_p-n)}^p\right)^{\top}\right).
\end{aligned}$$

The proof is therefore complete.

Remark 4.3: Assume that $\tau=\oplus_{p=1}^N\tau_p$ is formally *J*-symmetric, that is $\tau^+=J\tau J$, where *J* is the complex conjugation. Then the operator $T_0(\tau)$ is the *J*-symmetric and $T_0(\tau)$ and $T_0(\tau^+)=J\left[T_0(\tau)\right]J$ form an adjoint pair with:

$$\Pi\left[T_0(\tau),T_0(\tau^+)\right]=\Pi\left[T_0(\tau)\right]. \tag{4.22}$$

Since $\tau[u]=\lambda wu$ if and only if $\tau^+[\bar{u}]=\bar{\lambda}w\bar{u}$ ($\lambda\in\mathbb{C}$), it follows from Lemma 3.1 that for all $\lambda\in\Pi\left[T_0(\tau)\right]$, $def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]$ is constant ℓ, say, so in (3.8) and (3.9), $r=s=\ell$ with $\frac{n}{2}\le\ell\le n$.

5. Discussion

In [5] Everitt and Zettl discussed the possibility of generating self-adjoint operators which are not expressible as the direct sums of self-adjoint operators defined in the separate intervals. In this section we extend this case to the case of general ordinary differential operators, *i.e.*, we discuss the possibility of the regularly solvable operators which are not expressible as the direct sums of regularly solvable operators defined in the separate intervals $I_p=(a_p,b_p)$, $p=1,2,3,4$. We will refer to these operators as "New regularly solvable operators" if a_p is a regular end point and b_p is singular, then by ([26], Theorem III.10.13) the sum:

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda}I\right]=4n \quad \text{for all } \lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right],$$

If and only if the term in (3.11) at the end point b_p is zero, $p=1,2,3,4$. By Lemma 3.4, for:
$\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$, we get in all cases:

$$0 \le def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right] \le 8n. \tag{5.1}$$

When each interval has at most one singular end-point,

$$4n \le def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right] \le 8n. \tag{5.2}$$

In the case when all end-points are regular,

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right]=8n, \text{ for all } \lambda \in \Pi\left[T_0(\tau),T_0(\tau^+)\right]. \tag{5.3}$$

Let,

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right]=d$$

And:

$$def\left[T_0(\tau_p)-\lambda I\right]+def\left[T_0(\tau_p^+)-\bar{\lambda} I\right]=d_p,\ p=1,2,3,4.$$

Then by part 3) in Lemma 3.1, we have that $d=\sum_{p=1}^{4} d_p$.

We now consider some of the possibilities:

Example 1. $d=0$. This is the minimal case in (5.1) and can only occur when all four end-points are singular. In this case $T_0(\tau)$ is itself regularly solvable and has no proper regularly solvable extensions, see Edmunds and Evans ([26], Chapter III) [10] [13] [19] [20].

Example 2. $d=n$ with one of d_1,d_2,d_3 and d_4 is equal to n and all the others are equal to zero. We assume that $d_1=n$ and $d_2=d_3=d_4=0$. The other possibilities are entirely similar. In this case we must have seven singular end-points and one regular. There are no new regularly solvable extensions and we have that, $S=S_1\oplus_{p=2}^{4} T_0(\tau_p)$, where S_1 is regularly solvable extension of $T_0(\tau_1)$, *i.e.*, all regularly solvable extensions of $T_0(\tau)$ can be obtained by forming sums of regularly solvable extensions of $T_0(\tau_p)$, $p=1,2,3,4$. These are obtained as in the "one interval" case.

Example 3. Six singular end-points and $d=2n$. We consider two cases:

1) One interval has two regular end-points, say, I_1, and each one of the others has two singular end-points. Then, $S=S_1\oplus_{p=2}^{4} T_0(\tau_p)$, where S_1 is regularly solvable extension of $T_0(\tau_1)$, generates all regularly solvable extensions of $T_0(\tau)$.

2) There are two intervals, say, I_1 rand I_2 each one has one regular and one singular end-point and each one of the others has two singular end-points. In this case $S=S_1\oplus S_2\oplus_{p=3}^{4} T_0(\tau_p)$, and $S_1\oplus S_2$ generates all regularly solvable extensions of $T_0(\tau)$. The other possibilities in the cases 1) and 2) are entirely similar.

Example 4: Five singular end-points and $d=3n$. We consider two cases:

1) There are two intervals, say, I_1 and I_2, such that I_1 has two regular end-points and I_2 has one regular and one singular end-points, and each one of the others has two singular end-points. In this case $d_1=2n$ and $d_2=n$, then, $S=S_1\oplus S_2\oplus_{p=3}^{4} T_0(\tau_p)$, which is similar to 2) in Example 3.

2) There are three intervals, say, I_1,I_2 and I_3 each one has one regular and

one singular end-point, and the fourth has two singular end-points. In this case $d_1 = d_2 = d_3 = n$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus T_0(\tau_4)$, and hence $\oplus_{p=1}^{3} S_p$ generates all regularly solvable extensions of $T_0(\tau)$. The possibilities are entirely similar.

Example 5: Four singular end-points and $d = 4n$. We consider three cases:

1) There are two intervals, say, I_1 and I_2, such that each one has two regular end-points and each one of the others has two singular end-points. In this case $d_1 = d_2 = 2n$ and $d_3 = d_4 = 0$, then, $S = S_1 \oplus S_2 \oplus_{p=3}^{4} T_0(\tau_p)$.

2) There are two intervals, say, I_1 and I_2, such that each has one regular and one singular end-point, and the others I_3 and I_4 has two regular and two singular end-points respectively. In this case $d_1 = d_2 = n$, $d_3 = 2n$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus T_0(\tau_4)$ as in Example 4 2).

3) Each interval has one regular and one singular end-points. In this case $d_p = n, p = 1,2,3,4$. Then "mixing" can occur and we get new regularly solvable extensions of $T_0(\tau)$. For the sake of definiteness assume that the end-points a_1, b_2, a_3 and b_4 are singular end-points and b_1, a_2, b_3 and a_4 are regular end-points. The other possibilities are entirely similar.

For $u \in D[T(\tau)] = \oplus_{p=1}^{4} D[T(\tau_p)]$ and $\Phi_j \in D[T(\tau^+)] = \oplus_{p=1}^{4} D[T(\tau_p^+)]$ with $u = \{u_1, u_2, u_3, u_4\}$, $\Phi_k = \{\Phi_{k1}, \Phi_{k2}, \Phi_{k2}, \Phi_{k4}\}$, condition (3.11)) reads:

$$0 = [u, \Phi_j] = \sum_{p=1}^{4} \left\{ [u_p, \Phi_{jp}]_p (b_p) - [u_p, \Phi_{jp}]_p (a_p) \right\}, j = 1, \cdots, n. \quad (4.22)$$

Also, for $v \in D[T(\tau^+)] = \oplus_{p=1}^{4} D[T(\tau_p^+)]$ and $\Psi_j \in D[T(\tau)] = \oplus_{p=1}^{4} D[T(\tau_p)]$ with $v = \{v_1, v_2, v_3, v_4\}$, $\Psi_j = \{\Psi_{j1}, \Psi_{j2}, \Psi_{j3}, \Psi_{j4}\}$, condition (3.12)) reads:

$$0 = [\Psi_j, v] = \sum_{p=1}^{4} \left\{ [\Psi_{jp}, v_p]_p (b_p) - [\Psi_{jp}, v_p]_p (a_p) \right\}, j = 1, \cdots, n, \quad (4.23)$$

and condition 2) in Theorem 3.5 reads:

$$0 = [\Psi_j, \Phi_k] = \sum_{p=1}^{4} \left\{ [\Psi_{jp}, \Phi_{kp}]_p (b_p) - [\Psi_{jp}, \Phi_{kp}]_p (a_p) \right\}, j, k = 1, \cdots, n. \quad (4.24)$$

By ([3], Theorem III.10.13), the terms involving the singular end-points a_1, b_2, a_3 and b_4 are zero so that (4.22), (4.23) and (4.24) reduces to:

$$[u_1, \Phi_{k1}]_1 (b_1) - [u_2, \Phi_{k2}]_2 (a_2) + [u_3, \Phi_{k3}]_3 (b_3) - [u_4, \Phi_{k4}]_4 (a_4) = 0,$$

$$[\Psi_{j1}, v_1]_1 (b_1) - [\Psi_{j2}, v_2]_2 (a_2) + [\Psi_{j3}, v_3]_3 (b_3) - [\Psi_{j4}, v_4]_4 (a_4) = 0,$$

and:

$$[\Psi_{j1}, \Phi_{k1}]_1 (b_1) - [\Psi_{j2}, \Phi_{k2}]_2 (a_2) + [\Psi_{j3}, \Phi_{k3}]_3 (b_3) - [\Psi_{j4}, \Phi_{k4}]_4 (a_4) = 0,$$

$j, k = 1, \cdots, n$. Thus, the boundary conditions are not separated for the four intervals and hence, the regularly solvable operators cannot be expressed as a direct sum of regularly solvable operators defined in the separate intervals $I_p, p = 1,2,3,4$. We refer to Everitt and Zettl's papers [5] [10] [12] [13] [19] [20] for more examples and more details.

Conclusion: We have characterized that all regularly solvable operators and their adjoints are generated by a general ordinary quasi-differential expression τ_{jp} in the direct sum Hilbert spaces $L_w^2(a_p, b_p), p = 1, \cdots, N$. The domains of these operators are described in terms of boundary conditions involving $L_w^2(a_p, b_p)$-solutions of the equations $\tau_{jp}[y] = \bar{\lambda} wy$ and its adjoint $\tau_{jp}^+[z] = \bar{\lambda} wz \left(\bar{\lambda} \in \mathbb{C}\right)$ on the intervals $[a_p, b_p)$. This characterization is an extension of those obtained in the case of one interval with one and two singular endpoints of the interval (*a*, *b*), and is a generalization of those proved in the case of self-adjoint and *J*-self-adjoint differential operators as a special case, where *J* denotes complex conjugation.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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