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On the Spectra of General Ordinary Quasi-Differential Operators and Their L_w^2-Solutions

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Abstract

In this paper, we consider the general ordinary quasi-differential expression τ of order n with complex coefficients and its formal adjoint τ^+ on the interval $[a,b)$. We shall show in the case of one singular end-point and under suitable conditions that all solutions of a general ordinary quasi-differential equation $(\tau-\lambda w)u = wf$ are in the weighted Hilbert space $L_w^2(a,b)$ provided that all solutions of the equations $(\tau-\lambda w)u=0$ and its adjoint $(\tau^+-\overline{\lambda}w)v=0$ are in $L_w^2(a,b)$. Also, a number of results concerning the location of the point spectra and regularity fields of the operators generated by such expressions may be obtained. Some of these results are extensions or generalizations of those in the symmetric case, while the others are new.

Keywords

General Ordinary Quasi-Differential Expressions, Regular and Singular End-Points, Singular Differential Operators, Essential Spectra, Point Spectra and Regularity Fields

1. Introduction

Akhiezer and Glazman [1] studied that the self-adjoint extension S of the minimal operator $T_0(\tau)$ generated by a formally symmetric differential expression τ with maximal deficiency indices have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. The relationship between the square-integrable solutions for real values of the spectral parameter and the spectrum of self-adjoint ordinary differential operators of even order with real coefficients and arbitrary deficiency index are studied in [2] [3]. Sobhy E.I. has been extended their results for general ordinary quasi-differential expression τ of $n\,th$ order with complex coefficients [4] [5] [6] [7] [8].

The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression τ are those which are regularly solvable with respect to the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ generated by a general quasi-differential expression τ and its formal adjoint τ^+ respectively, the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ form an adjoint pair of closed, densely-defined operators in the underlying L_w^2-space, that is $T_0(\tau)\subset\left[T_0(\tau^+)\right]^*$. Such an operator S satisfies $T_0(\tau)\subset S\subset\left[T_0(\tau^+)\right]^*$ and for some $\lambda\in\mathbb{C}$, $(S-\lambda I)$ is a Fredholm operator of zero index, this means that S has the desirable Fredholm property that the equation $(S-\lambda I)u=f$ has a solution if and only if f is orthogonal to the solution space of $(S-\lambda I)u=0$ and furthermore the solution space of $(S-\lambda I)u=0$ and $(S^*-\bar{\lambda}I)v=0$ have the same finite dimension. This notion was originally due to Visik in [5].

Our objective in this paper is an extension of those results in [7] [9] [10] [11] [12] [13] [14] for the general ordinary quasi-differential operators $T_0(\tau),T_0(\tau^+)$ and to investigate (according to the spectral theory) the location of the point spectra and regularity fields of general ordinary quasi-differential operators in the case of one singular end-point and when all solutions of the equations $[\tau-\lambda w]u=0$ and $[\tau^+-\bar{\lambda}w]v=0$ are in $L_w^2(a,b)$ for some (and hence all $\lambda\in\mathbb{C}$).

We deal throughout this paper with a general quasi-differential expression τ of arbitrary order n defined by Shin-Zettl matrices [15] [16] [17] [18], and the minimal operator $T_0(\tau)$ generated by $w^{-1}\tau[.]$ in $L_w^2(I)$, where w is a positive weight function on the underlying interval I. The end-points a and b of I may be regular or singular.

2. Notation and Preliminaries

We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [2] [11] ([18], Chapter III) and [19] [20] [21] [22]. The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written $nul(T)$, is the dimension of $N(T)$ and the deficiency of T, written $def(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed, then $def(T)=nul(T^*)$. The Fredholm domain of T is (in the notation of [18]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda\in\mathbb{C}$ which are such that $(T-\lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus

$\lambda \in \Delta_3(T)$ if and only if $(T-\lambda I)$ has closed range and finite nullity and deficiency. The index of $(T-\lambda I)$ is the number $ind(T-\lambda I) = nul(T-\lambda I) - def(T-\lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators A and B acting in a Hilbert space H are said to form an adjoint pair if $A \subset B^*$ and, consequently, $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(.,.)$ denotes the inner-product on H.

Definition 2.1: The field of regularity $\Pi(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that:

$$\|(A-\lambda I)x\| \geq K(\lambda)\|x\| \quad \text{for all} \quad x \in D(A), \tag{2.1}$$

or, equivalently, on using the Closed Graph Theorem, $nul(A-\lambda I) = 0$ and $R(A-\lambda I)$ are closed.

The joint field of regularity $\Pi(A,B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, $\bar{\lambda} \in \Pi(B)$ and both $def(A-\lambda I)$ and $def(B-\bar{\lambda} I)$ are finite. An adjoint pair A and B is said to be compatible if $\Pi(A,B) \neq \phi$.

Definition 2.2: A closed operator S in H is said to be **regularly solvable** with respect to the compatible adjoint pair of A and B if $A \subset S \subset B^*$ and $\Pi(A,B) \cap \Delta_4(S) \neq \phi$, where $\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), ind(S-\lambda I) = 0\}$.

Definition 2.3: The resolvent set $\rho(S)$ of a closed operator S in H consists of the complex numbers λ for which $(S-\lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\rho(S)$ in $\mathbb{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$ (see [5] [7] [12] [14] [18]).

$$\sigma_p(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is not injective}\},$$

i.e., the set of eigenvalues of S,

$$\sigma_c(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is injective}, R(S-\lambda I) \subsetneq \overline{R(S-\lambda I)} = H\};$$

$$\sigma_r(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is injective}, \overline{R(S-\lambda I)} \neq H\}.$$

For a closed operator S we have:

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S). \tag{2.2}$$

An important subset of the spectrum of a closed densely defined operator S in H is the so-called essential spectrum. The various essential spectra of S are defined as in ([18], Chapter 9) to be the sets:

$$\sigma_{ek}(S) = \mathbb{C} \setminus \Delta_k(S), (k = 1,2,3,4,5), \tag{2.3}$$

where $\Delta_3(S)$ and $\Delta_4(S)$ have been defined earlier.

Definition 2.4: For two closed densely defined operators A and B acting in H, if $A \subset S \subset B^*$ and the resolvent set $\rho(S)$ of S is nonempty (see [18]), S is said to be **well-posed** with respect to A and B.

Note that, if $A \subset S \subset B^*$ and $\lambda \in \rho(S)$ then $\lambda \in \Pi(A)$ and

$\bar{\lambda} \in \rho(S^*) \subset \Pi(B)$ so that if $def(A-\lambda I)$ and $def(B-\bar{\lambda} I)$ are finite, then A and B are compatible, in this case S is regularly solvable with respect to A and B. The terminology "Regularly solvable" mentioned by Visik in [4] [5] [6] [7] [8] [19] [20] [22], while the notion of "well posed" was introduced by Zhikhar in [23].

3. Quasi-Differential Expressions

The general quasi-differential expressions are defined in terms of a Shin-Zettl matrix Q on an interval I. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $n\times n$-matrices $Q=\{q_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$\left.\begin{array}{lr} q_{rs} \in L^2_{loc}(I), & (1\le r,s\le n,\, n\ge 2) \\ q_{r,r+1} \neq 0, & \text{a.e., on } I\ (1\le r\le n-1) \\ q_{rs}=0, & \text{a.e., on } I,\ (2\le r+1< s\le n) \end{array}\right\} \tag{3.1}$$

For $Q\in Z_n(I)$, the quasi-derivatives associated with Q are defined by:

$$\left.\begin{array}{l} y^{[0]} := y, \\ y^{[r]} := (q_{r,r+1})^{-1}\left\{\left(y^{[r-1]}\right)' - \sum_{s=1}^{r} q_{rs} y^{[s-1]}\right\}, (1\le r\le n-1) \\ y^{[n]} := \left\{\left(y^{[n-1]}\right)' - \sum_{s=1}^{n} q_{ns} y^{[s-1]}\right\} \end{array}\right\} \tag{3.2}$$

where the prime $'$ denotes differentiation.

The general quasi-differential expression τ associated with Q is given by:

$$\tau[.] := i^n y^{[n]},\ (n\ge 2) \tag{3.3}$$

this being defined on the set:

$$V(\tau) := \left\{y : y^{[r-1]} \in AC_{loc}(I), r=1,2,\cdots,n\right\}$$

where $AC_{loc}(I)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I_p.

The formal adjoint τ^+ of τ is defined by the matrix Q^+ given by:

$$\tau^+[.] := i^n z_+^{[n]},\ \text{for all}\ z\in V(\tau^+) = \left\{z : z_+^{[r-1]} \in AC_{loc}(I), r=1,2,\cdots,n\right\}, \tag{3.4}$$

where $z_+^{[r-1]}$, the quasi-derivatives associated with the matrix Q^+ in $Z_n(I)$,

$$Q^+ = (q_{rs})^+ = (-1)^{r+s+1}\overline{q_{n-s+1,n-r+1}},\ \text{for each } r \text{ and } s; \tag{3.5}$$

are therefore:

$$\left.\begin{array}{l} z_+^{[0]} := z, \\ z_+^{[r]} := \left(\bar{a}_{n-r,n-r+1}\right)^{-1}\left(\left(z_+^{[r-1]}\right)' - \sum_{s=1}^{r} (-1)^{r+s+1}\bar{q}_{n-s+1,n-r+1} z_+^{[s-1]}\right) \\ z_+^{[n]} := \left(z_+^{[n-1]}\right)' - \sum_{s=1}^{n} (-1)^{n+s+1}\bar{q}_{n-s+1,1} z_+^{[s-1]}, (1\le r\le n-1) \end{array}\right\} \tag{3.6}$$

Note that: $\left(Q^{+}\right)^{+}=Q$ and so $\left(\tau^{+}\right)^{+}=\tau$. We refer to [11] [13] [16]-[22] [24] for a full account of the above and subsequent results on quasi-differential expressions.

For $u\in V(\tau)$, $v\in V\left(\tau^{+}\right)$ and $\alpha,\beta\in I$, we have Green's formula,

$$\int_a^b\left\{\bar{v}\tau[u]-u\overline{\tau^{+}[v]}\right\}\mathrm{d}x=[u,v](b)-[u,v](a), \tag{3.7}$$

where,

$$\begin{aligned}[u,v](x)&=i^n\left(\sum_{r=0}^{n-1}(-1)^{r+s+1}u^{[r]}(x)\overline{v_{+}^{[n-r-1]}}(x)\right)\\&=(-i)^n\left(u,u^{[1]},\cdots,u^{[n-1]}\right)\times J_{n\times n}\begin{pmatrix}\bar{v}\\ \vdots\\ \bar{v}_{+}^{[n-1]}\end{pmatrix}(x);\end{aligned} \tag{3.8}$$

see [4] [10]-[18] [24]. Let the interval I have end-points $a,b(-\infty\le a<b\le\infty)$, and let $w:I\to\mathbb{R}$ be a non-negative weight function with $w\in L_{loc}^1(I)$ and $w>0$ (for almost all $x\in I$). Then $H=L_w^2(I)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions\ such that $\int_I w|f|^2<\infty$; the inner-product is defined by:

$$(f,g):=\int_I wf(x)\overline{g(x)}\mathrm{d}x\quad\left(f,g\in L_w^2(I)\right).$$

The equation:

$$\tau[u]-\lambda wu=0\quad(\lambda\in\mathbb{C})\text{ on }I \tag{3.9}$$

is said to be **regular** at the left end-point $a\in\mathbb{R}$, if for all $X\in(a,b)$, $a\in\mathbb{R}$, $w,q_{rs}\in L^1(a,X)$, $r,s=1,2,\cdots,n$.

Otherwise (3.9) is said to be **singular** at *a*. If (3.9) is regular at both end-points, then it is said to be regular; in this case we have:

$$a,b\in\mathbb{R},\quad w,q_{rs}\in L^1(a,b),\quad r,s=1,2,\cdots,n.$$

We shall be concerned with the case when *a* is a regular end-point of (3.9), the end-point *b* being allowed to be either regular or singular. Note that, in view of (3.5), an end-point of *I* is regular for (3.9), if and only if it is regular for the equation:

$$\tau^{+}[v]-\bar{\lambda}wv=0\quad(\lambda\in\mathbb{C})\text{ on }I \tag{3.10}$$

Note that: At a regular end-point *a*, say, $u^{[r-1]}(a)$ $\left(v_{+}^{[r-1]}(a)\right)$, $r=1,2,\cdots,n$ is defined for all $u\in V(\tau)$ $\left(v\in V\left(\tau^{+}\right)\right)$. Set:

$$\left.\begin{aligned}D(\tau)&:=\left\{u:u\in V(\tau),u\text{ and }w^{-1}\tau[u]\in L_w^2(a,b)\right\}\\D\left(\tau^{+}\right)&:=\left\{v:v\in V\left(\tau^{+}\right),v\text{ and }w^{-1}\tau_p^{+}[v]\in L_{w_p}^2(a,b)\right\}\end{aligned}\right\} \tag{3.11}$$

The subspaces $D(\tau)$ and $D\left(\tau^{+}\right)$ of $L_w^2(a,b)$ are domains of the so-called maximal operators $T(\tau)$ and $T\left(\tau^{+}\right)$ respectively, defined by:

$$T(\tau)u:=w^{-1}\tau[u],\ (u\in D(\tau))\ \text{ and }\ T\left(\tau^{+}\right)v:=w^{-1}\tau^{+}[v],\ \left(v\in D\left(\tau^{+}\right)\right).$$

For the regular problem the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$, are the restrictions of $w^{-1}\tau[u]$ and $w^{-1}\tau^+[v]$ to the subspaces:

$$\left.\begin{array}{l} D_0(\tau) := \left\{u : u \in D(\tau), u^{[r-1]}(a) = u^{[r-1]}(b)\right\} \\ D_0(\tau^+) := \left\{v : v \in D(\tau^+), v_+^{[r-1]}(a) = v_+^{[r-1]}(b)\right\} \end{array}\right\} \quad (3.12)$$

respectively. The subspaces $D_0(\tau)$ and $D_0(\tau^+)$ are dense in $L_w^2(a,b)$ and $T_0(\tau)$ and $T_0(\tau^+)$ are closed operators (see ([6], Section 3) and [18] [19] [20] [21] [22]).

In the singular problem we first introduce the operators $T_0'(\tau)$ and $T_0'(\tau^+)$; $T_0'(\tau_p)$ being the restriction of $w_p^{-1}\tau[.]$ to the subspace:

$$D_0'(\tau) := \left\{u : u \in D(\tau), supp(u) \subset (a,b)\right\} \quad (3.13)$$

and with $T_0'(\tau^+)$ defined similarly. These operators are densely-defined and closable in $L_w^2(a,b)$; and we define the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ to be their respective closures (see [5] [11] ([13], Section 5) [18] [22]) and We denote the domains of $T_0(\tau)$ and $T_0(\tau^+)$ by $D_0(\tau)$ and $D_0(\tau^+)$ respectively. It can be shown that:

$$\left.\begin{array}{l} u \in D_0(\tau) \Rightarrow u^{[r-1]}(a) = 0, (r = 1, 2, \cdots, n) \\ v \in D_0(\tau^+) \Rightarrow v_+^{[r-1]}(a) = 0, (r = 1, 2, \cdots, n) \end{array}\right\} \quad (3.14)$$

because we are assuming that a is a regular end-point. Moreover, in both regular and singular problems, we have:

$$T_0^*(\tau) = T(\tau^+),\ T_0^*(\tau^+) = T(\tau); \quad (3.15)$$

see ([13], Section 5) in the case when $\tau = \tau^+$ and compare with treatment in ([18], Section III.10.3) and [20] in general case.

For the case of one singular end-point, we consider our interval to be $I = [a,b)$ and denote by $T_0(\tau)$ and $T(\tau)$ the minimal and maximal operators. We see from (3.15) that $T_0(\tau) \subset T(\tau) = \left[T_0(\tau^+)\right]^*$ and hence $T_0(\tau)$ and $T(\tau)$ form an adjoint pair of closed densely defined operators in $L_w^2(a,b)$.

Lemma 3.1: For $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$, $def\left[T_0(\tau) - \lambda I\right] + def\left[T_0(\tau^+) - \bar{\lambda} I\right]$ is constant and:

$$0 \le def\left[T_0(\tau) - \lambda I\right] + def\left[T_0(\tau^+) - \bar{\lambda} I\right] \le 2n.$$

In the problem with one singular end-point, $n \le def\left[T_0(\tau) - \lambda I\right] + def\left[T_0(\tau^+) - \bar{\lambda} I\right] \le 2n$ for all $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$.

In the regular problem, $def\left[T_0(\tau) - \lambda I\right] + def\left[T_0(\tau^+) - \bar{\lambda} I\right] = 2n$ for all $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$.

Proof: The proof is similar to that in [3] [4] [15] [16] [17] [18] [21] [22], and therefore omitted.

For $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$, we define r, s and m as follows:

$$r = r(\lambda) := def\left[T_0(\tau) - \lambda I\right],\quad s = s(\lambda) := def\left[T_0(\tau^+) - \bar{\lambda} I\right] \quad (3.16)$$

and:

$$m = r + s. \tag{3.17}$$

Also,

$$0 \le m \le 2n. \tag{3.18}$$

For $\Pi\left[T_0(\tau), T_0(\tau^+)\right] \ne \varnothing$ the operators which are regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ are characterized by the following theorem which proved for a general quasi-differential operator in [7] [11] [12] ([18], Theorem 10.15) [23] [24].

Theorem 3.2: For $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right]$. Let *r*, *s* and *m* be defined by (3.16) and (3.17), and let $\psi_j\,(j = 1, 2, \cdots, r)$, $\Phi_k\,(k = r+1, \cdots, m)$ be arbitrary functions satisfying:

1) $\psi_j\,(j = 1, 2, \cdots, r) \subset D\left[T(\tau)\right]$ are linearly independent modulo $D\left[T_0(\tau)\right]$ and $\Phi_k\,(k = r+1, \cdots, m) \subset D\left[T(\tau^+)\right]$ are linearly independent modulo $D\left[T_0(\tau^+)\right]$

2) $\left[\psi_j, \Phi_k\right](b) - \left[\psi_j, \Phi_k\right](a) = 0$, $(j = 1, 2, \cdots, r; k = r+1, \cdots, m)$.

Then the set:

$$\left\{u : u \in D\left[T(\tau)\right], \left[u, \Phi_k\right](b) - \left[u, \Phi_k\right](a) = 0, k = r+1, \cdots, m\right\} \tag{3.19}$$

is the domain of an operator *S* which is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ and the set:

$$\left\{v : v \in D\left[T(\tau^+)\right], \left[\psi_j, v\right](b) - \left[\psi_j, v\right](a) = 0, j = 1, 2, \cdots, r\right\} \tag{3.20}$$

is the domain of the operator S^*; moreover $\lambda \in \Delta_4(S)$.

Conversely, if *S* is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$ and $\lambda \in \Pi\left[T_0(\tau), T_0(\tau^+)\right] \cap \Delta_4(S)$, then with *r*, *s* and *m* defined by (3.16) and (3.17) there exist functions $\psi_j\,(j = 1, 2, \cdots, r), \Phi_k\,(k = r+1, \cdots, m)$ which satisfy 1) and 2) and are such that (3.19) and (3.20) are the domains of *S* and S^* respectively.

S is self-adjoint if, and only if, $\tau = \tau^+$, $r = s$ and $\Phi_k = \psi_{k-r}\,(k = r+1, \cdots, m)$; *S* is *J*-self-adjoint if $\tau = J\tau^+J$ (*J* is a complex conjugate), $r = s$ and $\Phi_k = \bar{\psi}_{k-r}\,(k = r+1, \cdots, m)$.

Proof: The proof is entirely similar to that of [7] [11] [13] [18] [23] [24] and therefore omitted.

4. The Square Integrable Solutions

The following Lemma is very important for the proof of the main results in this section.

Lemma 4.1 (cf. [8]: Gronwall's inequality). Let $u(t)$ and $v(t)$ be two continuous and non-negative functions on the interval $I = [0, b)$, $c \ge 0$ be a constant. The classical Gronwall's inequality states that, if:

$$u(t) \le c + \int_0^t v(s)u(s)\,\mathrm{d}x, \quad 0 \le t \le 1.$$

Then:

$$u(t) \le c\exp\left(\int_0^t v(s)\,\mathrm{d}s\right)\mathrm{d}s, \quad 0 \le t \le 1. \tag{4.1}$$

Theorem 4.2: Suppose $f \in L^1_{loc}(I)$, and suppose that the conditions (3.1) are satisfied. Then, given any complex numbers $c_r \in \mathbb{C}$, $r = 0,1,2,\cdots,n-1$ and $x_0 \in (a,b)$, there exists a unique solutions of $\tau[\varphi] = wf$ ($\lambda \in \mathbb{C}$) which satisfies:

$$\varphi^{[r]}(x_0) = c_j,\ r = 0,1,2,\cdots,n-1.$$

Proof: The proof is similar to that in ([21], part II, Theorem 16.2.2) and therefore omitted.

Theorem 4.3: (cf. [18] [21]). Let τ be a regular quasi-differential expression of order n on the interval $[a,b]$. For $f \in L^2_w(a,b)$, the equation $\tau[\varphi] = wf$ has a solution $\varphi \in V(\tau)$ satisfying:

$$\varphi^{[r]}(a) = \varphi^{[r]}(b) = 0,\ r = 0,1,2,\cdots,n-1$$

If and only if f is orthogonal in $L^2_w(a,b)$ to solution space of $\tau^+[\psi] = 0$, *i.e.*,

$$R[T_0(\tau) - \lambda I] = N\left[T(\tau^+) - \bar{\lambda} I\right]^{\perp}.$$

Corollary 4.4 (cf. [12]), As a result from Theorem 4.2, we have that:

$$R[T_0(\tau) - \lambda I]^{\perp} = N\left[T(\tau^+) - \bar{\lambda} I\right].$$

Let $\varphi_k(t,\lambda), k = 1,2,\cdots,n$ be the solutions of the homogeneous equation:

$$(\tau - \lambda I)u = 0\ (\lambda \in \mathbb{C}) \tag{4.2}$$

satisfying:

$$\varphi_j^{[k-1]}(t_0,\lambda) = \delta_{k,r+1} \text{ for all } t_0 \in [a,b]\ (j,k = 1,2,\cdots,n;\ r = 0,1,\cdots,n-1)$$

for fixed t_0, $a < t_0 < b$. Then $\varphi_j^{[r]}(t,\lambda)$ is continuous in (t,λ) for $a < t < b$, $|\lambda| < \infty$, and for fixed t it is entire in λ. Let $\varphi_k^+(t,\lambda), k = 1,2,\cdots,n$ denote the solutions of the adjoint homogeneous equation:

$$(\tau^+ - \bar{\lambda} I)v = 0\ (\lambda \in \mathbb{C}) \tag{4.3}$$

satisfying:

$$(\varphi_k^+)^{[r]}(t_0,\lambda) = (-1)^{k+r}\delta_{k,n^2-r} \text{ for all } t_0 \in [a,b]$$

$(j,k = 1,2,\cdots,n;\ r = 0,1,\cdots,n-1)$.

Suppose $a < c < b$, by [21], a solution of the product equation:

$$(\tau - \lambda I)u = wf\ (\lambda \in \mathbb{C}),\ f \in L^1_w(a,b) \tag{4.4}$$

satisfying $u^{[r]}(c) = 0, r = 0,1,\cdots,n-1$ is giving by:

$$\varphi(t) = \left(\frac{1}{i^n}\right)\sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t,\lambda)\int_a^t \overline{\varphi_k^+(t,\lambda)}\, f(s)w(s)\mathrm{d}s,$$

where $\varphi_k^+(t,\lambda)$ stands for the complex conjugate of $\varphi_k(t,\lambda)$ and for each j,k, ξ^{jk} is constant which is independent of t,λ (but does depend in general on t_0).

The next lemma is a form of the variation of parameters formula for a general quasi-differential equation is giving by the following Lemma.

Lemma 4.5: Suppose $f \in L_w^1(a,b)$ locally integrable function and $\varphi(t,\lambda)$ is the solution of the Equation (4.4) satisfying:

$$\varphi^{[r]}(t_0,\lambda) = \alpha_{r+1} \quad \text{for} \quad r = 0,1,\cdots,n^2-1, \quad t_0 \in [a,b)$$

is giving by:

$$\varphi(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t,\lambda_0) + \frac{1}{i^n}(\lambda-\lambda_0)\sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t,\lambda_0) \times \int_a^t \overline{\varphi_k^+(t,\lambda_0)} f(s)w(s)\,\mathrm{d}s \tag{4.5}$$

for some constants $\alpha_1(\lambda),\alpha_2(\lambda),\cdots,\alpha_n(\lambda) \in \mathbb{C}$, where $\varphi_j(t,\lambda_0)$ and $\varphi_k^+(t,\lambda_0)$, $j,k = 1,2,\cdots,n$ are solutions of the Equations (4.2) and (4.3) respectively, ξ^{jk} is a constant which is independent of *t*.

Proof: The proof is similar to that in [8]-[16] and [17]-[22].

Lemma 4.5: Contain the following lemma as a special case.

Lemma 4.6: Suppose $f \in L_w^1(a,b)$ locally integrable function and $\varphi(t,\lambda)$ be the solution of the Equation (4.4) satisfying:

$$\varphi^{[r]}(t_0,\lambda) = \alpha_{r+1} \quad \text{for} \quad r = 0,1,\cdots,n^2-1, \quad t_0 \in [a,b).$$

Then:

$$\varphi^{[r]}(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j^{[r]}(t,\lambda_0) + \frac{1}{i^{n^2}}(\lambda-\lambda_0)\sum_{j,k=1}^{n^2} \xi^{jk}\varphi_j^{[r]}(t,\lambda_0) \times \int_a^t \overline{\varphi_k^+(t,\lambda_0)} f(s)w(s)\,\mathrm{d}s \tag{4.6}$$

for $r = 1,\cdots,n-1$. We refer to [13] [22] for more details.

Lemma 4.7: Suppose that for some $\lambda_0 \in \mathbb{C}$ all solutions of the equations:

$$(\tau - \lambda_0 I)\varphi = 0, \quad \left(\tau^+ - \overline{\lambda_0} I\right)\varphi^+ = 0 \tag{4.7}$$

are in $L_w^2(a,b)$. Then all solutions of the Equations in (4.7) are in $L_w^2(a,b)$ for every complex number $\lambda \in \mathbb{C}$.

Proof: The proof is similar to that in [16]-[22].

Lemma 4.8: Suppose that for some complex number $\lambda_0 \in \mathbb{C}$ all solutions of the Equations in (4.7) are in $L_w^2(a,b)$. Suppose $f \in L_w^2(a,b)$. Then all solutions of the Equation (4.4) are in $L_w^2(a,b)$ for all $\lambda \in \mathbb{C}$.

Proof: The proof is similar to that in [8]-[16] [22].

Remark: Lemma 4.8 also holds if the function *f* is bounded on $[a,b)$.

Lemma 4.9: Suppose that for some $\lambda_0 \in \mathbb{C}$ all solutions of the Equations in (4.7) are in $L_w^2(a,b)$. Then all solutions of the Equations (4.2) and (4.3) are in $L_w^2(a,b)$ for every complex number $\lambda \in \mathbb{C}$.

Proof: The proof is similar to that in [16]-[22].

Lemma 4.10: If all solutions of the equation $(\tau - \lambda_0 w)u = 0$ are bounded on $[a,b)$ and $\varphi_k^+(t,\lambda_0) \in L_w^1(a,b)$ for some $\lambda_0 \in \mathbb{C}$, $k = 1,\cdots,n$. Then all solutions of the equation $(\tau - \lambda w)u = 0$ are also bounded on $[a,b)$ for every complex number $\lambda \in \mathbb{C}$.

Lemma 4.11: Suppose that for some complex number $\lambda_0 \in \mathbb{C}$ all solutions of the Equations in (4.7) are in $L_w^2(a,b)$. Suppose $f \in L_w^2(0,b)$, then all solutions of the Equation (4.4) are in $L_w^2(a,b)$ for all $\lambda \in \mathbb{C}$.

Proof: Let $\{\varphi_1(t,\lambda), \varphi_2(t,\lambda), \cdots, \varphi_n(t,\lambda)\}$, $\{\varphi_1^+(s,\lambda), \varphi_2^+(s,\lambda), \cdots, \varphi_n^+(s,\lambda)\}$ be two sets of linearly independent solutions of the Equations (4.7). Then for any solutions $\varphi(t,\lambda)$ of the equation $(\tau - \lambda I)\varphi = wf$ ($\lambda \in \mathbb{C}$) which may be written as follows:

$(\tau - \lambda_0 w)\varphi = (\lambda - \lambda_0)w\varphi + wf$ and it follows from (4.5) that:

$$\varphi(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j(t,\lambda_0) + \frac{1}{i^n}\sum_{j,k=1}^{n} \xi^{jk}\varphi_j(t,\lambda_0) \times \int_a^t \overline{\varphi_k^+(t,\lambda_0)}\left[(\lambda - \lambda_0)\varphi(s,\lambda) + f(s)\right]w(s)\mathrm{d}s, \tag{4.8}$$

for some constants $\alpha_1(\lambda), \alpha_2(\lambda), \cdots, \alpha_n(\lambda) \in \mathbb{C}$. Hence:

$$|\varphi(t,\lambda)| = \sum_{j=1}^{n}\left(|\alpha_j(\lambda)||\varphi_j(t,\lambda_0)|\right) + \sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)| \times \int_a^t \overline{\varphi_k^+(t,\lambda_0)}\left[|\lambda - \lambda_0||\varphi(s,\lambda)| + |f(s)|\right]w(s)\mathrm{d}s. \tag{4.9}$$

Since $f \in L_w^2(a,b)$ and $\varphi_k^+(.,\lambda_0) \in L_w^2(a,b)$ for some $\lambda_0 \in \text{£}$, then $\varphi_k^+(.,\lambda_0) f \in L_w^1(a,b)$, for some $\lambda_0 \in \text{£}$ and $k = 1,\cdots,n$.

Setting:

$$C_j(\lambda) = \sum_{j,k=1}^{n}|\xi^{jk}|\left|\int_a^b \left|\overline{\varphi_k^+(t,\lambda_0)}\right|\right||f(s)|w(s)\mathrm{d}s,\ j = 1,2,\cdots,n, \tag{4.10}$$

then:

$$|\varphi(t,\lambda)| \le \sum_{j=1}^{n}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)|\varphi_j(t,\lambda_0)| + |\lambda - \lambda_0|\sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)|\int_a^b \left|\overline{\varphi_k^+(t,\lambda_0)}\right||\varphi(s,\lambda)|w(s)\mathrm{d}s. \tag{4.11}$$

On application of the Cauchy-Schwartz inequality to the integral in (4.11), we get:

$$|\varphi(t,\lambda)| \le \sum_{j=1}^{n}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)|\varphi_j(t,\lambda_0)| + |\lambda - \lambda_0|\sum_{j,k=1}^{n}|\xi^{jk}||\varphi_j(t,\lambda_0)| \times \left(\int_a^b \left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_a^b |\varphi(s,\lambda)|^2 w(s)\mathrm{d}s\right)^{\frac{1}{2}}. \tag{4.12}$$

From the inequality $(u+v)^2 \le 2(u^2+v^2)$ it follows that:

$$|\varphi(t,\lambda)|^2 \le 4\sum_{j=1}^{n}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)^2|\varphi_j(t,\lambda_0)|^2 + 4|\lambda - \lambda_0|^2\sum_{j,k=1}^{n}|\xi^{jk}|^2|\varphi_j(t,\lambda_0)|^2 \times \left(\int_0^b \left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)\mathrm{d}s\right)\left(\int_0^b |\varphi(s,\lambda)|^2 w(s)\mathrm{d}s\right). \tag{4.13}$$

By hypothesis there exist positive constant K_0 and K_1 such that:

$$\|\varphi_j(t,\lambda_0)\|_{L_w^2(a,b)} \le K_0 \text{ and } \left\|\overline{\varphi_k^+(t,\lambda_0)}\right\|_{L_w^2(a,b)} \le K_1;\ j,k = 1,2,\cdots,n. \tag{4.14}$$

Hence:

$$|\varphi(t,\lambda)|^2 \le 4\sum_{j=1}^{n}\left(|\alpha_j(\lambda)| + C_j(\lambda)\right)^2|\varphi_j(t,\lambda_0)|^2 + 4K_1^2|\lambda - \lambda_0|^2\sum_{j,k=1}^{n}|\xi^{jk}|^2|\varphi_j(t,\lambda_0)|^2\left(\int_0^b |\varphi(s,\lambda)|^2 w(s)\mathrm{d}s\right). \tag{4.15}$$

Integrating the inequality in (4.15) between a and t, we obtain:

$$\int_a^t |\varphi(s,\lambda)|^2 w(s)\mathrm{d}s \le K_2 + \left(4|\lambda-\lambda_0|^2 \sum_{j,k=1}^n |\xi^{jk}|^2\right) \int_a^t |\varphi_j(s,\lambda_0)|^2 \left(\int_a^s |\varphi(x,\lambda)|^2 w(x)\mathrm{d}x\right) w(s)\mathrm{d}s, \tag{4.16}$$

where:

$$K_2 = 4K_0^2 \sum_{j=1}^n \left(|\alpha_j(\lambda)| + C_j(\lambda)\right)^2. \tag{4.17}$$

Now, on using Gronwall's inequality (**Lemma 4.1**), it follows that:

$$\int_a^t |\varphi(s,\lambda)|^2 w(s)\mathrm{d}s \le K_2 \exp\left(4K_1^2 |\lambda-\lambda_0|^2 \sum_{j,k=1}^n |\xi^{jk}|^2 \int_a^t |\varphi_j(t,\lambda_0)|^2 w(s)\mathrm{d}s\right). \tag{4.18}$$

Since, $\varphi_j(t,\lambda_0) \in L_w^2(a,b)$ for some $\lambda_0 \in £$ and for $j = 1,\cdots,n$, then $\varphi(t,\lambda) \in L_w^2(a,b)$.

Remark: Lemma 4.11 also holds if the function f is bounded on $[a,b)$.

Lemma 4.12: Let $f \in L_w^2(a,b)$. Suppose for some $\lambda_0 \in £$ that:

1) All solutions of $(\tau^+ - \overline{\lambda}_0 I)\varphi^+ = 0$ are in $L_w^2(a,b)$.

2) $\varphi_j(t,\lambda_0), j = 1,\cdots,n$ are bounded on $[a,b)$.

Then $\varphi^{[r]}(t,\lambda) \in L_w^2(a,b)$ for any solution $\varphi(t,\lambda)$ of the equation $(\tau - \lambda I)\varphi = wf$ for all $\lambda \in £$.

Proof: The proof is similar to that in ([20], Theorem 4.2).

Lemma 4.13: Let $f \in L_w^2(a,b)$. Suppose for some $\lambda_0 \in £$ that:

1) All solutions of $(\tau^+ - \overline{\lambda} I)\varphi^+ = 0$ are in $L_w^2(a,b)$.

2) $\varphi_j^{[r]}(t,\lambda_0), j = 1,\cdots,n$ are bounded on $[a,b)$ for some $r = 0,1,\cdots,n-1$.

Then $\varphi^{[r]}(t,\lambda) \in L_w^2(a,b)$ for any solution $\varphi(t,\lambda)$ of the equation $(\tau - \lambda I)\varphi = wf$ for all $\lambda \in £$.

Proof: The proof is the same up to (4.11). By using **Lemma 4.3**, (3.11) becomes:

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^n \left(|\alpha_j(\lambda)| + C_j(\lambda)\right) \left|\varphi_j^{[r]}(t,\lambda_0)\right| + |\lambda-\lambda_0| \sum_{j,k=1}^n \sum_{r=0}^{n-1} |\xi^{jk}| \left|\varphi_j^{[r]}(t,\lambda_0)\right| \int_a^b \left|\overline{\varphi_k^+(t,\lambda_0)}\right| \left|\varphi^{[r]}(t,\lambda)\right| w(s)\mathrm{d}s. \tag{4.19}$$

Applying the Cauchy Schwartz inequality to the integral in (4.19), we get:

$$\left|\varphi^{[r]}(t,\lambda)\right| \le \sum_{j=1}^n \left(C_j + |\alpha_j(\lambda)|\right) \left|\varphi_j^{[r]}(t,\lambda_0)\right| + |\lambda-\lambda_0| \sum_{j,k=1}^n \sum_{r=0}^{n-1} |\xi^{jk}| \left|\varphi_j^{[r]}(t,\lambda_0)\right| \times \left(\int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^t \left|\varphi^{[r]}(t,\lambda)\right|^2 w(s)\mathrm{d}s\right)^{\frac{1}{2}}. \tag{4.20}$$

From the inequality $(u+v)^2 \le 2(u^2+v^2)$ it follows that:

$$\left|\varphi^{[r]}(t,\lambda)\right|^2 \le 4\sum_{j=1}^n \left(C_j^2 + |\alpha_j(\lambda)|^2\right) \left|\varphi_j^{[r]}(t,\lambda_0)\right|^2 + 4|\lambda-\lambda_0|^2 \sum_{j,k=1}^n \sum_{r=0}^{n-1} |\xi^{jk}|^2 \left|\varphi_j^{[r]}(t,\lambda_0)\right|^2 \times \left(\int_0^t \left|\overline{\varphi_k^+(t,\lambda_0)}\right|^2 w(s)\mathrm{d}s\right) \left(\int_0^t \left|\varphi^{[r]}(s,\lambda)\right|^2 w(s)\mathrm{d}s\right). \tag{4.21}$$

Since $\varphi_k^+(t,\lambda_0) \in L_w^2(0,b)$ for some $\lambda_0 \in \mathbb{C}$ and $\varphi_j^{[r]}(t,\lambda_0), j=1,\cdots,n$ are bounded on $[a,b)$ for some $r=0,1,\cdots,n-1$ by hypothesis, then there exist a positive constants K_0 and K_1 such that:

$$\left|\varphi_j^{[r]}(t,\lambda_0)\right| \le K_0 \quad \text{and} \quad \left\|\overline{\varphi_k^+(s,\lambda_0)}\right\|_{L_w^2(0,b)} \le K_1. \tag{4.22}$$

Hence,

$$\begin{aligned}\left|\varphi^{[r]}(t,\lambda)\right|^2 \le{}& 4K_0^2 \sum_{j=1}^n \left(C_j^2 + \left|\alpha_j(\lambda)\right|^2\right) + 4K_0^2 K_1 \left|\lambda-\lambda_0\right|^2 \\ &\times \sum_{j,k=1}^n \sum_{r=0}^{n-1} \left|\xi^{jk}\right|^2 \left(\int_0^t \left|\varphi^{[r]}(s,\lambda)\right|^2 w(s)\mathrm{d}s\right).\end{aligned} \tag{4.23}$$

By integrating the inequality in (4.23) between *a* and *t*, and by using **Lemma 4.1** (Gronwall's inequality), we have the result.

5. The Spectra of Differential Operators

In this subsection we deal with the various components of the spectra of quasi-differential operators $T_0(\tau)$ and $T_0(\tau^+)$.

We see from (3.15) and Theorem 4.2 that $T_0(\tau) \subset T(\tau) = \left[T_0(\tau^+)\right]^*$ and hence $T_0(\tau)$ and $T_0(\tau^+)$ form an adjoint pair of closed, closed-densely operators in $L_w^2(a,b)$.

We shall now investigate in the case of one singular end-point that the resolvent of all well-posed extensions of the minimal operator $T_0(\tau)$ and we show that in the maximal case, *i.e.*, when:

$$def\left[T_0(\tau)-\lambda I\right] = def\left[T_0(\tau^+)-\bar{\lambda} I\right] = n \quad \text{for all} \quad \lambda \in \Pi\left[T_0(\tau),T_0(\tau^+)\right]$$

these resolvent are integral operators, in fact they are Hilbert-Schmidt integral operators by considering that the function f be in $L_w^2(a,b)$, *i.e.*, is quadratically integrable over the interval $[a,b)$.

Theorem 5.1: Suppose for an operator $T_0(\tau)$ with one singular end-point that,

$$def\left[T_0(\tau)-\lambda I\right] = def\left[T_0(\tau^+)-\bar{\lambda} I\right] = n \quad \text{for all} \quad \lambda \in \Pi\left[T_0(\tau),T_0(\tau^+)\right],$$

and let *S* be an arbitrary closed operator which is a well-posed extension of the minimal operator $T_0(\tau)$ and $\lambda \in \rho(S)$, then the resolvents R_λ and $R_{\bar{\lambda}}^*$ of *S* and S^* respectively are Hilbert-Schmidt integral operators whose kernels $K(t,s,\lambda)$ and $K^+(s,t,\bar{\lambda})$ are continuous functions on $[a,b)\times[a,b)$ and satisfy:

$$K(t,s,\lambda) = \overline{K^+(s,t,\bar{\lambda})} \quad \text{and} \quad \int_a^b\int_a^b \left|K(t,s,\lambda)\right|^2 w(s)w(t)\mathrm{d}s\mathrm{d}t < \infty. \tag{5.1}$$

where:

$$R_\lambda f(t) = \int_a^b K(t,s,\lambda) f(s) w(s)\mathrm{d}s \quad \text{and} \quad R_{\bar{\lambda}}^* g(t) = \int_a^b \overline{K^+(s,t,\bar{\lambda})} g(t) w(t)\mathrm{d}t,$$

for $f,g \in L_w^2(a,b)$, and for all $s,t \in [a,b)$.

Remark: An example of a closed operator which is a well-posed with respect to a compatible adjoint pair is given by the Visik extension ([7], Theorem 1) (see

([18], Theorem III.3.3) and [21]). Note that if S is well-posed, then $T_0(\tau)$ and $T_0(\tau^+)$ are compatible adjoint pair and S is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$.

Proof: Let $def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda} I\right]=n$ for all $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$, then we choose a fundamental system of solutions $\{\varphi_1(t,\lambda),\varphi_2(t,\lambda),\cdots,\varphi_n(t,\lambda)\}$, $\{\varphi_1^+(t,\lambda),\varphi_2^+(t,\lambda),\cdots,\varphi_n^+(t,\lambda)\}$ of the equations,

$$\left[T_0(\tau)-\lambda I\right]\varphi_j=0,\left[T_0(\tau^+)-\bar{\lambda} I\right]\varphi_k^+=0\ \ (j,k=1,\cdots,n)\text{ on }[a,b), \tag{5.2}$$

so that $\{\varphi_1(t,\lambda),\varphi_2(t,\lambda),\cdots,\varphi_n(t,\lambda)\}$, $\{\varphi_1^+(t,\lambda),\varphi_2^+(t,\lambda),\cdots,\varphi_n^+(t,\lambda)\}$ belong to $L_w^2(a,b)$ *i.e.*, they are quadratically integrable in the interval $[a,b)$. Let $R_\lambda=(S-\lambda I)^{-1}$ be the resolvent of any well-posed extension of the minimal operator $T_0(\tau)$. For $f\in L_w^2(a,b)$ we put $\varphi(t,\lambda)=R_\lambda f(t)$ then $\left[T(\tau)-\lambda I\right]\varphi=wf$ and consequently has a solution $\varphi(t,\lambda)$ in the form,

$$\begin{aligned}\varphi(t,\lambda)=\sum_{j=1}^n\alpha_j(\lambda)\varphi_j(t,\lambda_0)+\frac{1}{i^n}(\lambda-\lambda_0)\sum_{j,k=1}^n\xi^{jk}\varphi_j(t,\lambda_0)\\ \times\int_a^t\overline{\varphi_k^+(t,\lambda_0)}f(s)w(s)\mathrm{d}s\end{aligned} \tag{5.3}$$

for some constants $\alpha_1(\lambda),\alpha_2(\lambda),\cdots,\alpha_n(\lambda)\in\mathbb{C}$ (see **Lemma 4.5**). Since $f\in L_w^2(a,b)$ and $\varphi_k^+(.,\lambda_0)\in L_w^2(a,b)$ for some $\lambda_0\in\mathbb{C}$, then $\varphi_k^+(.,\lambda_0)f\in L_w^1(a,b)$, $k=1,\cdots,n$ for some $\lambda_0\in\mathbb{C}$ and hence the integral in the right-hand of (5.3) will be finite.

To determine the constants $\alpha_j(\lambda), j=1,\cdots,n$, let $\varphi_k^+(t,\lambda),k=1,\cdots,n$ be a basis for $\left\{D(S^*)/D_o\left[T(\tau^+)\right]\right\}$, then because $\varphi(t,\lambda)\in D(S)\subset\rho(S)\subset\Delta_4(S)$, we have from Theorem 3.2 that,

$$\left[\varphi,\varphi_k^+\right](b)-\left[\varphi,\varphi_k^+\right](a)=0,\ (k=1,2,\cdots,n)\text{ on }[a,b) \tag{5.4}$$

and hence from (5.3), (5.4) and on using **Lemma 4.7**, we have:

$$\begin{aligned}\left[\varphi,\varphi_k^+\right](b)=\sum_{j=1}^n\Big[\alpha_j(\lambda)+\frac{1}{i^n}(\lambda-\lambda_0)\sum_{j,k=1}^n\xi^{jk}\\ \times\int_a^t\overline{\varphi_k^+(t,\lambda_0)}f(s)w(s)\mathrm{d}s\Big]\left[\varphi_j,\varphi_k^+\right](b),\\ \left[\varphi,\varphi_k^+\right](a)=\sum_{j=1}^n\alpha_j(\lambda)\left[\varphi_j,\varphi_k^+\right](a),\ k=1,2,\cdots,n.\end{aligned} \tag{5.5}$$

By substituting these expressions into the conditions (5.4), we get:

$$\begin{aligned}&\sum_{j=1}^n\left[\alpha_j(\lambda)+\frac{1}{i^n}(\lambda-\lambda_0)\sum_{j,k=1}^n\xi^{jk}\int_a^t\overline{\varphi_k^+(t,\lambda_0)}f(s)w(s)\mathrm{d}s\right]\left[\varphi_j,\varphi_k^+\right](b)\\ &=\sum_{j=1}^n\alpha_j(\lambda)\left[\varphi_j,\varphi_k^+\right](a).\end{aligned}$$

This implies that the system:

$$\sum_{j=1}^n\alpha_j(\lambda)\left[\varphi_j,\varphi_k^+\right]_a^b=-\frac{\lambda-\lambda_0}{i^n}\left(\sum_{j,k=1}^n\xi^{jk}\int_a^t\overline{\varphi_k^+(t,\lambda_0)}f(s)w(s)\mathrm{d}s\right), \tag{5.6}$$

in the variable $\alpha_j(\lambda), j=1,2,\cdots,n$. The determinant of this system does not vanish (see [9] and [12]). If we solve the system (5.6) we obtain:

$$\alpha_j(\lambda) = \frac{\lambda - \lambda_0}{i^n}\left(\sum_{j,k=1}^{n} \xi^{jk} \int_a^b h_j(s,\lambda) f(s) w(s) \mathrm{d}s\right), \quad j = 1,2,\cdots,n \tag{5.7}$$

where $h_j(s,\lambda)$ is a solution of the system:

$$\sum_{j=1}^{n} h_j(s,\lambda)\left(\left[\varphi_j, \varphi_k^+\right]\right)_a^b = -\sum_{j,k=1}^{n} \xi^{jk} \overline{\varphi_k^+(t,\lambda_0)}\left[\varphi_j, \varphi_k^+\right](b). \tag{5.8}$$

Since, the determinant of the above system (5.8) does not vanish, and the functions $\varphi_k^+(s,\lambda_0), k = 1,2,\cdots,n$ are continuous in the interval $[a,b)$, then the functions $h_j(s,\lambda)$ are also continuous in the interval. By substituting in formula (5.3) for the expressions $\alpha_j(\lambda), j = 1,2,\cdots,n$ we get,

$$\begin{aligned} R_\lambda f(t) &= \varphi(t,\lambda) \\ &= \frac{\lambda-\lambda_0}{i^n}\left[\sum_{j,k=1}^{n} \alpha_j(\lambda)\varphi_j(t,\lambda_0)\int_a^t \left[\xi^{jk}\overline{\varphi_k^+(t,\lambda_0)} + h_j(s,\lambda)\right] f(s) w(s) \mathrm{d}s \right. \\ &\quad \left. + \sum_{j=1}^{n} \varphi_j(t,\lambda_0)\int_t^b h_j(s,\lambda) f(s) w(s) \mathrm{d}s\right] \end{aligned} \tag{5.9}$$

Now, we put:

$$K(t,s,\lambda) = \begin{cases} \dfrac{\lambda-\lambda_0}{i^n}\left(\sum_{j=1}^{n} \varphi_j(t,\lambda_0) h_j(s,\lambda)\right) & \text{for } t < s \\ \dfrac{\lambda-\lambda_0}{i^n}\left(\sum_{j,k=1}^{n} \xi^{jk} \varphi_j(t,\lambda_0)\left(\overline{\varphi_k^+(t,\lambda_0)} + h_j(s,\lambda)\right)\right) & \text{for } t > s \end{cases} \tag{5.10}$$

Formula (5.9) then takes the form:

$$R_\lambda f(t) = \int_a^b K(t,s,\lambda) f(s) w(s) \mathrm{d}s \quad \text{for all} \quad t \in [a,b), \tag{5.11}$$

i.e., R_λ is an integral operator with the kernel $K(s,t,\lambda)$ operating on the functions $f \in L_w^2(a,b)$. Similarly, the solutions $\varphi^+(t,\lambda)$ of the equation $\left[T(\tau^+) - \bar{\lambda} I\right]\varphi^+ = wg$ has the form:

$$\begin{aligned} \varphi^+(s,\lambda) &= \sum_{j=1}^{n} \alpha_j(\lambda)\varphi_j^+(s,\lambda_0) + \frac{\bar{\lambda}-\bar{\lambda}_0}{i^n}\sum_{j,k=1}^{n} \xi^{jk} \varphi_j^+(s,\lambda_0) \\ &\quad \times \int_a^s \overline{\varphi_k(t,\lambda_0)} g(t) w(t) \mathrm{d}t, \end{aligned} \tag{5.12}$$

where $\varphi_k(t,\lambda_0)$ and $\varphi_j^+(s,\lambda_0), k, j = 1,2,\cdots,n$ are solutions of the Equations in (5.2). The argument as before leads to,

$$R_{\bar{\lambda}}^* g(t) = \int_a^b \overline{K^+(s,t,\bar{\lambda})} g(t) w(t) \mathrm{d}t \quad \text{for} \quad g \in L_w^2(a,b), \tag{5.13}$$

i.e., $R_{\bar{\lambda}}^*$ is an integral operator with the kernel $K^+(t,s,\bar{\lambda})$ operating on the functions $g \in L_w^2(a,b)$, where:

$$K^+(s,t,\bar{\lambda}) = \begin{cases} \dfrac{\bar{\lambda}-\bar{\lambda}_0}{i^n}\left(\sum_{j=1}^{n} \varphi_j^+(s,\lambda_0) h_j^+(t,\lambda)\right) & \text{for } s < t \\ \dfrac{\bar{\lambda}-\bar{\lambda}_0}{i^n}\left(\sum_{j,k=1}^{n} \xi^{jk} \varphi_j^+(s,\lambda_0)\left(\overline{\varphi_k(t,\lambda_0)} + h_j^+(t,\lambda)\right)\right) & \text{for } s > t \end{cases} \tag{5.14}$$

and $h_j^+(t,\lambda)$ is a solution of the system:

$$\sum_{j=1}^{n} \overline{h_j(s,\lambda)}\left(\left[\varphi_j, \varphi_k^+\right]\right)_a^b = -\sum_{j,k=1}^{n} \xi^{jk} \varphi_j(t,\lambda_0)\left[\varphi_j, \varphi_k^+\right](b). \tag{5.15}$$

From definitions of R_λ and $R^*_{\bar{\lambda}}$, it follows that:

$$\begin{aligned}(R_\lambda f,g) &= \int_a^b\left\{\int_a^b K(t,s,\lambda)f(s)w(s)\mathrm{d}s\right\}\overline{g(t)}w(t)\mathrm{d}t \\ &= \int_a^b\left\{\int_a^b K(t,s,\lambda)\overline{g(t)}w(t)\mathrm{d}t\right\}f(s)w(s)\mathrm{d}s = \left(f,R^*_{\bar{\lambda}}g\right),\end{aligned} \tag{5.16}$$

for any continuous functions $f,g\in H$ and by construction (see (5.10) and (5.14)), $K(t,s,\lambda)$ and $K^+(s,t,\bar{\lambda})$ are continuous functions on $[a,b)\times[a,b)$ and (5.16) gives us:

$$K(t,s,\lambda)=\overline{K^+(s,t,\bar{\lambda})} \text{ for all } t,s\in[a,b)\times[a,b). \tag{5.17}$$

Since $\varphi_j(t,\lambda),\varphi_k^+(s,\lambda)\in L_w^2(a,b)$ for $j,k=1,2,\cdots,n$ and for fixed *s*, $K(t,s,\lambda)$ is a linear combination of $\varphi_j(t,\lambda)$ while, for fixed *t*, $K^+(s,t,\bar{\lambda})$ is a linear combination of $\varphi_k^+(s,\lambda)$ Then we have:

$$\int_a^b|K(t,s,\lambda)|^2 w(t)\mathrm{d}t<\infty,\ \int_a^b\left|K^+(s,t,\bar{\lambda})\right|^2 w(s)\mathrm{d}s<\infty,\ a<s,t<b$$

and (5.17) implies that,

$$\int_a^b|K(t,s,\lambda)|^2 w(s)\mathrm{d}s=\int_a^b\left|K^+(s,t,\bar{\lambda})\right|^2 w(s)\mathrm{d}s<\infty,$$

$$\int_a^b\left|K^+(s,t,\bar{\lambda})\right|^2 w(t)\mathrm{d}t=\int_a^b|K(t,s,\lambda)|^2 w(t)\mathrm{d}t<\infty.$$

Now, it is clear from (5.8) that the functions $h_j(s,\lambda),(j=1,2,\cdots,n)$ belong to $L_w^2(a,b)$ since $h_j(s,\lambda)$ is a linear combination of the functions $\varphi_j^+(s,\lambda)$ which lie in $L_w^2(a,b)$ and hence $h_j(t,\lambda)$ belong to $L_w^2(a,b)$. Similarly $h_j^+(t,\lambda)$ belong to $L_w^2(a,b)$. By the upper half of the formula (5.10) and (5.14), we have:

$$\int_a^b\left(\int_a^b|K(t,s,\lambda)|^2 w(s)\mathrm{d}s\right)w(t)\mathrm{d}t<\infty,$$

for the inner integral exists and is a linear combination of the products $\varphi_j(t,\lambda)\varphi_k^+(s,\lambda)$, $(j,k=1,2,\cdots,n)$ and these products are integrable because each of the factors belongs to $L_w^2(a,b)$. Then by (5.17), and by the upper half of (5.14),

$$\int_a^b\left(\int_a^b|K(t,s,\lambda)|^2 w(s)\mathrm{d}s\right)w(t)\mathrm{d}t=\int_a^b\left(\int_a^b\left|K^+(s,t,\bar{\lambda})\right|^2 w(s)\mathrm{d}s\right)w(t)\mathrm{d}t<\infty.$$

Hence, we also have:

$$\int_a^b\int_a^b|K(t,s,\lambda)|^2 w(t)w(s)\mathrm{d}t\mathrm{d}s<\infty,$$

and the theorem is completely proved for any well-posed extension.

Remark: It follows immediately from Theorem 5.1 that, if for an operator $T_0(\tau)$ with one singular end-point that $def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n$ for all $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$ and *S* is well-posed with respect to $T_0(\tau)$ and $T_0(\tau^+)$ with $\lambda\in\rho(S)$ then $R_\lambda=(S-\lambda I)^{-1}$ is a Hilbert-Schmidt integral operator. Thus it is a completely continuous operator, and consequently its spectrum is discrete and consists of isolated eigenvalues having finite algebraic (so geometric) multiplicity with zero as the only possible point of accumulation. Hence, the spectra of all well-posed operators S are discrete, *i.e.*,

$$\sigma_{ek}(S)=\varnothing\,,\ \text{for}\ \ k=1,2,3,4,5\,. \tag{5.18}$$

We refer to [3] [5] [7] [14] ([18], Theorem IX.3.1) [21] for more details.

An example of a closed operator which is a well-posed with respect to a compatible adjoint pair is given by the Visik extension ([7], Theorem 1) (see ([18], Theorem III.3.3) [21]). Note that if S is well-posed, then $T_0(\tau)$ and $T_0(\tau^+)$ are compatible adjoint pair and S is regularly solvable with respect to $T_0(\tau)$ and $T_0(\tau^+)$.

Lemma 5.2: The point spectra $\sigma_p\left[T_0(\tau)\right]$ and $\sigma_p\left[T_0(\tau^+)\right]$ of the operators $T_0(\tau)$ and $T_0(\tau^+)$ are empty.

Proof: Let $\lambda\in\sigma_p\left[T_0(\tau)\right]$. Then there exists a nonzero element $\varphi\in D\left[T_0(\tau)\right]$, such that:

$$\left[T_0(\tau)-\lambda I\right]\varphi=0.$$

In particular, this gives:

$$(\tau-\lambda w)\varphi=0,\ \varphi^{[r]}(a)=\varphi^{[r]}(b)=0,\ r=0,1,2,\cdots,n-1.$$

From Theorem 4.2, it follows that $\varphi\equiv 0$ and hence $\sigma_p\left[T_0(\tau)\right]=\varnothing$. Similarly $\sigma_p\left[T_0(\tau^+)\right]=\varnothing$.

Theorem 5.3: 1) $\rho\left[T_0(\tau)\right]=\varnothing$,

2) $\sigma_p\left[T_0(\tau)\right]=\sigma_c\left[T_0(\tau)\right]=\varnothing$,

3) $\sigma\left[T_0(\tau)\right]=\sigma_r\left[T_0(\tau)\right]=\mathbb{C}$.

Proof: 1) Since $R\left[T_0(\tau)-\lambda I\right]$ is a proper closed subspace of $L^2_w(a,b)$, then the resolvent set $\rho\left[T_0(\tau)\right]$ is empty.

2) Since $R\left[T_0(\tau)-\lambda I\right]$ is closed, then the continuous spectrum of $T_0(\tau)$ is empty set, *i.e.*, $\sigma_c\left[T_0(\tau)\right]=\varnothing$.

3) From 1) and 2) and **Lemma 5.2**, it follows that $\sigma\left[T_0(\tau)\right]=\sigma_r\left[T_0(\tau)\right]=\mathbb{C}$.

Corollary 5.4: 1) $\sigma_c\left[T(\tau)\right]=\sigma_r\left[T(\tau)\right]=\varnothing$,

2) $\sigma\left[T(\tau)\right]=\sigma_p\left[T(\tau)\right]=\mathbb{C}$ and $\rho\left[T(\tau)\right]=\varnothing$.

Proof: From Theorem 4.2 and since $T(\tau)=\left[T_0(\tau^+)\right]^*$, it follows that $R\left[T(\tau)-\lambda I\right]$ is closed for every $\lambda\in\mathbb{C}$, see ([3], Theorem 1.3.7). Also, we have:

$$null\left[T(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n,$$

and:

$$def\left[T(\tau)-\lambda I\right]=null\left[T_0(\tau^+)-\bar{\lambda}I\right]=n\,.$$

1) Since $R\left[T(\tau)-\lambda I\right]$ is closed and $def\left[T(\tau)-\lambda I\right]=0$, then $R\left[T(\tau)-\lambda I\right]=H$ and this yields that:

$$\sigma_c\left[T(\tau)\right]=\sigma_r\left[T(\tau)\right]=\varnothing.$$

2) Since $null\left[T(\tau)-\lambda I\right]=n$ for every $\lambda\in\mathbb{C}$, then we have $\sigma_p\left[T(\tau)\right]=\mathbb{C}$.

It also follows that $\sigma\left[T(\tau)\right]=\mathbb{C}$ and hence $\rho\left[T(\tau)\right]=\varnothing$.

Lemma 5.5: (cf. ([18], Lemma IX.9.1)). If $I=[a,b]$, with $-\infty<a<b<\infty$ then for any $\lambda\in\mathbb{C}$, the operator $T_0(\tau)$ has closed range, zero nullity and deficiency n. Hence,

$$\sigma_{ek}\left[T_0(\tau)\right]=\begin{cases}\varnothing & (k=1,2,3)\\ \mathbb{C} & (k=4,5)\end{cases} \tag{5.19}$$

Proof: The proof is similar to that in ([7], **Lemma 4.9**) and [18].

Corollary 5.6: Let $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$ with:

$$def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n. \tag{5.20}$$

Then,

$$\sigma_{ek}(S)=\varnothing\text{, for } k=1,2,3. \tag{5.21}$$

of all regularly solvable extensions S with respect to the compatible adjoint pair $T_0(\tau)$ and $T_0(\tau^+)$.

Proof: Since:

$$def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n,\quad \text{for all } \lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right].$$

Then we have from ([18], Theorem III.3.5) that,

$$dim\left\{D(S)/D_0\left[T_0(\tau)\right]\right\}=def\left[T_0(\tau)-\lambda I\right]=n,$$

$$dim\left\{D(S^*)/D_0\left[T_0(\tau^+)\right]\right\}=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n.$$

Thus S is an n-dimensional extension of $T_0(\tau)$ and so by [5] [7] and ([18], Corollary IX.4.2).

$$\sigma_{ek}(S)=\sigma_{ek}\left[T_0(\tau)\right],\ (k=1,2,3). \tag{5.22}$$

From **Lemma 5.2** and **Lemma 5.5**, we get,

$$\sigma_{ek}\left[T_0(\tau)\right]=\varnothing,\ (k=1,2,3). \tag{5.23}$$

Hence, by (5.22) and (5.23) we have that,

$$\sigma_{ek}(S)=\varnothing,\ (k=1,2,3).$$

Remark: If S is well-posed (say the Visik's extension, see [5] [6]) we get from (5.19) and (5.22) that:

$$\sigma_{ek}\left[T_0(\tau)\right]=\varnothing,\ (k=1,2,3).$$

On applying (5.22) again to any regularly solvable operator S under consideration, hence (5.21).

Corollary 5.7: Let $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$ with:

$$def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n. \tag{5.24}$$

Then,

$$\sigma_{ek}(S)=\varnothing\text{, for } k=1,2,3. \tag{5.25}$$

for all regularly solvable operators S with respect to the compatible adjoint pair $T_0(\tau)$ and $T_0(\tau^+)$.

Proof: Since:

$$def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n,\quad \text{for all}\quad \lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right].$$

Then we have from ([18], Theorem III.3.5) that,

$$dim\left\{D(S)/D_0\left[T_0(\tau)\right]\right\}=def\left[T_0(\tau)-\lambda I\right]=n,$$

$$dim\left\{D(S^*)/D_0\left[T_0(\tau^+)\right]\right\}=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n.$$

Thus S is an n-dimensional extension of $T_0(\tau)$ and so by [5] [7] [18],

$$\sigma_{ek}(S)=\sigma_{ek}\left[T_0(\tau)\right],\ (k=1,2,3). \tag{5.26}$$

From **Lemma 5.2** and **Lemma 5.5**, we get,

$$\sigma_{ek}\left[T_0(\tau)\right]=\varnothing,\ (k=1,2,3). \tag{5.27}$$

Hence, by (5.26) and (5.27) we have that,

$$\sigma_{ek}(S)=\varnothing,\ (k=1,2,3).$$

Remark: If S is well-posed (say the Visik extension, (see [5] [7])), we get from (5.21) and (5.26) that:

$$\sigma_{ek}\left[T_0(\tau)\right]=\varnothing,\ (k=1,2,3).$$

On applying (5.26) again to any regularly solvable extensions S under consideration, hence (5.25).

Corollary 5.8: If for some $\lambda_0\in\mathbb{C}$, there are n linearly independent solutions of the equations:

$$(\tau-\lambda_0 w)u=0,\ \left(\tau^+-\bar{\lambda}_0 w\right)v=0,\ \lambda_0\in\Pi\left[T_0(\tau),T_0(\tau^+)\right] \tag{5.28}$$

in $L^2_w(a,b)$, and hence,

$$\Pi\left[T_0(\tau),T_0(\tau^+)\right]=£\quad\text{and}\quad\sigma_{ek}\left[T_0(\tau),T_0(\tau^+)\right]=\varnothing,\ \ k=1,2,3,$$

where $\sigma_{ek}\left[T_0(\tau),T_0(\tau^+)\right]$ is the joint essential spectra of $T_0(\tau),T_0(\tau^+)$ defined as the joint field of regularity $\Pi\left[T_0(\tau),T_0(\tau^+)\right]$.

Proof: Since all solutions of the equations in (5.28) are in $L^2_w(a,b)$ for some $\lambda_0\in\mathbb{C}$ then, $def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n$, for some $\lambda_0\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$.

From **Lemma 4.12**, we have that $T_0(\tau)$ has no eigenvalues and so $\left[T_0(\tau)-\lambda_0 I\right]^{-1}$ exists and its domain $R\left[T_0(\tau)-\lambda_0 I\right]$ is a closed subspace of $L^2_w(a,b)$. Hence, since $T_0(\tau)$ is a closed operator, then $\left[T_0(\tau)-\lambda_0 I\right]^{-1}$ is bounded and hence $\Pi\left[T_0(\tau)\right]=£$. Similarly $\Pi\left[T_0(\tau^+)\right]=£$. Therefore $\Pi\left[T_0(\tau),T_0(\tau^+)\right]=£$ and hence, $def\left[T_0(\tau)-\lambda I\right]=def\left[T_0(\tau^+)-\bar{\lambda}I\right]=n$, for all $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$.

From Corollary 5.7, we have for any regularly solvable extension S of $T_0(\tau)$

that $\sigma_{ek}(S)=\varnothing$, $k=1,2,3$ and by (5.22), we get $\sigma_{ek}\left[T_0(\tau)\right]=\varnothing$, $k=1,2,3$. Similarly $\sigma_{ek}\left[T_0(\tau^+)\right]=\varnothing$, $k=1,2,3$. Hence,

$$\sigma_{ek}\left[T_0(\tau),T_0(\tau^+)\right]=\varnothing,\ k=1,2,3.$$

Remark: If there are n linearly independent solutions of the Equations (5.28) in $L^2_w(a,b)$ for some $\lambda_0\in\mathbb{C}$ then the complex plane can be divided into two disjoint sets:

$$\pounds=\Pi\left[T_0(\tau),T_0(\tau^+)\right]\cup\sigma_{ek}\left[T_0(\tau),T_0(\tau^+)\right],\ k=1,2,3.$$

We refer to [3] [4] [5] [7] [11] [12] [13] [14] [18] [19] [20] [21] [22] for more details.

Conclusion. We have investigated (according to the spectral theory) the location of the point spectra and regularity fields of general ordinary quasi-differential operators in the case of one singular end-point and when all solutions of the equations $[\tau-\lambda w]u=0$ and $\left[\tau^+-\bar{\lambda}w\right]v=0$ are in the space $L^2_w(a,b)$ for some (and hence all $\lambda\in\mathbb{C}$).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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