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On the Absence of Zeros of Riemann Zeta-Function Out of $\Re(z) = 1/2$

Jorge Julián Sánchez Martínez

ACESyD, S.L., Valencia, Spain
Email: jsanchez@acesyd.com

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Abstract

This work shows, after a brief introduction to Riemann zeta function $\zeta(z)$, the demonstration that all non-trivial zeros of this function lies on the so-called "critical line", $\Re(z) = 1/2$, the one Hardy demonstrated in his famous work that infinite countable zeros of the above function can be found on it. Thus, out of this strip, the only remaining zeros of this function are the so-called "trivial ones" $z = -2n$, $n \in \mathbb{N}$. After an analytical introduction reminding the existence of a germ from a generic zero lying in $\Re(z) = 1/2$, we show through a Weierstrass-Hadamard representation approach of the above germ that non-trivial zeros out of $\Re(z) = 1/2$ cannot be found.

Keywords

Riemann Zeta Function, Analyticity, Weierstrass-Hadamard Product, Representation

1. Introduction

The Riemann Hypothesis -RH- is considered to be one of the most relevant problems still unsolved. Bombieri's statement [1] is very accurate concerning its scope. Though many attempts lately achieved to demonstrate the existence of zeros in the so-called "critical line" (non-trivial zeros) come from a variety of approaches and disciplines even these days [2] [3] [4], actually the existence of infinite-isolated zeros, thus numerable in this line is due to Hardy [5] a long time ago. Nevertheless, the non-existence of zeros out of the critical line remains undemonstrated indeed.

In this contribution, we show that only in the critical line the existence of zeros for the Riemann function is true and confirm the topological character of these zeros as countable, following a pure analytic methodological approach.

In his famous book of 1748, Euler proved what has now named the Euler product formula [6]. This product is the result of the infinite sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\{P\}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \text{analytic for } \Re(s) > 1$$

where $\{P\}$ is the infinite set of primes. $s = \sigma + it$ is a complex variable. By definition, the above expression is the Riemann zeta-function, $\zeta(s)$.

Riemann extended Euler's result to continue zeta analytically in "s" variable and he established the Functional Relation:

$$\Lambda(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \Lambda(1-s)$$

Γ being the Gamma function and $\zeta(s)$ our Riemann zeta-function [7]. It is worth mentioning here that the most popular expression for the Functional Equation is:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$$

And it can be expressed in a variety of forms, as developed in [7]. Our choice for the functional equation in the form of $\Lambda(s)$ initially shown is for simplicity in the argumentation towards an analytic function having the same zeros as Riemann zeta-function $\zeta(s)$.

RH is the assertion that all the non-trivial zeros of $\Lambda(s)$ are on the line in the complex plane $\Re(s) = \frac{1}{2}$. For example, P. Sarnak [8] has claimed, "…elegant, crispy, falsifiable and far-reaching, this conjecture is the epitome of what a good conjecture should be…". Moreover, its generalizations to other zeta functions, TO number theory and mathematical physics applications have many striking consequences, making the conjecture even more relevant.

In 1914, Hardy demonstrated the existence of infinite zeros in $\Re(s) = \frac{1}{2}$ [5], and later in 1989, it was demonstrated that more than 2/5 of those zeros lied in $\Re(s) = \frac{1}{2}$ [9]. Nevertheless, the question about the absolute non-existence of zeros out of $\Re(s) = \frac{1}{2}$ is still remaining.

We will show through an analytical approximation that Riemann conjecture is true, while confirming simultaneously that the zeros are infinite numerable. For this, the following structure will be developed below:

- An analytical equivalent function $\xi(z)$ without $z = 1$ as the pole in the critical region $0 < \Re(s) < 1$ (existence and uniqueness of an analytical solution in $0 < \Re(s) < 1$).
- The non-existence of zeros of $\zeta(z)$ out of $\Re(z) = 1/2$. In two steps:
    - o First, we will show in a germ starting in a generic zero, be $[\zeta]_{z_0}$, in the critical line ($\Re(z) = 1/2$) that only at most one additional zero can be found in the boundary of any analytic extension of the germ.

- Then, three generic misaligned zeros in a germ extension of the primitive one defined above $[\zeta]_{z_0}$ cannot be found.

2. An Analytical Equivalent Function $\xi(z)$ without $z = 1$ as Pole in the Critical Region $0 < \Re(s) < 1$

Firstly, we assume here all the consideration will lie on the simply connected open set $\Omega \subset \mathbb{C}$ bounded by the lines $\Re(z) = 0$ and $\Re(z) = 1$:

$$\Omega = \{z \in \mathbb{C} : 0 < \Re(z) < 1\} \quad \text{(Figure 1)}$$

Let us remove the $z = 1$ pole in the above functional relation, $\Lambda(z)$, while maintaining its symmetry under "$z \to 1 - z$" by defining a new function:

$$\xi(z) := z(z-1)\Lambda(z) = z(z-1)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \xi(1-z)$$

This function is analytic in $\bar{\Omega}$. In fact, $z = 0$ and $z = 1$, strictly speaking, become avoidable discontinuities. Specifically, $\xi(z)$ is analytic in $\partial\Omega$.

Thus, we have the following analytical function in $\Omega \subset \mathbb{C}$:

$$\xi(z) := \Omega \subset \mathbb{C} \to \mathbb{C}$$
$$x + iy \to u(x,y) + iv(x,y)$$

Thus, each component $u(x,y), v(x,y)$ is a real function at least $C^2(\bar{\Omega})$. Cauchy-Riemann conditions being satisfied: $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. Consequently, both functions are harmonic in an open domain whose boundary has a smooth function "$\xi(z)|_{\partial\Omega}$" defined; thus we have identified for each component of $\xi(z)$ a Laplacian linear differential operator evaluated in Ω with prescribed values on $\partial\Omega$: *A* Dirichlet problem for each $u(x,y), v(x,y)$ component. In the following, we detail the evidence of existence and uniqueness for only one component, identical for the other one. For example, $u(x,y)$. Prior to this, we need to claim the Riemann's theorem for conformal transformation, where every open simple connected subset can be conformally mapped onto $B(0,1)$, thus onto an open bounded set [10]. For the frontier, applying monodromy theorem for analytic extensions in germs for open connected subsets, we can assure at least continuity in our $u(x,y), v(x,y)$. Let this open bounded connected set be Ω^* and *f* the analytical conformal mapping on it from our initial Ω. Immediately follows $u(x,y), v(x,y) \in C^2(\bar{\Omega}^*)$ using Riemann's theorem for conformal mapping and monodromy theorem to extend germ in the connected open set towards the border.

Now, we remind here both the Gauss divergence and Stoke's theorem for a $\omega \in \Lambda^k(\mathbb{R}^d)$ (*k* form) defined in a generic Ω^* *d*-manifold[1] open bounded set, where the first one can be seen as a variant of Stoke's [11] [12]:

$$\int_{\partial\Omega^*} \omega = \int_{\Omega^*} \mathrm{d}\omega \quad \text{(Stoke's Theorem);}$$

$$\int_{\partial\Omega^*} u(x)\frac{\mathrm{d}x_k}{\mathrm{d}n}\mathrm{d}S_x = \int_{\Omega^*} D_k u(x)\mathrm{d}x \quad \text{(Gauss variant divergence theorem);}$$

[1]In our case, d = 2 as seen in Figure 1.

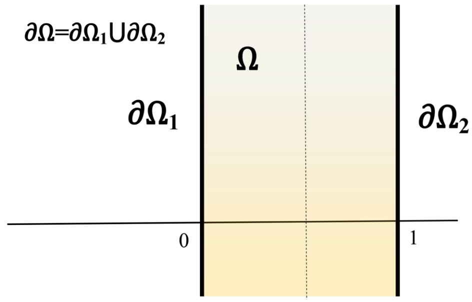

Figure 1. Region of analicity for $\xi(z)$. Ω is open and $\bar{\Omega}$ is convex, simple connected. Dotted line represents the "critical line", $\Re(z)=1/2$.

where $\frac{\mathrm{d}}{\mathrm{d}n}$ denotes differentiation in the direction of the exterior unit normal $\bar{n}=(n_1,\cdots,n_d)$ of $\partial\Omega^*$, $\mathrm{d}x=\mathrm{d}x_1\cdots\mathrm{d}x_d$ and $\mathrm{d}S_x$ = surface element over integration. Keeping in mind that $\Delta u=D_k^2u$ is easy to arrive to the specific Green's identity for Laplacian linear differential operator evaluated in Ω^*, thus assuring existence:

$$\int_{\Omega^*}\Delta u\mathrm{d}x=\int_{\partial\Omega^*}\frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}S\to 0=\int_{\partial\Omega^*}\frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}S$$

Green's identity for the Dirichlet problem finding *u* in Ω^* from prescribed values of $\Delta u=0$ in Ω^* and from *u* on $\partial\Omega^*$.

As always discussing uniqueness of linear problems, it is straightforward to verify that if two difference solutions $u_1(x,y),u_2(x,y)\in C^2(\bar{\Omega}^*)$ form its difference, this one is equal to 0.

Summarizing, Green's identity for the Dirichlet problem for the harmonic components $u(x,y),v(x,y)$ of $\zeta(z)$ assures existence and uniqueness of this function in Ω set, thus for $\zeta(z)$. This is of capital importance for the Taylor expansion series discussions described below.

3. The Non Existence of Zeros of $\zeta(z)$ Out of $\Re(z)=1/2$

3.1. About the Existence of at Most One Zero in the Boundary of an Analytical Extension of a Germ from the Vicinity of a General Zero on $\Re(z)=1/2$, $[\zeta]_{z_0}$

First, let's remind $\zeta(z)$ is analytic in $\Omega\subset\mathbb{C}$. So, the zeros are isolated points due to identity principle (non accumulated points in the subset "zeros of *ζ*").

Let $z_0=\left(\frac{1}{2},y_0\right)$ one generic zero for $\zeta(z)$ lying in the "critical line" $\Re(z)=\frac{1}{2}$. We can consider Taylor expansion in z_0 valid for radius until the minimum distance, say *ε*, to another general zero, say z_1 (**Figure 2**). This distance is well defined as isolated points as reminded in the above paragraph. Be

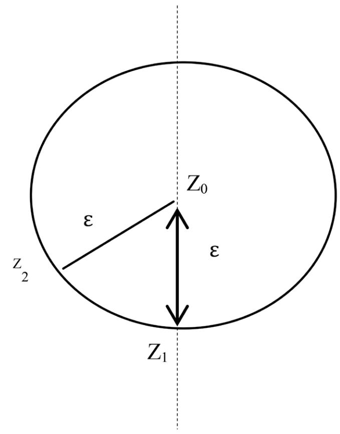

Figure 2. Schema of vicinity in z_0, z_1 as a zeros of $\zeta(z)$, including the hypothesis of an additional zero in the boundary of the disk, z_2 (**Figure 1**).

$\overline{B(z_0,\varepsilon)}$ the disk for the Taylor series around z_0; as $\zeta(z)$ is analytical and univocally defined, for the Dirichelt problem previously discussed, we will show that only one zero can be found in this disk, defined by the minimum distance above claimed.

Assuming z_0 a k-zero (zero of order "K"), the Taylor expansion is:

$$\zeta(z)=(z-z_0)^k P(z-z_0)=(z-z_0)^k\left[a_k+\sum\nolimits_{n>0}a_{n+k}(z-z_0)^n\right]$$

At z_1, only one possible solution of $\zeta(z)$ is admitted, and is zero. Considering the above Taylor expression:

$$0=(z_1-z_0)^k P(z_1-z_0)=(z_1-z_0)^k\left[a_k+\sum\nolimits_{n>0}a_{n+k}(z_1-z_0)^n\right]$$

Directly ones obtains, since $z_1\neq z_0$:

$$a_k=-\sum\nolimits_{n>0}a_{n+k}(z_1-z_0)^n \tag{1}$$

One could coherently points out, according to properties of modulus "triangular inequality":

$$\left||a_k|-\left|\sum\nolimits_{n>0}a_{n+k}(z_1-z_0)^n\right|\right|\leq 0\Rightarrow |a_k|=\left|\sum\nolimits_{n>0}a_{n+k}(z_1-z_0)^n\right| \tag{2}$$

This inequality is always satisfied if we multiply the left side by $\mathrm{e}^{i\theta},\theta\in\mathbb{R}$:

$$|a_k|=\left|\mathrm{e}^{i\theta}\right|\left|\sum\nolimits_{n>0}a_{n+k}(z_1-z_0)^n\right|=\left|\sum\nolimits_{n>0}a_{n+k}\mathrm{e}^{i\theta}(z_1-z_0)^n\right|,\ \theta\in\mathbb{R} \tag{3}$$

The above would define uncertainty for the definition of Taylor's coefficient $a_k=\dfrac{1}{k!}\dfrac{\mathrm{d}^k\zeta(z_0)}{\mathrm{d}z^k}$: is a constant function on all $\theta\in\mathbb{R}$, incompatible with uniqueness of Dirichelt problem for the harmonic components of $\zeta(z)$. Thus, uniqueness for the value of $\theta\in\mathbb{R}$ is claimed, thus also for z_1.

It is straightforward to conclude that further additional z_k zeros of $\zeta(z)$ in the direction defined by (z_1-z_0), the same argumentation is valid, so the affirmation remains: maximum one zero can be found in the border of disks centered at z_0.

3.2. Lemma: Impossibility to Find Three Misaligned Zeros in a Function Element $\{\zeta, A\}$ Belonging to $[\zeta]_{z_0}$ in $0<\Re(z)<1$

From $[\zeta]_{z_0}$, germ from $\{\zeta, B(z_0,\varepsilon)\}$, the corresponding analytic function in open path connected A ($B(z_0,\varepsilon)\subset A$) is immediately identified.

Being analytical $\zeta(z)$ in z_1, z_2 (and without loss of generality we can assume $z_0 = 0$)[2], it can be expressed in Weirestrass-Hadamard factorial representation in A-set:

$$\zeta(z)=z^m e^{g(z)}\prod_{\{p_n\}}E_{p_n}\left(\frac{z}{z_n}\right)$$

Since $\zeta(z_1)=0 \;\Rightarrow\; E_{p_1}\left(\frac{z}{z_1}\right)=\left(1-\frac{z}{z_1}\right)\exp\left(\frac{z}{z_1}+\frac{z^2}{2z_1^2}+\cdots+\frac{z^{p_1}}{z_1^{p_1}p_1}\right) \quad (4)$

Also, $\zeta(z_2)=0 \;\Rightarrow\; E_{p_2}\left(\frac{z}{z_2}\right)=\left(1-\frac{z}{z_2}\right)\exp\left(\frac{z}{z_2}+\frac{z^2}{2z_2^2}+\cdots+\frac{z^{p_2}}{z_2^{p_2}p_2}\right) \quad (5)$

($p_1, p_2 \neq 0$)[3]

From (5), for instance,

$$0=\left(1-\frac{z_1 z}{z_2 z_1}\right)\exp\left(\frac{z_1 z}{z_2 z_1}+\frac{z_1^2 z^2}{2z_2^2 z_1^2}+\cdots+\frac{z_1^{p_2} z^{p_2}}{p_2 z_2^{p_2} z_1^{p_2}}\right)\exp\left(\frac{z}{z_1}+\frac{z^2}{2z_1^2}+\cdots+\frac{z^{p_1}}{z_1^{p_1}p_1}\right)\exp-\left(\frac{z}{z_1}+\frac{z^2}{2z_1^2}+\cdots+\frac{z^{p_1}}{z_1^{p_1}p_1}\right) \quad (5^*)$$

$$0=\left(1-\frac{z^*}{z_1}\right)\exp\left(\frac{z^*}{z_1}+\frac{z^{*2}}{2z_1^2}+\cdots+\frac{z^{*p_2}}{z_1^{p_2}p_2}\right)\exp\left(\frac{z^*}{z_1}+\frac{z^{*2}}{2z_1^2}+\cdots+\frac{z^{*p_1}}{z_1^{p_1}p_1}\right)\exp-\left(\frac{z^*}{z_1}+\frac{z^{*2}}{2z_1^2}+\cdots+\frac{z^{*p_1}}{z_1^{p_1}p_1}\right) \quad (6)$$

($z^*=\frac{z_1}{z_2}z := \lambda z$, $\lambda\in\mathbb{C}$, biunivocal function)

Regrouping terms in (6) and taking into account both the exponential is never cancelled and $E_{p_1}\left(\frac{z^*}{z_1}\right)=0$, we conclude that z_1 is also zero of $E_{p_2}\left(\frac{z}{z_2}\right)$ implying $z_2 = z_1$, in contradiction with our hypothesis they were different (**Figure 3**). Finally, also notice aligning zeros parallel to z_0-z_1, thus $\alpha(z_0-z_1)$ - $\alpha\in\mathbb{R}$ - is congruent always taking in mind that zeros are isolated points; thus α are isolated infinite points, thus numerable. Be $\{\alpha_n\}_{n\in\mathbb{N}}$, accordingly to Weierstrass representation theorem.

Finally, it is worth pointing out that if ever the germ includes a function element where the corresponding vicinity includes $z = 1$. Same argumentation applies with the following conversion for $\zeta(z)$:

[2]Justification of this is based on the related *complete analytical function* associated to $[\zeta]_{z_0}$ defining a centered chart of the Riemann surface for the point z_0.

[3]$p_1 = p_2 = 0$ is a 2 degree polynomial with two distinct solutions after the Fundamental Theorem of Algebra.

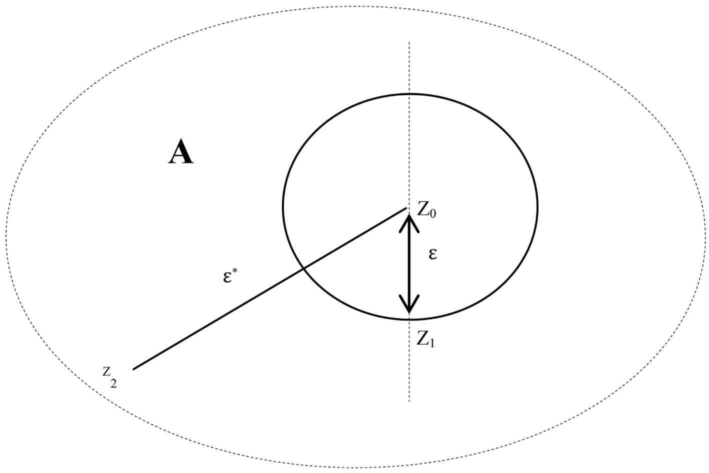

Figure 3. Schema showing the misalignment between three different zeros. In our argumentation, notice that $\varepsilon^* > \varepsilon$, by our definition of ε as minimum distance to a zero from z_0.

$\xi(z) = z(z-1)\zeta(z) = \xi(1-z)$, being analytic in $z = 1$.

The above procedure can be specifically applied and checked to Voro's result concerning the analytical function representation for a germ of $\zeta(z)$ in the vicinity of a non trivial zero in $0 < \Re(z) < 1$ [13]:

$$\zeta(z) = \frac{\exp\left(\log(2\pi) - 1 - \frac{\gamma}{2}\right)z}{2(z-1)\Gamma\left(1+\frac{z}{2}\right)} \prod_{\{z_n\}}\left(1-\frac{z}{z_n}\right)\exp\left(\frac{z}{z_n}\right)$$

$\{z_n\}$ zeros in the referred vicinity, γ the well known Euler constant and Γ gamma function.

Sumarizing, no misaligned zeros can be found in a germ generated by a function element $\{\zeta, B(z_0, \varepsilon)\}$, being z_0 a generic zero lying on $\Re(z) = 1/2$: Riemann conjecture is in consequence demonstrated as true.

4. Conclusion

The Riemann Hypothesis is demonstrated and specifically confirmed that in the critical strip $\Re(z) = 1/2$ the infinite zeros are numerable, according to Hardy's demonstration [5]. As a direct consequence, all remaining conjectures depending on the validity of Riemann's conjecture are also validated.

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During the typesetting of this article, the invasion of Ukraine has occurred, contravening law and all international civic norms by unilateral action of a foreign country.

This author dedicates this work to the noble, heroic and long-suffering Ukrainian people, especially to their researchers and science people. When bombs and violence speak, knowledge is muted. To the memory of each and every one of Ukrainian citizens, for peace in freedom.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Bombieri, E. (2000) Problems of the Millennium: The Riemann Hypothesis. CLAY.

[2] Mussardo, G. and LeClair, A. (2021) Randomness of Möbius Coefficients and Brownian Motion: Growth of the Mertens Function and the Riemann Hypothesis. *Journal of Statistical Mechanics: Theory and Experiment*, **2021**, Article No. 113106. https://doi.org/10.1088/1742-5468/ac22fb

[3] Durmagambetov, A.A. (2016) The Riemann Hypothesis-Millennium Prize Problem. *Advances in Pure Mathematics*, **6**, 915-920. https://doi.org/10.4236/apm.2016.612069

[4] Chen, C.M. (2021) Geometric Proof of Riemann Conjecture (Continued). *Advances in Pure Mathematics*, **11**, 771-783. https://doi.org/10.4236/apm.2021.119051

[5] Hardy, G.H. (1914) Sur les zéros de la fonction $\zeta(s)$ de Riemann. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, **158**, 1012-1014.

[6] Kyrala, A. (1972) Applied Functions of a Complex Variable. Wiley-InterScience.

[7] Titchmarsh, E.C. (1986) The Theory of the Riemann Zeta-Function. Oxford University Press, New York, 15-16.

[8] Sarnak, P. (2004) Problems of the Millennium: The Riemann Hypothesis. CMI Annual Report, 5-21.

[9] LMFDB Collaboration (2013) The Zeros of the Riemann Zeta-function. Database. http://www.lmfdb.org/zeros/zeta/

[10] Markusevich, A. (1979) Theory of Functions of a Complex Variable. Chelsea Publishing, Co., New York, 27-40.

[11] Guillemin, V. and Haine, P. (2019) Differentials Forms. World Scientific, New Jersey, 142-147. https://doi.org/10.1142/11058

[12] Fritz, J. (1991) Partial Differential Equations. Springer-Verlag, New York, 94-105.

[13] Voros, A. (1987) Spectral Functions, Special Functions and the Selberg Zeta Function. *Communications in Mathematical Physics*, **110**, 439-465. https://doi.org/10.1007/BF01212422