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Some Geometric Properties of the *m*-Möbius Transformations

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Abstract

Möbius transformations, which are one-to-one mappings of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ have remarkable geometric properties susceptible to be visualized by drawing pictures. Not the same thing can be said about *m*-Möbius transformations f_m mapping $\overline{\mathbb{C}}^m$ onto $\overline{\mathbb{C}}$. Even for the simplest entity, the pre-image by f_m of a unique point, there is no way of visualization. Pre-images by f_m of figures from $\mathbb{C}$ are like ghost figures in $\mathbb{C}^m$. This paper is about handling those ghost figures. We succeeded in doing it and proving theorems about them by using their projections onto the coordinate planes. The most important achievement is the proof in that context of a theorem similar to the symmetry principle for Möbius transformations. It is like saying that the images by *m*-Möbius transformations of symmetric ghost points with respect to ghost circles are symmetric points with respect to the image circles. Vectors in $\mathbb{C}^m$ are well known and vector calculus in $\mathbb{C}^m$ is familiar, yet the pre-image by f_m of a vector from $\mathbb{C}$ is a different entity which materializes by projections into vectors in the coordinate planes. In this paper, we study the interface between those entities and the vectors in $\mathbb{C}^m$. Finally, we have shown that the uniqueness theorem for Möbius transformations and the property of preserving the cross-ratio of four points by those transformations translate into similar theorems for *m*-Möbius transformations.

Keywords

Möbius Transformation, Conformal Mapping, Symmetry with Respect to a Circle, Symmetry Principle

1. Introduction

The bi-Möbius transformations are functions $f:\overline{\mathbb{C}}\times\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ of the form:

$$
\begin{aligned}
f_2(z_1,z_2) &= \frac{Az_1z_2 + a(1-z_1-z_2)}{a(z_1z_2 - z_1 - z_2) + A} \\
&= \frac{(Az_2 - a)z_1 + a(1-z_2)}{a(z_2-1)z_1 + (A - z_2)} \\
&= \frac{(Az_1 - a)z_2 + a(1-z_1)}{a(z_1-1)z_2 + (A - z_1)}
\end{aligned} \tag{1}
$$

where $a \in \mathbb{C}\setminus\{0,1\}$ and $A = a^2 - a + 1$. By denoting $\omega = a + 1/a - 1$ and $s_1 = z_1 + z_2$, $s_2 = z_1z_2$, we have:

$$
f_2(z_1,z_2) = \frac{\omega s_2 - s_1 + 1}{s_2 - s_1 + \omega} \tag{2}
$$

Let us notice that if $z_2 \in \overline{\mathbb{C}}\setminus\{a,1/a\}$, then $(Az_2 - a)(A - az_2) + a^2(z_2-1)^2 \neq 0$, thus $f_2(z_1,z_2)$ is a Möbius transformation in z_1 and if $z_1 \in \overline{\mathbb{C}}\setminus\{a,1/a\}$, then $(Az_1 - a)(A - az_1) + a^2(z_1-1)^2 \neq 0$, thus $f_2(z_1,z_2)$ is a Möbius transformation in z_2, which justifies the name of *bi-Möbius transformation* we have given to $w = f_2(z_1,z_2)$.

It is expected that the fixed points of f_2 as a Möbius transformation in z_1 depend on z_2. In reality, solving for z_1 the equation $f_2(z_1,z_2) = z_1$ we obtain $(z_2 - 1)\left[z_1^2 - (\omega+1)z_1 + 1\right] = 0$. Since $f_2(z_1,1) = z_1$, all the points z_1 are fixed points of the Möbius transformation $f_2(z_1,1)$. On the other hand, the equation $z_1^2 - (\omega+1)z_1 + 1 = 0$ does not depend on z_2, therefore its solutions which are the fixed points ξ_1 and ξ_2 do not depend on z_2 and are such that $\xi_1 + \xi_2 = \omega + 1$ and $\xi_1\xi_2 = 1$. Similarly, the fixed points of f_2 as a Möbius transformation in z_2 do not depend on z_1. It can be easily checked that in fact they are the same, in other words, $f_2(\xi_k, z_2) = f_2(z_1, \xi_k) = \xi_k, k = 1,2$ for every z_1 and z_2 in $\overline{\mathbb{C}}$. In particular $f_2(\xi_1,\xi_2) = \xi_1 = \xi_2 = \xi$, hence $2\xi = \omega + 1$ and $\xi^2 = 1$. When $\xi = 1$ we have $\omega = 1$, hence $a = 1$, which has been excluded. When $\xi = -1$, we have $\omega = -3$, thus f_2 defined by $\omega = -3$ has the double fixed point -1.

An easy computation shows that:

$$
f_3(z_1,z_2,z_3) = f_2(z_1, f_2(z_2,z_3)) = f_2(f_2(z_1,z_2), z_3) = \frac{(1+\omega)s_3 - s_2 + 1}{s_3 - s_1 + (1+\omega)} \tag{3}
$$

where $s_3 = z_1z_2z_3$, $s_2 = z_1z_2 + z_1z_3 + z_2z_3$ and $s_1 = z_1 + z_2 + z_3$. Moreover, $f_3(z_1,z_2,z_3)$ is a Möbius transformation in each one of the variables, if the other variables do not take the values a or $1/a$. Again, solving for z_1 the equation $f_3(z_1,z_2,z_3) = z_1$ we obtain $(z_2z_3 - 1)\left[z_1^2 - (1+\omega)z_1 + 1\right] = 0$, which indicates that $f_3(z_1,z_2,z_3)$ has the same fixed points ξ_1 and ξ_2 as the Möbius transformation in every variable when the other two z_j and z_k are such that $z_jz_k \neq 1$.

By using formula (3) repeatedly, we have computed in [1] several functions f_m. We provide here a list of them, which will be needed in Section 3 when dealing with the uniqueness of m-Möbius transformations. There is no harm to write $f_m(\mathbf{z})$ instead of $f_m(z_1,z_2,\cdots,z_m)$, where $\mathbf{z} = (z_1,z_2,\cdots,z_m)$.

$$f_4(\mathbf{z})=\frac{(\omega^2+\omega-1)s_4-\omega s_3+s_2-s_1+\omega}{\omega s_4-s_3+s_2-\omega s_1+(\omega^2+\omega-1)}$$

$$f_5(\mathbf{z})=\frac{\omega(\omega+2)s_5-(\omega+1)s_4+s_3-s_1+(\omega+1)}{(\omega+1)s_5-s_4+s_2-(\omega+1)s_1+\omega(\omega+2)}$$

$$f_6(\mathbf{z})=\frac{(\omega^3+2\omega^2-\omega-1)s_6-(\omega^2+\omega-1)s_5+\omega s_4-s_3+s_2-\omega s_1+(\omega^2+\omega-1)}{(\omega^2+\omega-1)s_6-\omega s_5+s_4-s_3+\omega s_2-(\omega^2+\omega-1)s_1+(\omega^3+2\omega^2-\omega-1)}$$

$$f_7(\mathbf{z})=\frac{(\omega^3+2\omega^2+\omega-1)s_7-\omega(\omega+2)s_6+(\omega+1)s_5-s_4+s_2-(\omega+1)s_1+\omega(\omega+2)}{\omega(\omega+2)s_7-(\omega+1)s_6+s_5-s_3+(\omega+1)s_2-\omega(\omega+2)s_1+(\omega^3+2\omega^2-\omega-1)}$$

$$f_8(\mathbf{z})=\frac{\omega(\omega^3+3\omega^2-3)s_8-(\omega^3+2\omega^2-\omega-1)s_7+(\omega^2+\omega-1)s_6-\omega s_5+s_4-s_3+\omega s_2-(\omega^2+\omega-1)s_1+(\omega^3+2\omega^2-\omega-1)}{(\omega^3+2\omega^2-\omega-1)s_8-(\omega^2+\omega-1)s_7+\omega s_6-s_5+s_4-\omega s_3+(\omega^2+\omega-1)s_2-(\omega^3+2\omega^2-\omega-1)s_1+\omega(\omega^3+3\omega^2-3)}$$

$$f_9(\mathbf{z})=\frac{(\omega^4+4\omega^3+3\omega^2-2\omega-1)s_9-(\omega^3+3\omega^2+\omega-1)s_8-\omega(\omega+2)s_7-(\omega+1)s_6+s_5-s_3+(\omega+1)s_2-\omega(\omega+2)s_1+(\omega^3+3\omega^2+\omega-1)}{(\omega^3+3\omega^2-\omega-1)s_9-\omega(\omega+2)s_8+(\omega+1)s_7-s_6+s_4-(\omega+1)s_3+\omega(\omega+2)s_2-(\omega^3+2\omega^2-\omega-1)s_1+(\omega^4+4\omega^3+3\omega^2-2\omega-1)}$$

When we checked for the fixed points of $f_4(\mathbf{z})$ as a Möbius transformation in z_4 we got the equation:

$$[\omega(s_3-1)+s_1-s_2][z_4^2-(\omega+1)z_4+1]=0$$

where s_k are the symmetric sums in z_1,z_2,z_3, which shows that f_4 has the same fixed points ξ_1 and ξ_2 as f_2 and f_3. Due to the symmetry of f_4, this is true when it is considered as a Möbius transformation in any one of its variables if the other three are such that $\omega(s_3-1)+s_1-s_2\neq 0$. For $f_5(\mathbf{z})$ the equation is:

$$[(\omega+1)(s_4-1)+s_1-s_3][z_5^2-(\omega+1)z_5+1]=0$$

where s_k are the symmetric sums in z_1,z_2,z_3,z_4 and again we find the same fixed points ξ_1 and ξ_2 when f_5 is treated as a Möbius transformation in any one of its variables and $(\omega+1)(s_4-1)+s_1-s_3\neq 0$.

It is expected similar properties to be true for any *m*-Möbius transformation, yet for higher values of *m*, the computation becomes too tedious.

We notice that the coefficients of s_k, which are polynomials in ω become more and more complicated as *k* increases. Yet, an interesting pattern should be noticed, namely that in every f_m the coefficient of s_k at the numerator is the same as the coefficient of s_{m-k} at the denominator. Also, if we compare the coefficients from f_m and from f_{m+2} we find another surprising pattern, which will be studied in detail in Section 3.

More generally, if $f_m(z_1,z_2,\cdots,z_m)=f_2(f_{m-1}(z_1,z_2,\cdots,z_{m-1}),z_m)$, then:

$$f_m(z_1,z_2,\cdots,z_m)=\frac{a_0(\omega)s_m+a_1(\omega)s_{m-1}+\cdots+a_m(\omega)}{a_m(\omega)s_m+a_{m-1}(\omega)s_{m-1}+\cdots+a_0(\omega)} \tag{4}$$

where s_j are symmetric sums of z_k and $a_j(\omega)$ are polynomials in ω. Moreover, $w=f_m(z_1,z_2,\cdots,z_m)$ is a Möbius transformation in every z_k if the other variables belong to $\overline{\mathbb{C}}\setminus\{a,1/a\}$. This is an m-Möbius transformation, see [1]

[2] [3] for more details about these transformations and their applications.

2. Images of Circles

The geometric properties of the m-Möbius transformations concern the way these mappings transform figures from each one of the planes (z_k) into figures situated in the (w)-plane. As Möbius transformations, they are obviously conformal mappings, hence they will preserve the angles of those figures, except at singular points. They will transform circles (including straight lines, which can be considered circles of infinite radius) into circles. However, there are details which need to be examined.

A circle C centered at w_0 and of radius r in the (w)-plane has the equation:

$$|w-w_0|=r \text{, or } |w|^2-2Re(\overline{w}_0 w)+|w_0|^2=r^2.$$

It is convenient to write this equation under the form:

$$\alpha|w|^2+2Re(bw)+\beta=0, \tag{5}$$

where $a,\beta\in\mathbb{R}$ and $b\in\mathbb{C}$. Indeed, we obtain (5) when we replace w_0 by $-\overline{b}/\alpha$ and r^2 by $|b|^2/\alpha^2-\beta/\alpha$ into the equation $|w-w_0|=r$. Then, when $\alpha=0$, this equation becomes that of a straight line. We will continue to call it circle (of infinite radius, or centered at infinity).

Theorem 1. The circle (5) is the image by $w=f_m(z_1,z_2,\cdots,z_m)$, $z_j\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ fixed, for $j\neq k$ of circles:

$$\alpha_k|z_k|^2+2Re(b_k z_k)+\beta_k=0 \tag{6}$$

from the (z_k)-planes, $k=1,2,\cdots,m$. More exactly, for every k, fixing $z_j, j\neq k$ there is a unique circle (6) which is mapped bijectively by $f_m(z_1,z_2,\cdots,z_m)$ onto the circle (5) from the (w)-plane.

Proof: Indeed, the coefficients α_k,b_k and β_k can be uniquely determined as functions of α,b and β by using f_m as follows. With fixed $z_j, j\neq k$, let us denote:

$$p_k=Af_{m-1}(z_1,z_2,\cdots,z_{k-1},z_{k+1},\cdots,z_m)-a,$$
$$q_k=a\left[f_{m-1}(z_1,z_2,\cdots,z_{k-1},z_{k+1},\cdots,z_m)-1\right],$$
$$r_k=A-f_{m-1}(z_1,z_2,\cdots,z_{k-1},z_{k+1},\cdots,z_m)$$

and:

$$w=f_m(z_1,z_2,\cdots,z_m)=\frac{p_k z_k-q_k}{q_k z_k+r_k}.$$

Then:

$$|w|^2=\frac{p_k z_k-q_k}{q_k z_k+r_k}\frac{\overline{p}_k\overline{z}_k-\overline{q}_k}{\overline{q}_k\overline{z}_k+\overline{r}_k}$$

hence, (5) becomes:

$$\alpha(p_k z_k-q_k)(\overline{p}_k\overline{z}_k-\overline{q}_k)/(q_k z_k+r_k)(\overline{q}_k\overline{z}_k+\overline{r}_k)$$
$$+b(p_k z_k-q_k)/(q_k z_k+r_k)+\overline{b}(\overline{p}_k\overline{z}_k-\overline{q}_k)/(\overline{q}_k\overline{z}_k+\overline{r}_k)+\beta=0,$$

or:

$$\begin{aligned}&\alpha\left[|p_k|^2|z_k|^2-2Re\left(p_k\overline{r}_k z_k\right)+|q_k|^2\right]\\&+b\left[p_k\overline{q}_k|z_k|^2+p_k\overline{r}_k z_k-|q_k|^2\overline{z}_k-q_k\overline{r}_k\right]\\&+\overline{b}\left[\overline{p}_k q_k|z_k|^2+\overline{p}_k r_k\overline{z}_k-|q_k|^2 z_k-\overline{q}_k r_k\right]\\&+\beta\left(|q_k|^2|z_k|^2+q_k\overline{r}_k z_k+\overline{q}_k r_k\overline{z}_k+|r_k|^2\right)=0,\end{aligned}$$

which gives:

$$\begin{aligned}\alpha_k&=\alpha|p_k|^2+\beta|q_k|^2+2Re\left(bp_k\overline{q}_k\right)\in\mathbb{R}\\b_k&=-\alpha p_k\overline{r}_k+bp_k\overline{r}_k-\overline{b}|q_k|^2+\beta q_k\overline{r}_k\in\mathbb{C}\\\beta_k&=\alpha|q_k|^2+\beta|r_k|^2-2Re\left(bq_k\overline{r}_k\right)\in\mathbb{R}\end{aligned}\tag{7}$$

Obviously, different values of $z_j, j\neq k$ will determine different circles (6) in the (z_k)-plane which are mapped bijectively by f_m onto the circle (5). The bijective nature is assured by the fact that the Möbius transformations are bijective and f_m is a Möbius transformation of the (z_k)-plane as long as every $z_j\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ remains fixed. Varying some of z_j, the coefficients p_k, q_k and r_k will all change and then α_k,β_k and b_k will be different, thus all circles (6) will change, despite of the fact that the circle (5) remains the same.

The concept of pre-image by f_m can be useful in order to describe this change. The pre-image by f_m of a point $w\in\overline{\mathbb{C}}$ is by definition $\left\{\mathbf{z}\in\overline{\mathbb{C}}^m \mid \mathbf{z}=(z_1,z_2,\cdots,z_m), f_m(z_1,z_2,\cdots,z_m)=w\right\}$. The pre-image by f_m of a figure $\mathfrak{F}\subset\overline{\mathbb{C}}$ is $\left\{\mathbf{z}\in\overline{\mathbb{C}}^m \mid f_m(\mathbf{z})\in\mathfrak{F}\right\}$. Since f_m depends only on the symmetric sums s_k the pre-image by f_m of any figure is invariant with respect to the permutations of the variables, *i.e.*, it is a *symmetric* figure in $\overline{\mathbb{C}}^m$. In particular, the pre-image W of a single point w is a symmetric figure, which means that if $(z_1,z_2,\cdots,z_m)$ belongs to W, then so does any point obtained by a permutation of these coordinates.

Let us deal for simplicity with the case of $f_2(z_1,z_2)$ given by (2). We can chose arbitrarily $z_1\in\overline{\mathbb{C}}\setminus\{a,1/a\}$. Solving for z_2 the equation:

$w=f_2(z_1,z_2)=\dfrac{\omega z_1 z_2-z_1-z_2+1}{z_1 z_2-z_1-z_2+\omega}$ we get:

$$z_2=\frac{(1-w)z_1+\omega w-1}{(\omega-w)z_1+w-1}\tag{8}$$

so, with this value of z_2 we have $f_2(z_1,z_2)=w$, *i.e.*, $(z_1,z_2)\in W$. We notice that z_2, as a function of z_1 is a Möbius transformation and therefore a bijective mapping of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$, which means that the pre-image by f_2 of a single point $w\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ is a subset of $\overline{\mathbb{C}}^2$ in one-to-one correspondence with $\overline{\mathbb{C}}$. Given z_1 in the (z_1)-plane there is a unique z_2 in the (z_2)-plane, namely that given by (8) such that $(z_1,z_2)\in W$. Due to the symmetry of f_2. this happens if and only if $(z_2,z_1)\in W$, which means that W is a symmetric figure in $\overline{\mathbb{C}}^2$.

When $w=0$, then:

$$z_2=\frac{z_1-1}{\omega z_1-1} \tag{8'}$$

hence every point $(z_1,z_2)\in\overline{\mathbb{C}}^2$, where z_2 is given by (8') is carried by f_2 into the origin. Due to the symmetry of f_2, the same is true for every point $(z_1,z_2)\in\overline{\mathbb{C}}^2$, where z_2 is arbitrary and $z_1=(z_2-1)/(\omega z_2-1)$.

Now, suppose that w belongs to the circle C of equation (5). Let us denote by $\mathbf{C}$ the pre-image by f_m of the circle C, *i.e.*, $\mathbf{C}=\{\mathbf{z}\in\mathbb{C}^m \mid f_m(\mathbf{z})\in C\}$. The formula (8) tells us that for every $w\in C$, we can pick up arbitrarily z_1 and if z_2 is given by (8), then $(z_1,z_2)\in\mathbf{C}$. But, for z_1 fixed (8) is a Möbius transformation in w and it maps the circle C onto a circle $C_2(z_1)$ into the (z_2)-plane. We will call it the projection onto the (z_2)-plane of the *section* of $\mathbf{C}$ by $z_1=const$. Analogously, a projection $C_1(z_2)$ onto the (z_1)-plane can be defined of the section of $\mathbf{C}$ by $z_2=const$. The circles $C_1(z_2)$ and $C_2(z_1)$ are mapped bijectively by $f_2(z_1,z_2)$ onto the circle C when we keep z_2, respectively z_1 fixed in $\overline{\mathbb{C}}\setminus\{a,1/a\}$. An easy computation shows that in terms of p_2,q_2 and r_2 the equation (8) is $z_2=(r_2w+q_2)/(-q_2w+p_2)$, where $p_2=Az_1-a$, $q_2=a(z_1-1)$ and $r_2=A-z_1$. When z_1 varies, all these parameters vary and we obtain different formulas (8) and therefore different circles $C_2(z_1)$. Hence, there are infinitely many circles into the planes (z_1) and (z_2) which are mapped bijectively by f_2 onto the circle C, one for every $z_2=const$, respectively $z_1=const$. They are all projections onto the two planes of sections of $\mathbf{C}$ by $z_2=const$, respectively $z_1=const$. However, we can prove:

Theorem 2. There is a unique Möbius transformation of the (z_1)-plane into the (z_2)-plane which maps bijectively every circle $C_1(z_2)$ onto a circle $C_2(z_1)$.

Proof. For a given $z_2\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ to every circle C in the (w)-plane corresponds uniquely a circle $C_1(z_2)$ which is the image of C by the Möbius transformation $z_1(w)=\frac{(\omega-z_2)w+z_2-1}{(1-z_2)w+\omega z_2-1}$ of the (w)-plane into the (z_1)-plane and for every given $z_1\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ a unique circle $C_2(z_1)$ exists, which is the image of C by the Möbius transformation $z_2(w)=\frac{(\omega-z_1)w+z_1-1}{(1-z_1)w+\omega z_1-1}$ of the (w)-plane into the (z_2)-plane. The function $z_2\circ z_1^{-1}$ is a Möbius transformation of the (z_1)-plane into the (z_2)-plane which carries the circle $C_1(z_2)$ into the circle $C_2(z_1)$. The affirmation of the theorem is not trivial since it asserts the uniqueness of such a Möbius transformation, while it is known that there are infinitely many Möbius transformations which map a given circle onto another given circle. They differ by rotations around the center of any one of these circles. The theorem states that the mapping $z_2\circ z_1^{-1}$ is uniquely determined by the circles $C_1(z_2)$ and $C_2(z_1)$. Indeed, the two circles are in turn uniquely determined by the circle C and the two fixed values of z_1 and z_2, which define uniquely the functions $z_1(w)$ and $z_2(w)$.

The general case can be treated similarly. We choose $(z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_m)$ arbitrarily in $\overline{\mathbb{C}}^{m-1}$ and let:

$$z_k(w) = \frac{(1-w) f_{m-1}(z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_m) + \omega w - 1}{(\omega - w) f_{m-1}(z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_m) + w - 1} \tag{9}$$

such that $f_m(z_1, z_2, \cdots, z_m) = w$. Due to the symmetry of f_m, this happens if and only if $f_m(z_1', z_2', \cdots, z_m') = w$ where $(z_1', z_2', \cdots, z_m')$ is an arbitrary permutation of $(z_1, z_2, \cdots, z_m)$. The formula (9) represents a Möbius transformation in w which maps a circle C of equation (5) onto a circle C_k into the (z_k)-plane when all the other variables are kept constant in $\overline{\mathbb{C}} \setminus \{a, 1/a\}$. The theorem 2 in the general form states that for every k and j, $j \neq k$ there is a unique Möbius transformation of the (z_k)-plane into the (z_j)-plane which carries every circle C_k into a circle C_j.

Let us deal now with the symmetry with respect to a circle (see [4], page 80) of Equation (5). As we have seen, that equation represents a proper circle when $\alpha \neq 0$ or a line when $\alpha = 0$. When that line is the real axis, we say that the points w and $\overline{w}$ are symmetric with respect to it. Yet, for any line it is known what symmetric points with respect to that line mean, namely z and z^* are symmetric points with respect to the line L if and only if L is the bisecting normal of the segment determined by z and z^*. This concept can be extended to the case when $\alpha \neq 0$. In that case the Equation (5) is $|w - w_0| = r$, where w_0 is the center of the circle and r is its radius. As shown in [4], page 81 by using the tool of cross ratios, w and w^* are symmetric with respect to this circle if and only if:

$$w^* = r^2 / (\overline{w} - \overline{w}_0) + w_0 . \tag{10}$$

The symmetry principle states that if a Möbius transformation carries a circle C_1 into a circle C_2, then it transforms any pair of symmetric points with respect to C_1 into a pair of symmetric points with respect to C_2. Here circle means proper circle or line. This principle can be extended to m-Möbius transformations in the following way.

Theorem 3 (The Main Theorem). Let w and w^* be *symmetric points* with respect to the circle (5) and let W and W^* be the pre-images by f_m of w and respectively w^*. Then the projection onto the (z_k)-plane of any section of W and W^* obtained by keeping $z_j \in \overline{\mathbb{C}} \setminus \{a, 1/a\}$ fixed, $j \neq k$ are points symmetric with respect to the circle (6) corresponding to that section.

Proof: By the **Theorem 1**, the circle (5) is the image by Möbius transformations of every circle (6) from the (z_k)-plane when $z_j \in \overline{\mathbb{C}} \setminus \{a, 1/a\}$, $j \neq k$ are kept fixed. Then the symmetry principle applied to these circles is exactly this theorem.

This theorem describes in fact a phenomenon happening in $\overline{\mathbb{C}}^m$ related to m-Möbius transformations. If we consider $\mathbf{C}$ as a *generalized circle* in $\overline{\mathbb{C}}^m$ and W as the equivalent of a point from $\overline{\mathbb{C}}$, then it makes sense to say that W^* is the symmetric of W with respect to $\mathbf{C}$, since this is true for the projections on every

(z_k)-plane of any section of them obtained by keeping $z_j \in \overline{\mathbb{C}} \setminus \{a, 1/a\}$ fixed, $j \neq k$. The projection of the respective section of $\mathbf{C}$ is C_k and that of the sections of W and W^* is z_k and z_k^*. The symmetric of z_k with respect to C_k. The theorem states that a m-Möbius transformation carries symmetric points with respect to $\mathbf{C}$ into symmetric points with respect to C.

There is a one-to-one mapping of W onto W^* assigning to every $\mathbf{z} = (z_1, z_2, \cdots, z_m) \in W$ the point $\mathbf{z}^* = (z_1^*, z_2^*, \cdots, z_m^*)$, where z_k^* is the symmetric of z_k with respect to C_k. It can be called reflection in $\mathbf{C}$, extending to $\overline{\mathbb{C}}^m$ a concept pertinent to $\overline{\mathbb{C}}$ (see [4], page 81). The reflection in a circle is an involution and the composition of two reflections with respect to different circles result in a Möbius transformation. More generally, the composition of an odd numbers of reflections with respect to different circles is again a reflection and the composition of an even number of reflections is a Möbius transformation. Moreover, as shown in [4], page 172, every Möbius transformation can be obtained as the composition of two or four reflections in circles.

The symmetry principle states that if a Möbius transformation $w = M(z)$ carries the circle C_1 into the circle C_2, then M composed with reflection in C_1 is reflection in C_2. In other words $[M(z)]^* = M(z^*)$, where $*$ on the left hand side means reflection in C_2 and on the right hand side reflection in C_1.

We notice that (10) does not represent a Möbius transformation since it is anticonformal, while Möbius transformations are conformal mappings. Yet, in terms of mapping properties there are some similarities between the reflections in circles and Möbius transformations. One of them is that of preservation of circles ([5], page 126). Even more can be said: if a line L does not pass through the center w_0 of the circle C, then its reflection in C is a circle C_1 passing through w_0 and vice-versa, if a circle C_1 passes through w_0 then its reflection in C is a line L not passing through w_0. If the proper circle C_1 does not pass through w_0 then its reflection in C is another proper circle not passing through w_0.

Next we will investigate some similar properties of the reflections with respect to $\mathbf{C}$ and m-Möbius transformations of $\mathbf{C}$.

Theorem 4. Let $\mathbf{C}$ be the pre-image by a m-Möbius transformation $w = f_m(\mathbf{z})$ of the circle $C: |w - w_0| = r$. Then f_m composed with reflection in $\mathbf{C}$ is reflection in C. If L is a line not passing through w_0, then $\mathbf{L} \cap \mathbf{w}_0 = \varnothing$, where $\mathbf{L}$ and $\mathbf{w}_0$ are the pre-images of L and w_0. Moreover, if $\mathbf{L}^*$ is the image of $\mathbf{L}$ by reflection in $\mathbf{C}$, then $\mathbf{w}_0 \subset \mathbf{L}^*$. Reciprocally, if C_1 is a proper circle which passes through w_0 then $\mathbf{w}_0 \subset \mathbf{C}_1$ and $\mathbf{C}_1^* \cap \mathbf{w}_0 = \varnothing$.

Proof: By the formula (9) and the symmetry principle we have $z_k(w^*) = [z_k(w)]^*$ for every $k = 1, 2, \cdots, m$. Then:

$$w^* = f_m(z_1(w^*), z_2(w^*), \cdots, z_m(w^*)) = f_m([z_1(w)]^*, [z_2(w)]^*, \cdots, [z_m(w)]^*)$$

and the first term shows reflection in C, while the last one is f_m composed with reflection in $\mathbf{C}$. The second affirmation is true since the pre-image of the intersection of two sets is equal to the intersection of the pre-images of those sets. Next, f_m moves $\mathbf{L}^*$ into the reflection of L in C which passes through w_0, hence the pre-image of w_0 is included in $\mathbf{L}^*$. Finally, if a proper circle C_1 passes through w_0, then C_1^* should pass through ∞, which is the reflection in C of w_0. Thus C_1^* is a line not passing through w_0 and then $\mathbf{C}_1^* \cap \mathbf{w}_0 = \varnothing$.

Theorem 5. Given a circle $C_k : \alpha_k |z_k|^2 + 2Re(b_k z_k) + \beta_k = 0$ in the (z_k)-plane, for every $j \neq k$ there are infinitely many circles C_j in every (z_j)-plane such that C_k and C_j have the same image by f_m when all the other variables are kept fixed.

Proof: Suppose that a circle C of Equation (5) is given in the (w)-plane. We are looking for a circle (6) which is mapped bijectively by f_m onto the circle C when $z_j, j \neq k$ are kept all fixed. With the notations of Theorem 1, we have that if z_k is on C_k and w is on C where $w = f_m(z_1, z_2, \cdots, z_m)$, then $z_k = \frac{r_k w + q_k}{-q_k w + p_k}$, as in Theorem 1, thus $\overline{z}_k = \frac{\overline{r}_k \overline{w} + \overline{q}_k}{-\overline{q}_k \overline{w} + \overline{p}_k}$, and the equation of:

$C_k : \alpha_k z_k \overline{z}_k + 2Re(b_k z_k) + \beta_k$ becomes:

$$\alpha_k \frac{r_k w + q_k}{-q_k w + p_k} \frac{\overline{r}_k \overline{w} + \overline{q}_k}{-\overline{q}_k \overline{w} + \overline{r}_k} + b_k \frac{r_k w + q_k}{-q_k w + r_k} + \overline{b}_k \frac{\overline{r}_k \overline{w} + \overline{q}_k}{-\overline{q}_k \overline{w} + \overline{r}_k} + \beta_k = 0, \text{ or}$$

$$\begin{aligned} &\alpha_k (r_k w + q_k)(\overline{r}_k \overline{w} + \overline{q}_k) + b_k (r_k w + q_k)(-\overline{q}_k \overline{w} + \overline{r}_k) \\ &+ \overline{b}_k (\overline{r}_k \overline{w} + \overline{q}_k)(-q_k w + r_k) + \beta_k (q_k w - r_k)(\overline{q}_k \overline{w} - \overline{r}_k), \end{aligned}$$

which is:

$$C : \alpha w \overline{w} + 2Re(bw) + \beta,$$

where:

$$\begin{aligned} \alpha &= \alpha_k |r_k|^2 + \beta_k |q_k|^2 - 2Re(b_k r_k \overline{q}_k) \\ \beta &= \alpha_k |q_k|^2 + \beta_k |r_k|^2 + 2Re(b_k q_k \overline{r}_k) \\ b &= \alpha_k \overline{q}_k r_k - \beta_k q_k \overline{r}_k - \overline{b}_k |q_k|^2 + b_k |r_k|^2 \end{aligned} \tag{11}$$

Hence the circle C_k is mapped bijectively by f_m onto the circle C when p_k, q_k and r_k are kept fixed. On the other hand, the projection onto the (z_j)-plane of any section of the pre-image $\mathbf{C}$ of C by $z_l = const, l \neq j$ is a circle C_j which is mapped by f_m bijectively onto C when all the variables except z_j are kept fixed in $\overline{\mathbb{C}} \setminus \{a, 1/a\}$, therefore there are infinitely many circles C_j in every (z_j)-plane such that C_k and C_j have the same image by f_m.

The theory of Apollonius circles and of Steiner nets ([4], page 85) can be extended word by word to m-Möbius transformations. Let us deal first with the case $m = 2$. With the notation $k(z_2) = (Az_2 - a)/a(z_2 - 1)$, $\varphi(z_2) = a(z_2 - 1)/(Az_2 - a)$ and $\psi(z_2) = (z_2 - A)/a(z_2 - 1)$ the formula (1) becomes $w = k(z_2)[z_1 - \varphi(z_2)]/[z_1 - \psi(z_2)]$, which shows that $z_1 = \varphi(z_2)$ cor-

responds to $w=0$ and $z_1=\psi(z_2)$ corresponds to $w=\infty$, thus straight lines through the origin of the (w)-plane are images by $f_2(z_1,z_2)$ of circles C_l through the limit points $\varphi(z_2)$ and $\psi(z_2)$ from the (z_1)-plane for every $z_2\in\overline{\mathbb{C}}$. On the other hand, the concentric circles about the origin $|w|=\rho$, are the images by $w=f_2(z_1,z_2)$ of circles with equation
$\left|\left[z_1-\varphi(z_2)\right]/\left[z_1-\psi(z_2)\right]\right|=\rho/\left|k(z_2)\right|$ for every $z_2\in\overline{\mathbb{C}}$. These are the *Apollonius circles* C_a with the limit poins $\varphi(z_2)$ and $\psi(z_2)$. Together with the C_a circles they form the Steiner net. Every C_l circle is orthogonal to all C_a circles and every C_a circle is orthogonal to all C_l circles.There is exactly one C_a circle and one C_l circle from the Seiner net defined by z_2 passing through every point of the (z_1)-plane except the points $\varphi(z_2)$ and $\psi(z_2)$. Reflection in a C_a circle switch $\varphi(z_2)$ and $\psi(z_2)$ and transforms every C_l circle into itself and every C_a circle into another C_a circle.

The pre-image by f_2 of any Steiner net from the (w)-plane is an object in $\overline{\mathbb{C}}^2$ whose sections by $z_2=const$ and $z_1=const$ are Steiner nets in the (z_1)-plane, respectively the (z_2)-plane. Indeed, this is true due to the formula (8) and the preservation of circles by Möbius transformations. The Theorem 2 implies the following:

Corollary 1. There is a unique Möbius transformation of the (z_1)-plane into the (z_2)-plane which carries the Steiner net determined by $z_2=const$ into the Steiner net determined by $z_1=const$.

Proof: Indeed, by the Theorem 2 there is a unique Möbius transformation $z_1=M(z_2)$ which carries an Apollonius circle $C_a(z_2)$ into another Apollonius circle $C_a(z_1)$. Then, by the symmetry principle every $C_l(z_2)$ circle is transformed into a $C_l(z_1)$ circle. Yet the family of these circles determines uniquely the family of the $C_a(z_1)$ circles, which are all the orthogonal circles to them. Finally, the whole Steiner net from the (z_2)-plane is mapped by $z_1=M(z_2)$ into the Steiner net determined by $z_2=const$.

The pre-image by f_m of a Steiner net from the (w)-plane is an object in $\overline{\mathbb{C}}^m$ whose sections obtained by keeping all z_j fixed for $j\neq k$ is a Steiner net in the (z_k)-plane, due to the formula (9) and the preservation of circles by Möbius transformations. Every point of $\overline{\mathbb{C}}^m$, except the pre-image of the limit points of the net, belongs to the pre-image of both families of circles belonging to the Steiner net from the (w)-plane.

Given a Steiner net in the (w)-plane, its pre-image by $f_m(\mathbf{z})$ into $\overline{\mathbb{C}}^m$ projects into Steiner nets in every (z_k)-plane, the image of which by f_m is the original Steiner net from the (w)-plane. By Theorem 2, there is a unique Möbius transformation of any (z_j)-plane into the (z_k)-plane carying one such Steiner net into the other.

The question arises: what is the pre-image by f_m of a triangle, rectangle, or in general of an arbitrary polygon? We cannot describe these pre-images, yet we can imagine their sections when keeping constant all the variables except one. Since such a section of the pre-image of a line segment is an arc of a circle or a half line, when one of the ends is sent to infinity, the answer to that question is the following. The projection onto the (z_k)-plane of the sections as previously defined of the pre-image by f_m of a triangle is a curvilinear triangle having the same angles as the original one. Some of these triangles can be infinite, in the sense that one side is an arc of a circle and the other two are half-lines. An analogous situation appears for the pre-image of an arbitrary polygon.

3. Uniqueness of *m*-Möbius Transformations

It is known that there is a unique Möbius transformation in the plane moving three distinct points into other three distinct points. In what follows, we will study similar properties of *m*-Möbius transformations.

Theorem 6. There is a unique *m*-Möbius transformation f_m moving a given point $\mathbf{z} \in \overline{\mathbb{C}}^m, m = 2,3$ into a given point $w \in \overline{\mathbb{C}}$.

Proof: Indeed, such transformations have the form see ([1]) $f_2(z_1, z_2) = \dfrac{\omega s_2 - s_1 + 1}{s_2 - s_1 + \omega}$, respectively $f_3(z_1, z_2, z_3) = \dfrac{(1+\omega)s_3 - s_2 + 1}{s_3 - s_1 + (1+\omega)}$. Then, in the first case, $w = f_2(\mathbf{z})$ implies $\omega s_2 - s_1 + 1 = w(s_2 - s_1 + \omega)$, hence ω is uniquely determined by $\omega = \dfrac{w(s_2 - s_1) + s_1 - 1}{s_2 - w}$. Analogously, in the second case, $w = f_3(\mathbf{z})$ implies $(1+\omega)s_3 - s_2 + 1 = w(s_3 - s_1 + 1 + \omega)$, thus ω is uniquely determined by $\omega = \dfrac{(s_3 + 1)(w - 1) + s_2 - ws_1 - 1}{s_3 - w}$.

This theorem is a particular case of the following:

Theorem 7. For $m = 2k$ and $m = 2k + 1$, $k = 1, 2, 3, \cdots$ the equation $f_m(\mathbf{z}) = w$ has the degree *k* in ω and therefore, with the exception of multiple roots, it has *k* solutions, which means that with those exceptions there are *k* different *m*-Möbius transformations $f_m(\mathbf{z})$ moving a given point $\mathbf{z} \in \overline{\mathbb{C}}^m$ into a given point $w \in \overline{\mathbb{C}}$.

Proof: We need a pretty elaborate induction argument. It can be made more obvious if we write the *m*-Möbius transformations as matrices $\begin{bmatrix} r_{m,1} \\ r_{m,2} \end{bmatrix}$ whose entries are the polynomials in ω appearing at the numerator and at the denominator of $f_m(\mathbf{z})$ and arrange the coefficients of these polynomials also as matrices. The examples of f_m which follow can be found in [1].

For $f_2(\mathbf{z})$ we have the matrix expression:

$$\begin{bmatrix} r_{2,1} \\ r_{2,2} \end{bmatrix} = \begin{bmatrix} \omega & -1 & 1 \\ 1 & -1 & \omega \end{bmatrix}$$

For $f_3(\mathbf{z})$ the matrix is:

$$\begin{bmatrix} r_{3,1} \\ r_{3,2} \end{bmatrix} = \begin{bmatrix} 1+\omega & -1 & 0 & 1 \\ 1 & 0 & -1 & 1+\omega \end{bmatrix}$$

We notice that starting with $m = 4$ every matrix $\begin{bmatrix} r_{m,1} \\ r_{m,2} \end{bmatrix}$ is built around the matrix $\begin{bmatrix} -r_{m-2,1} \\ -r_{m-2,2} \end{bmatrix}$ by adding a first and a last column as follows:

$$\begin{bmatrix} r_{4,1} \\ r_{4,2} \end{bmatrix} = \begin{bmatrix} \omega^2+\omega-1 & -r_{2,1} & \omega \\ \omega & -r_{2,2} & \omega^2+\omega-1 \end{bmatrix}$$

$$\begin{bmatrix} r_{5,1} \\ r_{5,2} \end{bmatrix} = \begin{bmatrix} \omega^2+2\omega & -r_{3,1} & \omega+1 \\ \omega+1 & -r_{3,2} & \omega^2+2\omega \end{bmatrix}$$

$$\begin{bmatrix} r_{6,1} \\ r_{6,2} \end{bmatrix} = \begin{bmatrix} \omega^3+2\omega^2-\omega-1 & -r_{4,1} & \omega^2+\omega-1 \\ \omega^2+\omega-1 & -r_{4,2} & \omega^3+2\omega^2-\omega-1 \end{bmatrix}$$

$$\begin{bmatrix} r_{7,1} \\ r_{7,2} \end{bmatrix} = \begin{bmatrix} \omega^3+3\omega^2+\omega-1 & -r_{5,1} & \omega^2+2\omega \\ \omega^2+2\omega & -r_{5,2} & \omega^3+3\omega^2+\omega-1 \end{bmatrix}$$

$$\begin{bmatrix} r_{8,1} \\ r_{8,2} \end{bmatrix} = \begin{bmatrix} \omega^4+3\omega^3-3\omega & -r_{6,1} & \omega^3+2\omega^2-\omega-1 \\ \omega^3+2\omega^2-\omega-1 & -r_{6,2} & \omega^4+3\omega^3-3\omega \end{bmatrix}$$

$$\begin{bmatrix} r_{9,1} \\ r_{9,2} \end{bmatrix} = \begin{bmatrix} \omega^4+4\omega^3+3\omega^2-2\omega-1 & -r_{7,1} & \omega^3+3\omega^2+\omega-1 \\ \omega^3+3\omega^2+\omega-1 & -r_{7,2} & \omega^4+4\omega^3+3\omega^2-2\omega-1 \end{bmatrix}$$

Let us notice that a simplification with $\omega-1$ occurs in every $f_{2k+1}(\mathbf{z})$ such that $f_{2k}(\mathbf{z})$ and $f_{2k+1}(\mathbf{z})$ have the same degree as rational functions of ω. We need to prove this affirmation thoroughly by induction. It is clear that the induction hypothesis should be:

$$\begin{bmatrix} r_{2k,1} \\ r_{2k,2} \end{bmatrix} = \begin{bmatrix} p_k(\omega) & -r_{2k-2,1} & p_{k-1}(\omega) \\ p_{k-1}(\omega) & -r_{2k-2,2} & p_k(\omega) \end{bmatrix}$$

$$\begin{bmatrix} r_{2k+1,1} \\ r_{2k+1,2} \end{bmatrix} = \begin{bmatrix} q_k(\omega) & -r_{2k-1,1} & q_{k-1}(\omega) \\ q_{k-1}(\omega) & -r_{2k-1,2} & q_k(\omega) \end{bmatrix}$$

where $p_k(\omega)$ and $q_k(\omega)$ are polynomials of degree k and $p_{k-1}(\omega)$ and $q_{k-1}(\omega)$ are polynomials of degree $k-1$. Moreover, it can be easily checked that for every $k=1,2,3,4$ we have $q_k(\omega)-q_{k-1}(\omega)=p_k(\omega)$, $p_k(\omega)-p_{k-1}(\omega)=(\omega-1)q_k(\omega)$, $\omega p_k(\omega)-p_{k-1}(\omega)=(\omega-1)q_k(\omega)$ and $\omega q_k(\omega)-q_{k-1}(\omega)=p_{k+1}(\omega)$ and these equalities should be a part of the induction hypothesis.

The proof consists in showing that the formula:

$$f_{m+1}(z_1,z_2,\cdots,z_{m+1}) = \frac{(\omega z_{m+1}-1) f_m(z_1,z_2,\cdots,z_m)+1-z_{m+1}}{(z_{m+1}-1) f_m(z_1,z_2,\cdots,z_m)+\omega-z_{m+1}}$$

produces matrices obtained from the last two by replacing k with $k+1$, *i.e.*:

$$\begin{bmatrix} r_{2k+2,1} \\ r_{2k+2,2} \end{bmatrix} = \begin{bmatrix} p_{k+1}(\omega) & -r_{2k,1} & p_k(\omega) \\ p_k(\omega) & -r_{2k,2} & p_{k+1}(\omega) \end{bmatrix}$$

$$\begin{bmatrix} r_{2k+3,1} \\ r_{2k+3,2} \end{bmatrix} = \begin{bmatrix} p_{k+2}(\omega) & -r_{2k+1,1} & p_{k+1}(\omega) \\ p_{k+1}(\omega) & -r_{2k+1,2} & p_{k+2}(\omega) \end{bmatrix}$$

We have:

$$f_{2k+2}(z_1,z_2,\cdots,z_{2k+2})=\frac{(\omega z_{2k+2}-1)\dfrac{q_k(\omega)s_{2k+1}+\cdots+q_{k-1}(\omega)}{q_{k-1}(\omega)s_{2k+1}+\cdots+q_k(\omega)}+1-z_{2k+2}}{(z_{2k+2}-1)\dfrac{q_k(\omega)s_{2k+1}+\cdots+q_{k-1}(\omega)}{q_{k-1}(\omega)s_{2k+1}+\cdots+q_k(\omega)}+\omega-z_{2k+2}}$$

$$=\frac{\left[\omega q_k(\omega)-q_{k-1}(\omega)\right]z_{2k+2}s_{2k+1}+\cdots+q_k(\omega)-q_{k-1}(\omega)}{\left[q_k(\omega)-q_{k-1}(\omega)\right]z_{2k+2}s_{2k+1}+\cdots+\omega q_k(\omega)-q_{k-1}(\omega)}$$

$$=\frac{p_{k+1}(\omega)s_{2k+2}+\cdots+p_k(\omega)}{p_k(\omega)s_{2k+2}+\cdots+p_{k+1}(\omega)},$$

which shows that indeed, the matrix corresponding to $f_{2k+2}(z_1,z_2,\cdots,z_{2k+2})$ is:

$$\begin{bmatrix} r_{2k+2,1} \\ r_{2k+2,2} \end{bmatrix}=\begin{bmatrix} p_{k+1}(\omega) & -r_{2k,1} & p_k(\omega) \\ p_k(\omega) & -r_{2k,2} & p_{k+1}(\omega) \end{bmatrix}$$

The computation for the second matrix is similar. This proof provides more information than that about the degree of the equation $f_m(\mathbf{z})=w$, namely it shows the structure of $f_m(\mathbf{z})$.

4. Vectors in $\mathbb{C}^m$

The orthogonality of two vectors in $\mathbb{C}^m$ is expressed by the cancellation of their *inner product* (see [5], page 151). With the help of *m*-Möbius transformations we can say more, namely an angle of two arbitrary two vectors in $\mathbb{C}^m$ can be defined, such that the respective angle is $\pi/2(\bmod 2\pi)$ when their inner product is zero. For simplicity, we deal first with the case $m=2$. Let $\mathbf{z},\zeta\in\mathbb{C}^2$ be arbitrary points. Following a tradition (see [4], page 12), we will keep the same notation for their *position vectors, i.e.*, vectors pointing from the origin to those points. The inner product of the vectors $\mathbf{z}$ and ζ is by definition $\langle\mathbf{z},\zeta\rangle=z_1\overline{\zeta}_1+z_2\overline{\zeta}_2$. We say that $\mathbf{z}$ and ζ are orthogonal if and only if $\langle\mathbf{z},\zeta\rangle=0$. Suppose that $\mathbf{z}$ and ζ are not those given by (8'), *i.e.*, their images by f_2 is not zero. We denote by $\mathbf{z}_0$ a point which is mapped by f_2 into zero, *i.e.*, $\mathbf{z}_0=(z_1,z_2)$, for an arbitrary z_1 and $z_2=(z_1-1)/(\omega z_1-1)$. Then $\mathbf{z}-\mathbf{z}_0$ and $\zeta-\mathbf{z}_0$ are vectors with the initial point $\mathbf{z}_0$ and the final points respectively $\mathbf{z}$ and ζ. Their images by f_2 are position vectors *u* and *v* in the (*w*)-plane. They make an angle $\alpha=\sphericalangle(u,v)$ which remains invariant to a conformal mapping.

Theorem 8. Let f_2 be a bi-Möbius transformation of parameter ω moving the points $\mathbf{z}-\mathbf{z}_0$ and $\zeta-\mathbf{z}_0$ into *u* and *v*. Then the projection of the section of the pre-image of *u* and *v* by $z_2=1$ does not depend on ω.

Proof. Indeed, let us deal with the mapping $z_1(w)=\dfrac{(\omega-z_2)w+z_2-1}{(1-z_2)w+\omega z_2-1}$ obtained by solving for z_1 the equation $w=f_2(z_1,z_2)$. For z_2 fixed this is a conformal mapping of the (*w*)-plane onto the (z_1)-plane. For $z_2=1$ we have $z_1(w)=w$, which implies that $\sphericalangle(z_1(u),z_1(v))=\sphericalangle(u,v)$. Due to the symmetry of f_2 we obtain a similar result for the projection onto the (z_2)-plane.

If u and v are orthogonal, so are $z_1(u)$ and $z_1(v)$, respectively $z_2(u)$ and $z_2(v)$. Then $z_1(u)\overline{z_1(v)}=0$ and $z_2(u)\overline{z_2(v)}=0$, thus $z_1(u)\overline{z_1(v)}+z_2(u)\overline{z_2(v)}=0$, which means that $\mathbf{z}$ and ζ are orthogonal. We can put by definition $\sphericalangle(\mathbf{z},\zeta)=\sphericalangle(u,v)$, which agrees with the the definition of orthogonality. These concepts generalize trivially to $\mathbb{C}^m$.

5. The Cross-Ratio

The cross-ratio of four points used by Desargues in his studies of projective geometry (see [5], page 154), reappears in complex analysis as a means of dealing with Möbius transformations. It is known that there is a unique Möbius transformation which carries three arbitrary distinct points $q,r,s\in\overline{\mathbb{C}}$ into $0,1,\infty$. This is $[z,q,r,s]=\frac{(z-q)(r-s)}{(z-s)(r-q)}$ and it is called the cross-ratio of the four points z,q,r,s. In other words, the cross-ratio $[z,q,r,s]$ is the image of z by the Möbius transformation which carries q,r,s into $0,1,\infty$. It is also known (see [4], page 79) that for any Möbius transformation $w=M(z)$ and for any four distinct points $z_1,z_2,z_3,z_4\in\overline{\mathbb{C}}$ we have $[w_1,w_2,w_3,w_4]=[z_1,z_2,z_3,z_4]$, where $w_j=M(z_j)$, $j=1,2,3,4$, hence the cross-ratio of four points is an invariant with respect to Möbius transformations.

What can be said if $w=M(\mathbf{z})$ is a m-Möbius transformation? Obviously, the cross-ratio cannot be defined in $\overline{\mathbb{C}}^m$. Yet, we can use the pre-image by M of the four points $w_j\in\overline{\mathbb{C}}$ and the fact that the projections of the sections obtained by keeping z_l fixed, $l\neq k$ of this pre-image onto any (z_k)-plane are complex numbers. More exactly, keeping fixed all z_l, $l\neq k$, the section of the pre-image of w_j is a unique point. Let us denote by $z_k^{(j)}$ the projection onto the (z_k)-plane of the section of the pre-image of w_j obtained by keeping z_l fixed, $l\neq k$. We have:

$$w_j=\frac{p_k z_k^{(j)}-q_k}{q_k z_k^{(j)}+r_k} \tag{12}$$

as in Section 2. Thus, we can state:

Theorem 9. For every m-Möbius transformation $w=M(\mathbf{z})$ the cross-ratio of four points from every (z_k)-plane is preserved.

Proof: Let us take $z_k^{(j)}, j=1,2,3,4$ four distinct points in the (z_k)-plane and for $z_l^{(j)}$ arbitrary in the (z_l)-planes, $l\neq k$ denote $w_j=M\left(z_1^{(j)},z_2^{(j)},\cdots,z_m^{(j)}\right)=\frac{p_k z_k^{(j)}-q_k}{q_k z_k^{(j)}+r_k}$, $j=1,2,3,4$. The projections onto the (z_k)-plane of the sections of the pre-image by M of w_j obtained when we keep $z_l=z_l^{(j)}, l\neq k$ fixed are exactly the points $z_k^{(j)}$. Since (9) is a Möbius transformation, we have $\left[z_1^{(k)},z_2^{(k)},z_3^{(k)},z_4^{(k)}\right]=[w_1,w_2,w_3,w_4]$, which proves the theorem.

Theorem 10. The cross-ratio $\left[z_1^{(k)},z_2^{(k)},z_3^{(k)},z_4^{(k)}\right]$ of four points in any (z_k)-

plane is real if and only if for arbitrary $z_l, l \neq k$, the points $w_j = M(\mathbf{z}_j)$, where $\mathbf{z}_j = \left(z_1, z_2, \cdots, z_{k-1}, z_j^{(k)}, z_{k+1}, \cdots, z_m\right)$ lie on a circle in the (w)-plane.

Proof: It is known that cross-ratio of four points in the complex plane is real if and only if the four points lie on a circle in that plane (see [4], page 79). Let the plane be a (z_k)-plane and let $z_j^{(k)}, j = 1,2,3,4$ lie on a circle. Then $\left[z_1^{(k)}, z_2^{(k)}, z_3^{(k)}, z_4^{(k)}\right]$ is real and $\left[z_1^{(k)}, z_2^{(k)}, z_3^{(k)}, z_4^{(k)}\right] = [w_1, w_2, w_3, w_4]$. The images of that circle by $M(\mathbf{z})$ for different choices of $z_l, l \neq k$ are circles in the (w)-plane which contain the points w_j. Reciprocally, if w_j lie on a circle in the (w)-plane, let us take the pre-image by $M(\mathbf{z})$ of that circle. Keeping $z_l, l \neq k$ constant we obtain a section of that pre-image whose projection onto the (z_k)-plane is a circle containing the points $z_j^{(k)}$, therefore, the cross-ratio of these points must be real.

6. Conclusion

The purpose of this work was to extend to $\overline{\mathbb{C}}^m$ the geometric concepts pertinent to Möbius transformations in the plane. The introduction of a function $f_m : \overline{\mathbb{C}}^m \to \overline{\mathbb{C}}$, which is a Möbius transformation in each one of the variables, when keeping the others constant allowed us to perform this task. The most remarkable achievement was the extension to $\mathbb{C}^m$ of the symmetry principle. The concept of the angle of two vectors in $\mathbb{C}^m$ has been also dealt with. However, just a few properties have been visited, so the potential for other developments is obvious. In particular, visualization in the style done in [6] might be possible having in view the fact that f_m is a classic Möbius transformation in every one of its variables when others are kept fixed. For the same reason, characterizations as those made by Haruki and Rassias in [7] [8] [9] [10] are expected.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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