

# A Procedure for Trisecting an Acute Angle

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## Abstract

This paper presents a graphical procedure, *using an unmarked straightedge and compass only*, for trisecting an arbitrary acute angle. The procedure, when applied to the  $30^\circ$  angle that has been “proven” to be not trisectable, produced a construction having the **identical angular relationship** with Archimedes’ Construction, in which the required trisection angle was found to be exactly one-third of the given angle (or  $\angle EMA = 1/3\angle ECG = 10^\circ$ ), as shown in **Figure 1(D)** and **Figure 1(E)** and Section 4 PROOF in this paper. Hence, based on this **identical angular relationship** between the construction presented and Archimedes’ Construction, one can only conclude that geometric requirements for arriving at an exact trisection have been met, notwithstanding the theoretical proofs of Wantzel, Dudley, and others.

## Keywords

Archimedes’ Construction, College Geometry, Angle Trisection, Trisectors, Famous Problems in Mathematics, History of Mathematics

## 1. Introduction

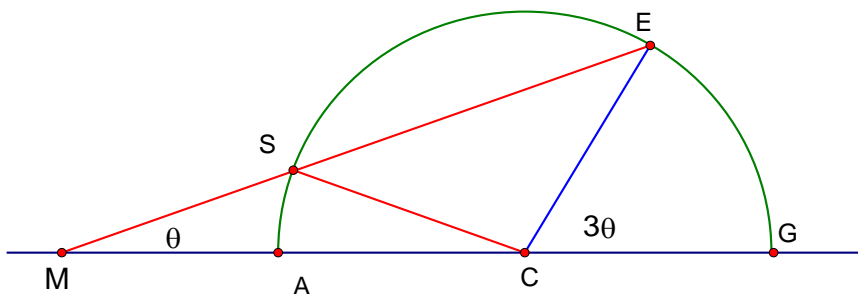
The trisection of an angle problem has been one of the most intriguing geometric challenges for mathematicians for centuries [1] [2]. Simply stated and also “proven”, *the trisection of an arbitrary acute angle (except  $45^\circ$ ) cannot be achieved using an unmarked straightedge and compass only* [3] [4]. Or, as stated by Underwood Dudley, author of *A Budget of Trisections* [4] “*There is no procedure, using only an unmarked straightedge and compasses to construct one-third of an arbitrary angle*”. Yet there have been countless attempts by a number of mathematicians to either disprove this assertion or devise a construction that is as close as possible to the exact solution. Some of the more notable attempts, in both cases, that can be found in the literature, besides *A Budget of Trisections* [4], include *The Trisectors* by Underwood Dudley [5], web article on “*The Trisection of an Angle*” by Jim Loy [6], and many more [7] [8] [9].

The object of this paper is not to debate the established proofs alluded to, but simply to present a comprehensive graphical procedure, *using an unmarked straightedge and compass only*, that will demonstrate one approach on how an arbitrary acute angle (of any measure) can be trisected. The procedure is based on the article entitled “*Mechanism Analysis of a Trisector*” published earlier [7]. In which a working model of a trisector [A1] was analyzed using principles of kinematics, instead of conventional mathematics and plane geometry, to study the trisection problem. The basis for employing this approach was the fact that *while the angle trisection could not be achieved using an unmarked straightedge and compass, yet a mechanism can be built to perform this task perfectly* [8]. Hence, performing a motion analysis on an actual trisector (See **Figure A1**) seemed a logical rationale for seeking to obtain fresh insight into understanding the trisection problem.

To be clear, *Kinematics* is the study of motion, and the purpose of the trisector model was simply to study and gain an understanding of its motion. Therefore, it is *not* a violation of the unmarked ruler and compass rule. For details on the motion analysis, see reference article [7]. It should also be noted that nowhere in the literature has this author encountered any attempt at tackling this *age-old problem of the angle trisection* as is demonstrated in this paper; *i.e.* applying the fundamental principles of kinematics (in this case motion analysis instead of conventional mathematics and plane geometry).

## 2. Theory

The procedure being presented is based on the well-known Archimedes’ Construction [1] represented in the diagram below, which illustrates the geometric requirements that must be met in order to arrive at an exact trisection, and the general theorem relating to arcs and angles.



Let  $\angle ECG$  (or  $3\angle\theta$ ) be the required angle to be trisected. With center at C and radius CE describe a semicircle. Given that a line from point E can be drawn to cut the semicircle at S and intersect the extended side GC at some point M such that the distance SM is equal to the radius SC, then from the general theorem relating to arcs and angles,

$$\angle EMG = 1/2(\angle ECG - \angle SCM) \tag{1}$$

$$2\angle EMG + \angle SCM = \angle ECG \tag{2}$$

$$\text{Since } \triangle CSM \text{ is an isosceles } \triangle, \angle SCM = \angle EMG = \angle \theta \quad (3)$$

Therefore,

$$3\angle EMG = \angle ECG \text{ or } 3\angle \theta = \angle ECG \text{ or } \angle EMA = 1/3\angle ECG. \quad (4)$$

**TO SUMMARIZE:**

**Geometric Requirements for EXACT TRISECTION:are:**

(1) Segments **SM, SC, EC,** and **CG** are all equal, and (2)  $\angle SMA = \angle SCA$

**ONCE**, these requirements are met,

**THEN**, the EXACT trisection of the given angle ( $\angle ECG$ ) is achieved **or**  $\angle EMA = 1/3\angle ECG$ .

**NOTE also** that, except for the given angle ( $\angle ECG$ ) being an acute angle, there are no other restrictions on **measure** of this angle.

**THEREFORE** the measure of  $\angle ECG$  can be **ANY REAL NUMBER (or ARBITRARY)**.

### 3. Procedure

To illustrate the procedure, we consider the  $30^\circ$  angle to be trisected (*i.e., divided into three exactly equal parts, using an unmarked straightedge and compass only*).

The construction for this angle is given in the following **Figures 1(A)-(C)**:<sup>1</sup>

**STEP 1** See **Figure 1(A)**

- 1) Using CG as the base, erect a perpendicular  $CC'$  at C.
- 2) With center at C and any convenient radius, describe a semicircle from point G cutting perpendicular  $CC'$  at E, and terminating at A on GC (extended).
- 3) Using CE as a base, form an equilateral triangle CEV, where V is the vertex.
- 4) Extend segment EV to meet GC (extended) at a point F.

**STEP 2** See **Figure 1(B)**

- 1) Using point E as center and CE as radius, describe an arc cutting the semicircle in STEP 1 at point  $E'$  to form the given angle  $\angle E'CG = 30^\circ$
- 2) Construct a ray from  $E'$  through point V and locate point L on said ray such that  $VL = VC$ .
- 3) Construct segment EF, cutting the semicircle at  $V'$ .
- 4) Join  $V'$  to C with segment  $V'C$  and extend segment  $V'F$  to  $V'F'$  such that  $V'F' = V'C$ .
- 5) Construct a ray from  $E'$  through point A and locate point N on said ray such that  $AN = AC$ .

**STEP 3** See **Figure 1(C)**

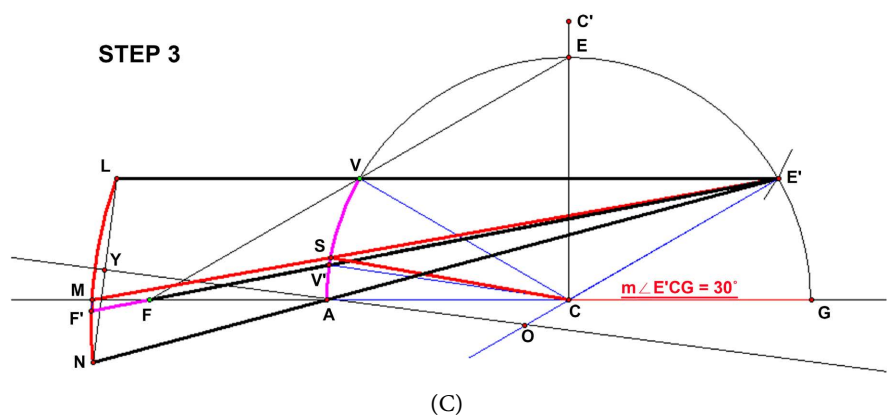
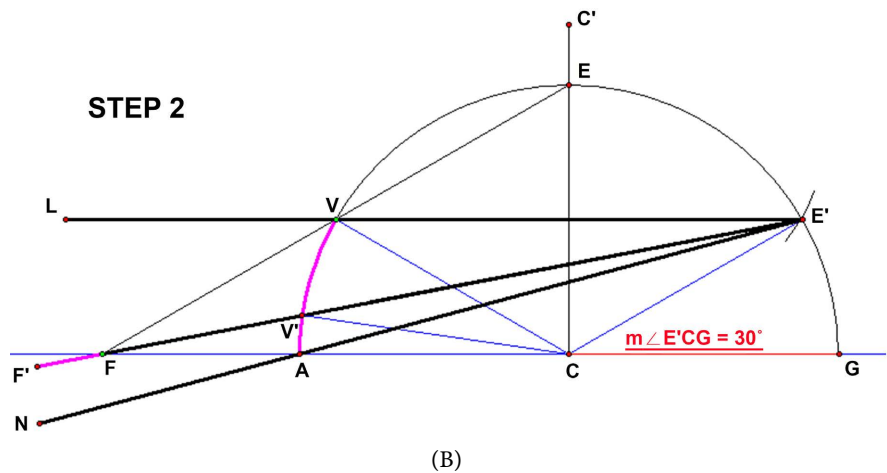
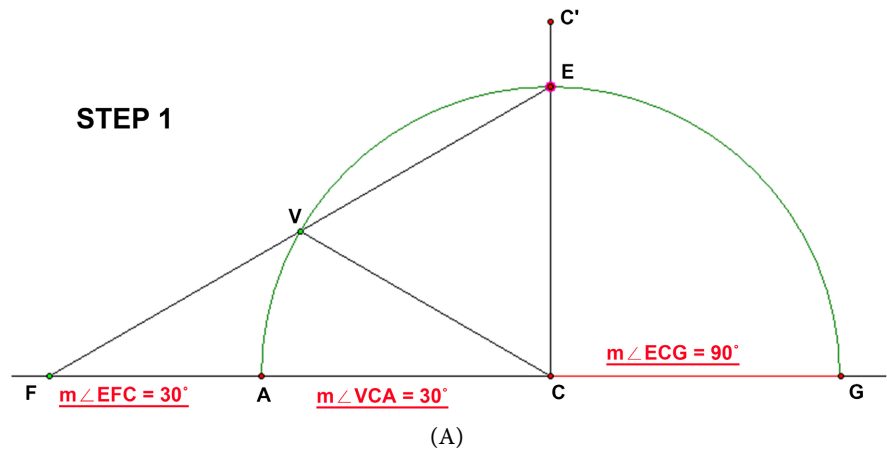
- 1) Join L to N with segment NL.
- 2) Locate midpoint Y of segment NL.
- 3) At midpoint Y, construct a perpendicular line to segment NL meeting line EC (extended) at point O.

<sup>1</sup>Note: While the Geometer's Sketch Pad [10] was employed in this procedure, the use of this software is not a violation of the unmarked straightedge and compass rule, since its sole purpose was strictly for 1) the layout of lines and arcs (for precision and not measurements, except at the end for final results), and 2) for color coding, which is appropriate for an effective graphical presentation. Otherwise, the construction can easily be hand drawn.

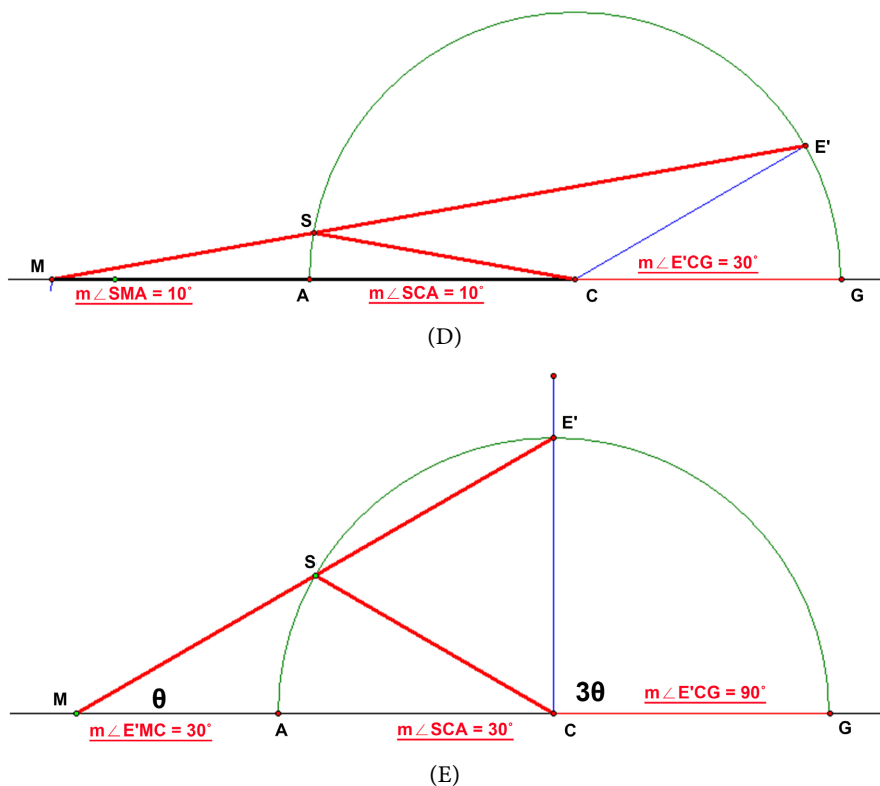
4) With center at O and radius ON describe an arc  $\langle NF'L \rangle$  from N through F' to L, cutting the baseline GF (extended) at a point M.

5) Join E' to M with EM cutting AV at a point S to form the required trisection angle,  $\angle EMG$ , and making  $\angle EMA = 1/3\angle ECG$ , in compliance with the Archimedes' Construction [1].

6) Join S to C with a segment SC to complete the construction<sup>2</sup>, which makes segment  $SM = SC$ . See **Figure 1(D)** and **Figure 1(E)** on Angular Relationship.



<sup>2</sup>See **Figure A2** in **Appendix** for comparison when the identical construction procedure is applied to benchmark angle  $45^\circ$  that is known to be trisectable.



**Figure 1.** (A) Construction Step 1; (B) Construction Step 2; (C) Composite construction showing trisection of the 30° angle:  $\angle E'MA = 1/3\angle E'CG$ ; (D) Resultant angles for 30° trisection:  $\angle E'MA = 1/3\angle E'CG$ ; (E) Archimedes’ model for the 90° angle:  $E'MA = 1/3\angle E'CG$ .

### 4. Proof

Referring to **Figure 1(D)** above, and applying the general theorem relating to arcs and angles (See Section 2 on THEORY of this paper), we get

$$\begin{aligned} \angle E'MG &= 1/2(\angle E'CG - \angle SCM) \quad \text{or} \quad \angle E'MA = 1/2(\angle E'CG - \angle SCM) \\ 2\angle E'MA &= \angle E'CG - \angle SCM \\ 2\angle E'MA + \angle SCM &= \angle E'CG \\ \text{Since } \angle SCM &= \angle E'MA \\ \text{Then } 3\angle E'MA &= \angle E'CG \end{aligned}$$

Therefore  $\angle E'MA = 1/3\angle E'CG = 1/3 \times 30^\circ = 10^\circ$  (QED)

### 5. Summary

A comprehensive graphical procedure, *using an unmarked straightedge and compass only*, for the trisection of an acute angle has been presented. The procedure, when applied to the “non-trisectable” 30° angle, produced a construction having an **identical angular relationship** with Archimedes’ Construction [1], in which the trisection angle has been found to be exactly one-third of the given angle (or  $\angle E'MA = 1/3\angle E'CG = 10^\circ$ ), as shown in **Figure 1(D)** and **Figure 1(E)**

as well as Section 4 PROOF in this paper. Therefore, based on the **identical angular relationship** between the construction presented and Archimedes' Construction [1], one can only conclude that the geometric requirements for arriving at an exact trisection have been met, notwithstanding the theoretical proofs of Wantzel, Dudley, and others [3] [4] [8].

To be specific, *the construction presented has achieved the desired objective of dividing an arbitrary acute angle (of any measure) into three exactly equal parts, using an unmarked straightedge and compass only.*

### Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix

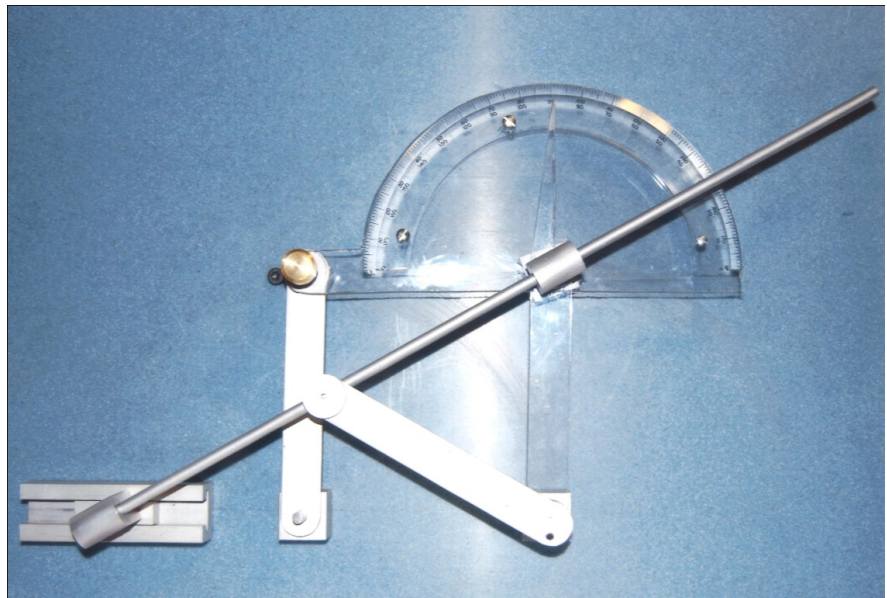


Figure A1. Angle trisector model.

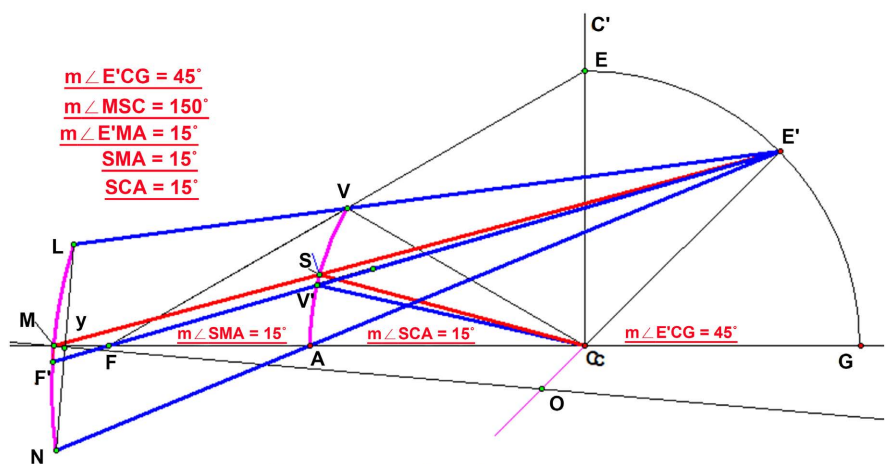


Figure A2. Composite construction showing trisection of the  $45^\circ$  angle  $\angle E'CG$ . Yielding  $\angle EMA = 15^\circ$ .