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The Proof of the 3X + 1 Conjecture

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Abstract

In this paper, we use two new effective tools and ingenious methods to prove the 3X + 1 conjecture. By using the recursive method, we firstly prove that any positive integer can be turned into an element of fourth column of the infinite-row-six-column-matrix after a finite times operation, thus we convert "the 3X + 1 conjecture" into an equivalent conjecture, which is: Any positive integer n must become 1 after finite operations under formation of $\sigma(n)$, where $\sigma(n)=\begin{cases}\sigma_1(n)=\frac{3}{2}n & (n\equiv 0 \text{ or } n\equiv 2 \pmod 4);\\ \sigma_2(n)=\frac{3n+1}{4} & (n\equiv 1 \pmod 4); \\ \sigma_3(n)=\frac{n+1}{4} & (n\equiv 3 \pmod 4).\end{cases}$ Then, with the help of the infinite-row-four-column-matrix, we continue to use the recursive method to prove this conjecture strictly.

Keywords

The 3X + 1 Conjecture, $(Z^+)_{\infty\times 6}$, $(Z^+)_{\infty\times 4}$, Transformation, Module

1. Introduction

3X + 1 conjecture: Take a positive integer X freely, if it is an even, divide it by 2 into X/2, if it is an odd, multiply it with 3 then add 1 on the product into 3X + 1, the ends operate again and again according to the above-mentioned rules, the final end inevitably is 1 after limited times.

The 3x + 1 conjecture is a world-famous unsolved number theory problem in recent 100 years [1]. It is also known as the 3x + 1 problem, the Collatz problem, the Syracuse problem, Kakutani's problem, Hasse's logarithm, and Ulam's prob-

lem. J. C. Lagarias wrote a review paper titled "The 3x + 1 problem and its generalizations", saying: "in this paper I describe the history of the 3x + 1 problem and survey all the literature I am aware of about this problem and its generalizations" [2].

"The exact origin of the 3x + 1 problem is obscure" [2]. It probably came into being between the 1920s and 1930s. In his review paper, J. C. Lagarias said: "It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz, at the University of Hamburg. In his student days in the 1930s, stimulated by the lectures of Edmund Landau, Oskar Perron, and Issai Schur, he became interested in number-theoretic functions" [2].

Since it was put forward, the conjecture has never been stopped studying on it. Up to now, many papers on this conjecture have been published at home and abroad [2]-[11], we can see from these papers [2] [3] [4] [5] that many people limited and stayed on the idea of function iteration. After all, the key is that infinite numerical iteration is quite difficult. Some scholars used the computer to verify the 3x + 1 conjecture to be correct for the numbers less than $100 \times 2^{50} = 11{,}258{,}990{,}684{,}262{,}400$ [2]. Some scholars used the computer science methods to study it [9]. "The 3x + 1 problem has been shown to be a computationally unsolvable problem" [2]. After all, limited verification is just a drop in the ocean for infinite numbers, no matter what a powerful verification is, it can't replace any theoretical proof. To solve it, people must use new powerful tools and ingenious ideas.

By doing researches, we find that a special tool can be used to solve it completely, which is a matrix with infinite rows and six columns in all positive integers arranged in the order of natural numbers, named as an infinite-row-six-column-matrix, denoted as $\left(Z^+\right)_{\infty\times 6}$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 6(n-1)+1 & 6(n-1)+2 & 6(n-1)+3 & 6(n-1)+4 & 6(n-1)+5 & 6(n-1)+6 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

We denote r-th column of $\left(Z^+\right)_{\infty\times 6}$ by $C_r\left(r=1,2,3,4,5,6\right)$, and denote the elements of C_r by e_r.

The overall idea of proof is simply as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 6n-5 & 6n-4 & 6n-3 & 6n-2 & 6n-1 & 6n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 10 \\ \cdots \\ 6n-2 \\ \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ \\ \\ \\ \end{pmatrix} \Rightarrow 1$$

By proving proposition 1: $4^r \in C_4$ ($r \in Z^+$), and its row number $n = 4^{r-1} - \dfrac{4^{r-1}-1}{3}$, we have a clear direction. Then we prove proposition 2: If $e \in C_1 \bigcup C_2 \bigcup C_3 \bigcup C_5 \bigcup C_6$, then $F^{(m)}(e) \in C_4$. All of a sudden, 6 columns are reduced to 1 column, which is equivalent to subtracting 5 columns, which makes us see the light and hope. People have never achieved such great results before.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 6n-5 & 6n-4 & 6n-3 & 6n-2 & 6n-1 & 6n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 10 \\ \cdots \\ 6n-2 \\ \cdots \end{pmatrix}$$

After proving the first step, we prove the second step.

$$\begin{pmatrix} 4 \\ 10 \\ \cdots \\ 6n-2 \\ \cdots \end{pmatrix} \Rightarrow (4)$$

By studding it, we find that there are three rules of the transformation from C_4 to C_4 (itself):

1) $e_4 \xrightarrow{\text{one time operation}} e_5 \xrightarrow{\text{two times operation}} e_4'$ $(n \equiv 0 \text{ or } n \equiv 2 \pmod 4)$;

$e_4 = 6(n-1)+4 \to e_4' = 6\left(\dfrac{3n}{2}-1\right)+4$.

2) $e_4 \xrightarrow{\text{one time operation}} e_2 \xrightarrow{\text{one time operation}} e_1 \xrightarrow{\text{two times operation}} e_4'$ $(n \equiv 1 \pmod 4)$;

$e_4 = 6(n-1)+4 \to e_4' = 6\left(\dfrac{3n+1}{4}-1\right)+4$.

3) $e_4 \xrightarrow{\text{one time operation}} e_2 \xrightarrow{\text{one time operation}} e_4'$ $(n \equiv 3 \pmod 4)$;

$e_4 = 6(n-1)+4 \to e_4' = 6\left(\dfrac{n+1}{4}-1\right)+4$.

e_4' is another element in C_4 that's different from the original one.

Seen in essence, any transformation of $F(x)$ is a linear transformation of the row ordinal number n of C_4, then the 3x + 1 conjecture is converted into the equivalent conjecture:

$$\sigma^{(t)}(n) = 1 \ \left(n \in Z^+, n \neq 1, t \in Z^+\right).$$

where

$$\sigma(n) = \begin{cases} \sigma_1(n) = \dfrac{3}{2}n & (n \equiv 0 \text{ or } n \equiv 2 \pmod 4); \\ \sigma_2(n) = \dfrac{3n+1}{4} & (n \equiv 1 \pmod 4); \\ \sigma_3(n) = \dfrac{n+1}{4} & (n \equiv 3 \pmod 4). \end{cases}$$

That is every positive integer n can become 1 by a finite timestrans formation

of $\sigma(n)$.

Then, we prove the equivalence conjecture.

Take the row numbers (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ···) of C_4 in $\left(Z^+\right)_{\infty\times 6}$ which are made into a matrix with infinite rows and 4 columns arranged in the order of natural numbers, named as an infinite-row-four-column-matrix, denoted it with $\left(Z^+\right)_{\infty\times 4}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \cdots & \cdots & \cdots & \cdots \\ 4k-3 & 4k-2 & 4k-1 & 4k \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

By proving proposition 4, proposition 5 and proposition 6, we prove the equivalent conjecture.

Introduce simply as follows:

Proposition 4: if $n \in C_1 \bigcup C_2 \bigcup C_4$, then $\sigma^{(m)}(n) \in C_3 \left(n \in Z^+, n \neq 1, m \in Z^+\right)$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \cdots & \cdots & \cdots & \cdots \\ 4k-3 & 4k-2 & 4k-1 & 4k \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4k-1 \\ & \cdots \end{pmatrix}$$

Proposition 5: if $e_3 \neq 3$, then $\sigma^{(m)}(e_3) = 3 \left(m \in Z^+\right)$.

$$\begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4k-1 \\ & \cdots \end{pmatrix} \Rightarrow (1 \quad 3)$$

Proposition 6: $\sigma_3(3) = 1$.

$$(1 \quad 3) \Rightarrow (1)$$

Up to now, we prove the conjecture of equivalence.

Finally, we give a summary proof:

1) If $X \neq 6(n-1)+4$, according to proposition 2.5, any positive integer that is not $6(n-1)+4$ can be turned to $6(n-1)+4$ after a finite transformation according to $F(x)$.

2) If $X = 6(n-1)+4 \left(n \neq 1\right)$, we can know from the equivalent conjecture that x can be turned to 4 after a finite operation.

3) If $X = 6(n-1)+4 \left(n = 1\right)$, then $x = 4$, according to $F_2(X)$, $F_2(4) = \frac{4}{2} = 2$, $F_2(2) = \frac{2}{2} = 1$.

Therefore, the 3x + 1 conjecture is correct.

The research of the 3X + 1 conjecture has promoted the development of some

branches of computer science and higher mathematics. Some results obtained in the research of the 3X + 1 conjecture have been applied to the research of related sciences [10]-[16].

Proving this conjecture by the tool and idea has great theoretical significance. It provides a new tool, a new idea and a new method to solve infinite problems. From then on, people can use this tool, idea and method to solve other number theory problems involving infinite numbers, such as the 3x − 1 conjecture, the 5x + 1 conjecture, Niuman conjecture.

The 3X + 1 conjecture is changed from a conjecture to a theorem here. From then on, people rest assured to use it to understand and solve relevant problems.

2. Main Results

The whole process contains three parts. Part one includes proposition 1, proposition 2, and proposition 3, which convert the 3x + 1 conjecture into an equivalent conjecture. Part two includes proposition 4, proposition 5 and proposition 6, which prove that the equivalent conjecture is correct. Part three is summative proof, by using proposition 2 and the equivalent conjecture with two situations proving the 3x + 1 conjecture is correct.

The overall idea of proof is simply as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 6n-5 & 6n-4 & 6n-3 & 6n-2 & 6n-1 & 6n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 10 \\ \cdots \\ 6n-2 \\ \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ \\ \\ \\ \end{pmatrix} \Rightarrow 1$$

Part One

In this part, we convert the 3x + 1 conjecture into an equivalent conjecture:

$$\sigma^{(t)}(n)=1 \ \left(n \in Z^{+}, n \neq 1, t \in Z^{+}\right).$$

where

$$\sigma(n)=\begin{cases} \sigma_1(n)=\dfrac{3}{2}n & \left(n \equiv 0 \text{ or } n \equiv 2 \pmod 4\right); \\ \sigma_2(n)=\dfrac{3n+1}{4} & \left(n \equiv 1 \pmod 4\right); \\ \sigma_3(n)=\dfrac{n+1}{4} & \left(n \equiv 3 \pmod 4\right). \end{cases}$$

That is every positive integer n that can become 1 by a finite transformation of $\sigma(n)$.

3X + 1 conjecture: Take a positive integer X freely, if it is an even, divide it by 2 into X/2, if it is an odd, multiply it with 3 then add 1 on the product into 3X + 1, the ends operate again and again according to the above-mentioned rules, the final end is inevitably 1 after limited times.

Now, we define the following transformation according to the 3X + 1 conjec-

ture.

$$F(X)=\begin{cases}F_1(X)=3X+1 & (X\equiv 1\ (\mathrm{mod}\,2));\\ F_2(X)=\dfrac{X}{2} & (X\equiv 0\ (\mathrm{mod}\,2)).\end{cases}$$

Every transformation of $F_1(X)$ is one time multiply and one time to add, every transformation of $F_2(X)$ is one time to divide. Therefore, one transformation of $F_1(X)$ is twice operation; one transformation of $F_2(X)$ is one time operation.

Now, by proving proposition 1, proposition 2 and proposition 3, we convert the 3X + 1 conjecture into the above equivalent conjecture.

Proposition 1: $4^r\in C_4\ (r\in Z^+)$, and its row number $n=4^{r-1}-\dfrac{4^{r-1}-1}{3}$.

Proof: $\because\ 4^r-4=4\left(4^{r-1}-1\right)=4\left[\left(2^{r-1}\right)^2-1\right]=4\left(2^{r-1}+1\right)\left(2^{r-1}-1\right)$.

$3\mid\left(2^{r-1}+1\right)\cdot 2^{r-1}\cdot\left(2^{r-1}-1\right)$, but $3\nmid 2^{r-1}$.

$\therefore\ 3\mid\left(2^{r-1}-1\right)\left(2^{r-1}+1\right)$.

$\therefore\ 6\mid 4\left(2^{r-1}-1\right)\left(2^{r-1}+1\right)$.

Let $4^r-4=6(n-1)$ $(n\in Z^+)$.

$\therefore\ 4^r=6(n-1)+4$.

$\therefore\ 4^r\in C_4$.

The following mathematical induction proves that row number of 4^r is $4^{r-1}-\dfrac{4^{r-1}-1}{3}$ $(r\in Z^+)$.

Proof: 1) As $r=1$, $n=4^{1-1}-\dfrac{4^{1-1}-1}{3}=1$, the conclusion is correct.

2) It is assumed that the conclusion is correct as $r=s$ $(s\in Z^+,\ s\geq 1)$, that is

$$4^s=6\left(4^{s-1}-\frac{4^{s-1}-1}{3}-1\right)+4.$$

3) As $r=s+1$ $(s\in Z^+,\ s\geq 1)$,

$\because\ 4^s=6\left[\left(4^{s-1}-\dfrac{4^{s-1}-1}{3}\right)-1\right]+4$.

$\therefore\ 4\times 4^s=4\left[6\left(4^{s-1}-\dfrac{4^{s-1}-1}{3}-1\right)+4\right]$.

$\therefore\ 4^{s+1}=6\left(4^s-\dfrac{4^s-4}{3}-4\right)+16=6\left[\left(4^{(s+1)-1}-\dfrac{4^{(s+1)-1}-1}{3}\right)-1\right]+4$.

According to the hypothesis, the conclusion is also correct as $r=s+1$ $(s\in Z^+,\ s\geq 1)$.

By combining (1) and (2), we know that the statement is correct for all positive integers.

By the above proof, proposition 1 is correct.

Proposition 2: If $e\in C_1\cup C_2\cup C_3\cup C_5\cup C_6$, then $F^{(m)}(e)\in C_4$.

Lemma 2.1: If $e \in C_1 \bigcup C_3 \bigcup C_5$, then $F(e) \in C_4$.

Prove: $\because e \in C_1 \bigcup C_3 \bigcup C_5$, then $e \equiv 1 \pmod 2$, according to transformation of $F(x)$,

$F_1(e_1) = 3(6n-5)+1 = 6(3n-3)+4 \in C_4$;

$F_1(e_3) = 3(6n-3)+1 = 6(3n-2)+4 \in C_4$;

$F_1(e_5) = 3(6n-1)+1 = 6(3n-1)+4 \in C_4$.

$\therefore F(e) \in C_4$.

Therefore, lemma 2.1 is correct.

Lemma 2.2: If $e \in C_2$, then $F^{(m)}(e) \in C_4 \ (m = 1,2)$.

Prove: $\because$ If $e \in C_2$, then $e = e_2 = 6(n-1)+2 = 6n-4$.

$$\therefore F_2(e_2) = F_2(6n-4) = \frac{6n-4}{2} = \begin{cases} 6\left(\frac{n+1}{2}-1\right)+1 & (n \equiv 1 \pmod 2); \\ 6\left(\frac{n}{2}-1\right)+4 & (n \equiv 0 \pmod 2). \end{cases}$$

$\because 6\left(\frac{n+1}{2}-1\right)+1 \equiv 1 \pmod 2$.

According to lemma 2.1,

$F(e_1) \in C_4$.

$\because F\left[6\left(\frac{n+1}{2}-1\right)+1\right] \in C_4$, $6\left(\frac{n}{2}-1\right)+4 \in C_4$.

Therefore, lemma 2.2 is correct.

Lemma 2.3: If $e \in C_6$, then $F^{(a+1)}(e) \in C_4 \left(a \in Z^+\right)$.

Proof: If $e \in C_6$, then $e = e_6$, Let $e_6 = 2^a \cdot 3(2h-1)$, (a is an exponent of factor 2 of unique decomposition of e_6, $a \in Z^+$, $h \in Z^+$).

According to the transformation of $F(x)$, we get

$$F_2^{(a)}\left[2^a 3(2h-1)\right] = 3(2h-1).$$

And $\because 3(2h-1) \equiv 1 \pmod 2$, according to lemma 2.1, we get

$$F_1\left[3(2h-1)\right] \in C_4.$$

Therefore, lemma 2.3 is correct.

So far, proposition 2 is proved.

Proposition 3: $F^{(m)}(e_4) = e_4'$ ($e_4 \in C_4$, $e_4 \neq 4$, $e_4' \in C_4$, $e_4 \neq e_4'$, $m = 2,3$).

Proof: $\because e_4 = 6(n-1)+4 = 6n-2 \ (n \neq 1)$, by the transformation of $F(X)$, we get

$$\begin{aligned} F_2(e_4) &= F_2(6n-2) \\ &= \frac{6n-2}{2} \\ &= \begin{cases} 6\left(\frac{n}{2}-1\right)+5 \in C_5 & (n \equiv 0 \text{ or } n \equiv 2 \pmod 4); \\ 6\left(\frac{n+1}{2}-1\right)+2 \in C_2 & (n \equiv 1 \text{ or } n \equiv 3 \pmod 4). \end{cases} \end{aligned}$$

$$F_1(e_5)=F_1\left[6\left(\frac{n}{2}-1\right)+5\right]$$
$$=3\left[6\left(\frac{n}{2}-1\right)+5\right]+1$$
$$=6\left(\frac{3n}{2}-1\right)+4\in C_4\quad (n\equiv 0 \text{ or } n\equiv 2\ (\mathrm{mod}\ 4)).$$

$$F_2(e_2)=F_2\left[6\left(\frac{n+1}{2}-1\right)+2\right]$$
$$=\frac{6\left(\frac{n+1}{2}-1\right)+2}{2}$$
$$=\begin{cases}6\left(\frac{n+3}{4}-1\right)+1\in C_1 & (n\equiv 1\ (\mathrm{mod}\ 4));\\ 6\left(\frac{n+1}{4}-1\right)+4\in C_4 & (n\equiv 3\ (\mathrm{mod}\ 4)).\end{cases}$$

$$F_1(e_1)=F_1\left[6\left(\frac{n+3}{4}-1\right)+1\right]$$
$$=3\left[6\left(\frac{n+3}{4}-1\right)+1\right]+1$$
$$=6\left(\frac{3n+1}{4}-1\right)+4\in C_4\quad (n\equiv 1\ (\mathrm{mod}\ 4)).$$

The operating process above can be simply expressed below:

1) $e_4 \xrightarrow{\text{one time operation}} e_5 \xrightarrow{\text{two times operation}} e_4'$ $(n\equiv 0 \text{ or } n\equiv 2\ (\mathrm{mod}\ 4))$;

$e_4=6(n-1)+4\to e_4'=6\left(\frac{3n}{2}-1\right)+4$.

2) $e_4 \xrightarrow{\text{one time operation}} e_2 \xrightarrow{\text{one time operation}} e_1 \xrightarrow{\text{two times operation}} e_4'$ $(n\equiv 1\ (\mathrm{mod}\ 4))$;

$e_4=6(n-1)+4\to e_4'=6\left(\frac{3n+1}{4}-1\right)+4$.

3) $e_4 \xrightarrow{\text{one time operation}} e_2 \xrightarrow{\text{one time operation}} e_4'$ $(n\equiv 3\ (\mathrm{mod}\ 4))$;

$e_4=6(n-1)+4\to e_4'=6\left(\frac{n+1}{4}-1\right)+4$.

e_4' is another element in C_4 that's different from the original one.

Therefore, proposition 3 is correct.

In the following, the 3x + 1 conjecture is converted into an equivalent conjecture.

Seen in essence, any transformation of $F(x)$ is a linear transformation of the row ordinal number n of C_4.

Therefore, we can define the transformation as follows:

$$\sigma(n)=\begin{cases}\sigma_1(n)=\frac{3}{2}n & (n\equiv 0 \text{ or } n\equiv 2\ (\mathrm{mod}\ 4));\\ \sigma_2(n)=\frac{3n+1}{4} & (n\equiv 1\ (\mathrm{mod}\ 4));\\ \sigma_3(n)=\frac{n+1}{4} & (n\equiv 3\ (\mathrm{mod}\ 4)).\end{cases}$$

Thus the 3x + 1 conjecture is converted into an equivalent conjecture:

$$\sigma^{(t)}(n)=1 \ \left(n\in Z^+, n\neq 1, t\in Z^+\right).$$

That is every positive integer n can be became 1 by a finite transformation of $\sigma(n)$.

Part Two

In this part, we prove the equivalent conjecture:

$$\sigma^{(t)}(n)=1 \ \left(n\in Z^+, n\neq 1, t\in Z^+\right).$$

where

$$\sigma(n)=\begin{cases}\sigma_1(n)=\dfrac{3}{2}n & \left(n\equiv 0 \text{ or } n\equiv 2 \pmod 4\right);\\ \sigma_2(n)=\dfrac{3n+1}{4} & \left(n\equiv 1 \pmod 4\right);\\ \sigma_3(n)=\dfrac{n+1}{4} & \left(n\equiv 3 \pmod 4\right).\end{cases}$$

Take the row numbers (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ···) of C_4 in $\left(Z^+\right)_{\infty\times 6}$ made into a matrix with infinite rows and 4 columns arranged in the order of natural numbers, named as an infinite-row-four-column-matrix, denoted as $\left(Z^+\right)_{\infty\times 4}$.

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ \cdots & \cdots & \cdots & \cdots\\ 4k-3 & 4k-2 & 4k-1 & 4k\\ \cdots & \cdots & \cdots & \cdots\end{pmatrix}$$

We denote k-th column of $\left(Z^+\right)_{\infty\times 4}$ by $C_k\left(k=1,2,3,4\right)$, and denote the elements of C_k by e_k.

The idea of proof in the following is simply described as follows:

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ \cdots & \cdots & \cdots & \cdots\\ 4k-3 & 4k-2 & 4k-1 & 4k\\ \cdots & \cdots & \cdots & \cdots\end{pmatrix}\Rightarrow\begin{pmatrix}1 & 3\\ 5 & 7\\ \cdots & \cdots\\ 4k-3 & 4k-1\\ \cdots & \cdots\end{pmatrix}\Rightarrow\begin{pmatrix}1 & 3\\ & 7\\ & \cdots\\ & 4k-1\\ & \cdots\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1 & 3\\ & 7\\ & \cdots\\ & 4^m j-\dfrac{4^m-1}{3}\\ & \cdots\end{pmatrix}\Rightarrow\begin{pmatrix}1 & 3\\ & 11\\ & \cdots\\ & 4^m-\dfrac{4^m-1}{3}\\ & \cdots\end{pmatrix}\Rightarrow(1\quad 3)\Rightarrow(1)$$

Now we prove these propositions one by one.

Proposition 4: If $n\in C_1\bigcup C_2\bigcup C_4$, then $\sigma^{(m)}(n)\in C_3\left(n\in Z^+, n\neq 1, m\in Z^+\right)$.

Lemma 4.1: $\sigma(n) \neq n \left(n \in Z^+, n \neq 1\right)$.

Prove: n is valuated with four consecutive positive integers: $4t-3$, $4t-2$, $4t-1$, $4t\,(t \in Z^+)$.

$$\sigma(n) = \begin{cases} \sigma_1(n) = \dfrac{3}{2}n & \left(n \equiv 0 \text{ or } n \equiv 2 \pmod 4\right); \\ \sigma_2(n) = \dfrac{3n+1}{4} & \left(n \equiv 1 \pmod 4\right); \\ \sigma_3(n) = \dfrac{n+1}{4} & \left(n \equiv 3 \pmod 4\right). \end{cases}$$

$$\therefore \sigma_2(4t-3) = \frac{3(4t-3)+1}{4} = 3t-2;\ \ \sigma_1(4t-2) = \frac{3(4t-2)}{2} = 6t-3;$$

$$\sigma_3(4t-1) = \frac{(4t-1)+1}{4} = t;\ \ \sigma_1(4t) = \frac{3(4t)}{2} = 6t.$$

1) As $t=1$, $4t-3=1$ (except), $4t-2=2$, $4t-1=3$, $4t=4$, then $\sigma_2(1)=1$ (except), $\sigma_1(2)=3$, $\sigma_3(3)=1$, $\sigma_1(4)=6$.

$\therefore \sigma(n) \neq n \left(n \neq 1\right)$.

(2) As $t \geq 2$, the following inequality group is correct:

$$t < 3t-2 < 4t-3 < 4t-2 < 4t-1 < 4t < 6t-3 < 6t.$$

That is,

$$\sigma(4t-1) < \sigma(4t-3) < 4t-3 < 4t-2 < 4t-1 < 4t < \sigma(4t-2) < \sigma(4t).$$

$\therefore \sigma(n) \neq n \left(n \in Z^+, n \neq 1\right)$.

Therefore, lemma 4.1 is correct.

Lemma 4.2: If $n \in C_2 \bigcup C_4$, then $\sigma_1^{(b)}(n) \in C_1 \bigcup C_3 \left(b \in Z^+\right)$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \cdots & \cdots & \cdots & \cdots \\ 4k-3 & 4k-2 & 4k-1 & 4k \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ \cdots & \cdots \\ 4k-3 & 4k-1 \\ \cdots & \cdots \end{pmatrix}$$

Prove: If $n \in C_2 \bigcup C_4$, then any even number can always be written as $2^b(2t-1)$ (b is the power of factor 2 of n, $b \in Z^+$, $t \in Z^+$), according to the transformation

$$\sigma_1(n) = \frac{3}{2}n \left(n \equiv 0 \text{ or } n \equiv 2 \pmod 4\right).$$

We get

$$\sigma_1^{(b)}\left[2^b(2t-1)\right] = 3^b(2t-1).$$

And $3^b(2t-1) \equiv 1 \pmod 2$, Then $3^b(2t-1) \in C_1 \bigcup C_3$.

Therefore, lemma 4.2 is correct.

Lemma 4.3 $\sigma_2^{(m)}(e_1) = e_3 \left(e_1 \neq 1, m \in Z^+\right)$.

$$\begin{pmatrix} 1 & 3 \\ 5 & 7 \\ \cdots & \cdots \\ 4k-3 & 4k-1 \\ \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4k-1 \\ & \cdots \end{pmatrix}$$

Prove: $e_1 = 4k-3$, when $e_1 \neq 1$, $k \neq 1$, $\sigma_2(n) = \frac{3n+1}{4}$, then

$$\sigma_2(e_1) = \frac{3(4k-3)+1}{4} = 3k-2.$$

Observe the following facts:

a) As $e_1 = 4k-3$, $\sigma_2(e_1) = 3k-2$,

1) As $k = 4q \in C_4 \left(q \in Z^+\right)$, $e_1 = 4(4q)-3 = 4^2 q - 4\times 1 + 1$,

$\sigma_2(e_1) = 3(4q)-2 = 4(3q)-2 \in C_2$;

2) As $k = 4q-1 \in C_3 \left(q \in Z^+\right)$, $e_1 = 4(4q-1)-3 = 4^2 q - 4\times 2 + 1$,

$\sigma_2(e_1) = 3(4q-1)-2 = 4(3q-1)-1 \in C_3$;

3) As $k = 4q-2 \in C_2 \left(q \in Z^+\right)$, $e_1 = 4(4q-2)-3 = 4^2 q - 4\times 3 + 1$,

$\sigma_2(e_1) = 3(4q-2)-2 = 4(3q-2) \in C_4$;

4) As $k = 4q-3 \in C_1 \left(q \in Z^+, q \neq 1\right)$, $e_1 = 4(4q-3)-3 = 4^2 q - 4^2 + 1$,

$\sigma_2(e_1) = \sigma_2\left(4^2 q - 4^2 + 1\right) = \sigma_2\left[4(4q-3)-3\right] = 3(4q-3)-2 = 4(3q-2)-3 \in C_1$.

At the same time, we find that as we replace the q in $e_1 = 4^2 q - 4^2 + 1$ by $4q-3$, we get

$$e_1 = 4^2(4q-3) - 4^2 + 1 = 4^3 q - 4^3 + 1 \in C_1.$$

And

$$\begin{aligned} \sigma_2^{(2)}\left(4^3 q - 4^3 + 1\right) &= \sigma_2^{(2)}\left[4\left(4^2 q - 4^2 + 1\right) - 3\right] = \sigma_2\left[3\left(4^2 q - 4^2 + 1\right) - 2\right] \\ &= \sigma_2\left[4(3\times 4q - 3\times 4 + 1) - 3\right] = 3(3\times 4q - 3\times 4 + 1) - 2 \\ &= 4\left(3^2 q - 3^2 + 1\right) - 3 \in C_1. \end{aligned}$$

This shows that in the above four situations:

1) As $k = 4q-1$, the result of e_1 becomes an element in C_3 after one time $\sigma_2(n)$ transformation;

2) As $k = 4q-2$ or $k = 4q$, according to lemma 4.2 of proposition 4, after a finite $\sigma_1(n)$ transformation, e_1 either becomes an element in C_3 or an element in C_1. In this case, according to $\sigma_2^{(2)}\left(4^3 q - 4^3 + 1\right) = 4\left(3^2 q - 3^2 + 1\right) - 3 \in C_1$, the element of C_1 must be able to be in the form $4^2 q - 4^2 + 1$.

3) As $k = 4q-3$, according to lemma 4.1, after one time $\sigma_2(n)$ transformation, e_1 becomes another element in C_1.

b) In this case, *i.e.*, when $k = 4q-3$ ($q \in Z^+$, $q \neq 1$),

$$\sigma_2(e_1) = 4(3q-2) - 3$$

$$\sigma_2^{(2)}(e_1) = 3(3q-2) - 2 = 9q - 8.$$

And

1)As $q=4j\in C_4\left(j\in Z^+\right)$, $e_1=4^3j-4\times4+1$,
$\sigma_2^{(2)}\left(e_1\right)=9\left(4j\right)-8=4\left(9j-2\right)\in C_4$;
2) As $q=4j-1\in C_3\left(j\in Z^+\right)$, $e_1=4^3j-4\times8+1$,
$\sigma_2^{(2)}\left(e_1\right)=9\left(4j-1\right)-8=4\left(9j-4\right)-1\in C_3$;
3) As $q=4j-2\in C_2\left(j\in Z^+\right)$, $e_1=4^3j-4\times12+1$,
$\sigma_2^{(2)}\left(e_1\right)=9\left(4j-2\right)-8=4\left(9j-6\right)-2\in C_2$;
4) As $q=4j-3\in C_1\left(j\in Z^+, j\neq1\right)$, $e_1=4^3j-4^3+1$,
$\sigma_2^{(2)}\left(e_1\right)=9\left(4j-3\right)-8=4\left(9j-8\right)-3\in C_1$.

At the same time, we find out:

$$4\left(4^3p-4^3+1\right)-3=4^4p-4^4+1\in C_1.$$

$$\begin{aligned}\sigma_2^{(3)}\left(4^4p-4^4+1\right)&=\sigma_2^{(3)}\left[4\left(4^3p-4^3+1\right)-3\right]=\sigma_2^{(2)}\left[4^3p-4^3+1\right]\\&=\sigma_2^{(2)}\left[4\left(4^2p-4^2+1\right)-3\right]=\sigma_2\left[3\left(4^2p-4^2+1\right)-2\right]\\&=\sigma_2\left[4\left(3\times4p-3\times4+1\right)-3\right]=3\left(3\times4p-3\times4+1\right)-2\\&=4\left(3^2p-3^2+1\right)-3\in C_1.\end{aligned}$$

It is inferred that:

a) As $e_1=4^mj-4^m+1\left(m\in Z^+, m\geq2,\ j\in Z^+\right)$,
$\sigma_2^{(m-1)}\left(4^mj-4^m+1\right)\in C_1$; $\sigma_2^{(m-m1)}\left(4^mj-4^m+1\right)\notin C_1\ \ \left(j\neq4k-3\right)$.
b) As $e_1=4^mj-4c+1\left(c\in Z^+, 1\leq c\leq4^{m-1}-1\right)$,
$\sigma_2^{(t)}\left(4^mj-4c+1\right)\notin C_1\ \ \left(t\in Z^+, t<m\right)$.

Now we prove the conclusion by mathematical induction:

1) As $m=2$, it is known that the conclusion is correct from the above facts.

2) It is assumed that the conclusion is correct as $m=s$ ($s\in Z^+$, $s\geq2$), that is

a) $e_1=4^sj-4^s+1\left(j\neq4k-3\right)$,

$$\sigma_2^{(s-1)}\left(4^sj-4^s+1\right)\in C_1;\ \ \sigma_2^{(s)}\left(4^sj-4^s+1\right)\notin C_1.$$

But

$$\sigma_2\left(4^sj-4^s+1\right)=\sigma_2\left[4\left(4^{s-1}j-4^{s-1}+1\right)-3\right]=4\left(4^{s-2}3j-3\times4^{s-2}+1\right)-3\in C_1.$$

then

$$\sigma_2^{(s-2)}\left[4\left(4^{s-2}3j\ -3\times4^{s-2}+1\right)-3\right]\in C_1,$$

$$\sigma_2^{(s-1)}\left[4\left(4^{s-2}3j-3\times4^{s-2}+1\right)-3\right]\notin C_1.$$

b) $e_1=4^sj-4c+1\left(c\in Z^+, 1\leq c\leq4^{s-1}-1\right)$,

$$\sigma_2^{(t)}\left(4^sj-4c+1\right)\notin C_1\ \ \left(t\in Z^+, t<s\right).$$

But

$$\sigma_2\left(4^sj-4c+1\right)=\sigma_2\left[4\left(4^{s-1}j-c+1\right)-3\right]=4^s3j-3c+1.$$

then

$$\sigma_2^{(t-1)}\left(4^s j-3c+1\right)\notin C_1.$$

3) As $m=s+1$, then

a) $e_1=4^{s+1}j-4^{s+1}+1=4\left(4^s j-4^s+1\right)-3$.

$\because\ \sigma_2\left[4\left(4^s j-4^s+1\right)-3\right]=4^s 3j-3\times 4^s+1=4\left(4^{s-1}3j-3\times 4^{s-1}+1\right)-3$.

We know from the hypothesis,

$$\sigma_2^{(s-1)}\left(4^s 3j-3\times 4^s+1\right)\in C_1;\ \ \sigma_2^{(s)}\left(4^s 3j-3\times 4^s+1\right)\notin C_1.$$

$$\therefore\ \sigma_2^{\left[1+(s-1)\right]}\left(4^{s+1}j-4^{s+1}+1\right)=\sigma_2^{\left[(s+1)-1\right]}\left[4\left(4^s j-4^s+1\right)\right]\in C_1;$$

$$\sigma_2^{(1+s)}\left(4^{s+1}j-4^{s+1}+1\right)=\sigma_2^{(s+1)}\left(4^{s+1}j-4^{s+1}+1\right)\notin C_1.$$

b) $e_1=4^{s+1}j-4c+1\left(c\in Z^+,1\le c\le 4^s-1\right)$,

$$\because\ \sigma_2\left(4^{s+1}j-4c+1\right)=\sigma_2\left[4\left(4^s j-c+1\right)-3\right]=4^s 3j-3c+1.$$

We know from the hypothesis,

$$\sigma_2^{(t)}\left(4^s 3j-3c+1\right)\notin C_1\ \left(t\in Z^+,t<s\right).$$

$$\therefore\ \sigma_2^{\left[1+(s-1)\right]}\left(4^{s+1}j-4c+1\right)=\sigma_2^{\left[(s+1)-1\right]}\left(4^{s+1}j-4c+1\right)\notin C_1.$$

Therefore, as $m=s+1$, the conclusion is also correct.

By combining (1), (2) and (3), the conclusion is correct for any m ($m\in Z^+$, $m\ge 2$).

Therefore, lemma 4.3 is correct.

We proved lemma 4.1, lemma 4.2 and lemma 4.3, therefore proposition 4is correct.

Proposition 5: If $e_3\ne 3$, then $\sigma_3^{(m)}\left(e_3\right)=3\left(m\in Z^+\right)$.

$$\begin{pmatrix}1 & 3\\ & 7\\ & \cdots\\ & 4k-1\\ & \cdots\end{pmatrix}\Rightarrow\begin{pmatrix}1 & 3\end{pmatrix}$$

Lemma 5.1: $4^x-\dfrac{4^x-1}{3}\in C_3\left(x\in Z^+\right)$.

Prove: $\because\ \left(4^x-\dfrac{4^x-1}{3}\right)+1=4^x-\dfrac{4^x-4}{3}=4\left(4^{x-1}-\dfrac{4^{x-1}-1}{3}\right)$.

$\therefore\ 4^x-\dfrac{4^x-1}{3}=4\left(4^{x-1}-\dfrac{4^{x-1}-1}{3}\right)-1\in C_3$.

$\therefore\ 4^x-\dfrac{4^x-1}{3}\in C_3$.

Therefore, lemma 5.1 is correct.

Lemma 5.2: If $e_3 \neq 4^m j - \frac{4^m - 1}{3} \left(m \in Z^+, m \geq 2,\ j = 1,2,3,4\right)$, then

$$\sigma_3^{(t)}\left(e_3\right) = 4^m j - \frac{4^m - 1}{3}.$$

$$\begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4k-1 \\ & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4^m j - \frac{4^m - 1}{3} \\ & \cdots \end{pmatrix}$$

Prove: $\because\ e_3 = 4k-1 \in C_3$, $\sigma_3\left(n\right) = \frac{n+1}{4}$, thus

$$\sigma_3\left(e_3\right) = \sigma_3\left(4k-1\right) = \frac{\left(4k-1\right)+1}{4} = k.$$

Observe the following facts:

As $k = 4d, 4d-1, 4d-2, 4d-3$ in $4k-1$, we can get:

1) $e_3 = 16d-1$, $\sigma_3\left(16d-1\right) = 4d \in C_4$;

2) $e_3 = 16d-5$, $\sigma_3\left(16d-5\right) = 4d-1 \in C_3$;

3) $e_3 = 16d-9$, $\sigma_3\left(16d-9\right) = 4d-2 \in C_2$;

4) $e_3 = 16d-13$, $\sigma_3\left(16d-13\right) = 4d-3 \in C_1$.

As $d = 4j, 4j-1, 4j-2, 4j-3$ in $16d-5$, we can get:

1) $e_3 = 64j-5$, $\sigma_3\left(64j-5\right) = \sigma_3\left[4\left(16j-1\right)-1\right] = 16j-1 = 4\left(4j\right)-1 \in C_3$;
$\sigma_3^{(2)}\left(64j-5\right) = \sigma_3\left[4\left(4j\right)-1\right] = 4j \notin C_3$.

2) $e_3 = 64j-21$,
$\sigma_3\left(64j-21\right) = \sigma_3\left[4\left(16j-5\right)-1\right] = 16j-5 = 4\left(4j-1\right)-1 \in C_3$;
$\sigma_3^{(2)}\left(64j-21\right) = \sigma_3\left[4\left(4j-1\right)-1\right] = 4j-1 \in C_3$.

3) $e_3 = 64j-37$,
$\sigma_3\left(64j-37\right) = \sigma_3\left[4\left(16j-9\right)-1\right] = 16j-9 = 4\left(4j-2\right)-1 \in C_3$;
$\sigma_3^{(2)}\left(64j-37\right) = \sigma_3\left[4\left(4j-2\right)-1\right] = 4j-2 \notin C_3$.

4) $e_3 = 64j-53$,
$\sigma_3\left(64j-53\right) = \sigma_3\left[4\left(16j-13\right)-1\right] = 16j-13 = 4\left(4j-3\right)-1 \in C_3$;
$\sigma_3^{(2)}\left(64j-53\right) = \sigma_3\left[4\left(4j-3\right)-1\right] = 4j-3 \notin C_3$.

At the same time, we find that

$e_3 = 16j-1, 16j-5, 16j-9, 16j-13$ can be written as a uniform formula as:

$$e_3 = 4^m j - \frac{\left(3c-2\right)4^{m-1} - 1}{3}\left(c = 1,2,3,4\right).$$

As $c = 2$,

$$e_3 = 4^m j - \frac{4^m - 1}{3}.$$

It is inferred that:

a) As $e_3 = 4^m j - \frac{4^m - 1}{3}\left(m \in Z^+, m \geq 2,\ j = 1,2,3,4\right)$,

$$\sigma_3^{(m-1)}(e_3)=\sigma_3^{(m-1)}\left(4^m j-\frac{4^m-1}{3}\right)\in C_3;$$

b) As $e_3=4^m j-\frac{(3c-2)4^{m-1}-1}{3}(c=1,3,4)$,

$$\sigma_3^{(t)}(e_3)=\sigma_3^{(t)}\left[4^m j-\frac{(3c-2)4^{m-1}-1}{3}\right]\notin C_3\left(t\in Z^+, t<m-1,\ j=1,2,3,4\right).$$

Here is the proof by mathematical induction:

1) As $m=2$, it is known that the conclusion is correct from the above facts.

2) It is assumed that the assumption is correct as $m=s$ ($s\in Z^+$, $s\geq 2$), *i.e.*

a) $e_3=4^s j-\frac{4^s-1}{3}\left(s\in Z^+, s\geq 2,\ j=2,4\right)$,

$$\sigma_3^{(s-1)}(e_3)=\sigma_3^{(s-1)}\left(4^s j-\frac{4^s-1}{3}\right)\in C_3;$$

b) $e_3=4^s j-\frac{(3c-2)4^{s-1}-1}{3}(c=1,3,4)$,

$$\sigma_3^{(t)}(e_3)=\sigma_3^{(t)}\left[4^s j-\frac{(3c-2)4^{s-1}-1}{3}\right]\notin C_3$$

$$\left(s\in Z^+, s\geq 2,\ j=1,2,3,4, t\in Z^+, t<s-1\right).$$

3) As $m=s+1$, then

a) $e_3=4^{s+1}j-\frac{4^{s+1}-1}{3}=4\left(4^s j-\frac{4^s-1}{3}\right)-1$.

$$\because\ \sigma_3(e_3)=\sigma_3\left[4\left(4^s j-\frac{4^s-1}{3}\right)-1\right]=4^s j-\frac{4^s-1}{3}\in C_3.$$

We know from the hypothesis,

$$\sigma_3^{(s-1)}(e_3)=\sigma_3^{(s-1)}\left(4^s j-\frac{4^s-1}{3}\right)\in C_3,$$

$$\therefore\ \sigma_3^{[1+(s-1)]}\left(4^{s+1}j-\frac{4^{s+1}-1}{3}\right)=\sigma_3^{[(s+1)-1]}\left[4^{s+1}j-\frac{4^{s+1}-1}{3}\right]\in C_3.$$

b) $e_3=4^{s+1}j-\frac{(3c-2)4^s-1}{3}=4\left(4^s j-\frac{(3c-2)4^{s-1}-1}{3}\right)-1(c=1,3,4)$.

$$\because\ \sigma_3(e_3)=\sigma_3\left[4\left(4^s j-\frac{(3c-2)4^{s-1}-1}{3}\right)-1\right]=4^s j-\frac{(3c-2)4^{s-1}-1}{3}\in C_3.$$

We know from the hypothesis,

$$\sigma_3^{(t)}\left[4^s j-\frac{(3c-2)4^{s-1}-1}{3}\right]\notin C_3,$$

$$\therefore\ \sigma_3^{[1+(s-1)]}\left[4^{s+1}j-\frac{(3c-2)4^s-1}{3}\right]=\sigma_3^{[(s+1)-1]}\left[4^{s+1}j-\frac{(3c-2)4^s-1}{3}\right]\notin C_3.$$

So as $m = s+1$, the assumption is also correct.

By combining (1), (2) and (3), it can be inferred that assumption is correct for any positive integer ($m \in Z^+$, $m \geq 2$).

Therefore, lemma 5.2 is correct.

Lemma 5.3: If $e_3 = 4^m j - \dfrac{4^m - 1}{3} \left(m \in Z^+, j \in Z^+, j = 2,3,4 \right)$, then

$$\sigma_3^{(m)} \left(4^m j - \frac{4^m - 1}{3} \right) = 3.$$

$$\begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4^m j - \dfrac{4^m - 1}{3} \\ & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ & 11 \\ & \cdots \\ & 4^m - \dfrac{4^m - 1}{3} \\ & \cdots \end{pmatrix} \Rightarrow (1 \quad 3)$$

Prove: $\because 4^m j - \dfrac{4^m - 1}{3} = 4\left(4^{m-1} j - \dfrac{4^{m-1} - 1}{3} \right) - 1 \in C_3$, $\sigma_3(n) = \dfrac{n+1}{4}$.

$$\therefore \sigma_3 \left[4\left(4^{m-1} j - \frac{4^{m-1} - 1}{3} \right) - 1 \right] = 4^{m-1} j - \frac{4^{m-1} - 1}{3}.$$

$$\vdots$$

$$\sigma_3 \left[4\left(4^{(m-s)-1} j - \frac{4^{(m-s)-1} - 1}{3} \right) - 1 \right] = 4^{(m-s)-1} j - \frac{4^{(m-s)-1} - 1}{3} \left(s \in Z^+ \right).$$

$$\therefore \sigma_3^{(m)} \left(4^m j - \frac{4^m - 1}{3} \right) = 4^0 j - \frac{4^0 - 1}{3} = j.$$

1) As $j = 3$, the statement is correct;

2) As $j = 2$, $\sigma_1(2) = 3$, the statement is yet correct;

3) As $j = 4$, $\sigma^{(5)}(4) = 3$, the statement is also correct.

Therefore, lemma 5.3 is correct.

Lemma 5.4: If $e_3 = 4^m j - \dfrac{4^m - 1}{3} \left(m \in Z^+, j \in Z^+, j > 4 \right)$, then

$$\sigma_3^{(m)} \left(4^m j - \frac{4^m - 1}{3} \right) = 3.$$

$$\begin{pmatrix} 1 & 3 \\ & 7 \\ & \cdots \\ & 4^m j - \dfrac{4^m - 1}{3} \\ & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ & 11 \\ & \cdots \\ & 4^m - \dfrac{4^m - 1}{3} \\ & \cdots \end{pmatrix} \Rightarrow (1 \quad 3)$$

Prove: $\because$ As $j \in Z^+$, $j > 4$, let $j = X$, $X = 6(n-1) + 4 \left(n \neq 1 \right)$, according to the above process of proof, X must be turned into the number in the form $4^m j' - \dfrac{4^m - 1}{3}$ ($m \in Z^+$, $j' \in Z^+$, $j' \neq 1$), the more X bigger, the more number

of times of operation. With *j'* becoming smaller and smaller, by a finite transformation, until *j'* = 3, 2, 4, by lemma 5.3, the last result must be 3.

Therefore, lemma 5.4 is correct.

According to lemma 5.1, lemma 5.1, lemma 5.3 and lemma 5.4, proposition 5 is correct.

Proposition 6: $\sigma_3(3)=1$.

$$(1 \quad 3)\Rightarrow(1)$$

Prove: $\because \sigma_3(n)=\dfrac{n+1}{4}$,

$\therefore \sigma_3(3)=\dfrac{3+1}{4}=1$.

Therefore, proposition 6 is correct.

According to proposition 4, proposition 5 and proposition 6, the equivalent conjecture:

$$\sigma^{(t)}(n)=1 \ \left(n\in Z^+, n\neq 1, t\in Z^+\right)$$

is correct.

That is, every positive integer *n* can became 1 by a finite times transformation of $\sigma(n)$.

Part Three

Summative proof:

1) If $X\neq 6(n-1)+4$, according to proposition 2, any positive integer that is not $6(n-1)+4$ can be turned to $6(n-1)+4$ after a finite transformation according to *F*(*x*).

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 6 \\ 7 & 8 & 9 & 11 & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 6n-5 & 6n-4 & 6n-3 & 6n-1 & 6n \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ 10 \\ \cdots \\ 6n-2 \\ \cdots \end{pmatrix}$$

2) If $X=6(n-1)+4 \ (n\neq 1)$, we can know from the equivalent conjecture that *X* can be turned to 4 after a finite operation.

$$\begin{pmatrix} \\ 10 \\ \cdots \\ 6n-2 \\ \cdots \end{pmatrix} \Rightarrow (4)$$

3) If $X=6(n-1)+4 \ (n=1)$, then *X* = 4, according to $F_2(X)$, $F_2(4)=\dfrac{4}{2}=2$, $F_2(2)=\dfrac{2}{2}=1$.

$$(4)\Rightarrow 1$$

So far, we proved the 3x + 1 conjecture correct.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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