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A Note on *m*-Möbius Transformations

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Abstract

Lie groups of bi-Möbius transformations are known and their actions on non orientable *n*-dimensional complex manifolds have been studied. In this paper, *m*-Möbius transformations are introduced and similar problems as those related to bi-Möbius transformations are tackled. In particular, it is shown that the subgroup generated by every *m*-Möbius transformation is a discrete group. Cyclic subgroups are exhibited. Vector valued *m*-Möbius transformations are also studied.

Keywords

Möbius Transformation, Complex Manifold, Lie Group

1. Introduction

When investigating Lie groups of Möbius transformations of the Riemann sphere, we were brought in [1] [2] and [3] to the study of some bi-Möbius transformations. These are functions $f:\overline{\mathbb{C}}\times\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ of the form:

$$f(z_1,z_2)=\frac{Az_1z_2+a(1-z_1-z_2)}{a(z_1z_2-z_1-z_2)+A}, \text{ where } a\in\mathbb{C}\setminus\{0,1\} \text{ and } A=a^2-a+1$$

Proposition 1: The function $f(z_1,z_2)=z_1\circ z_2$ is a composition law in $\overline{\mathbb{C}}$ satisfying:

a) $z_1\circ z_2=z_2\circ z_1$ for every $z_1,z_2\in\overline{\mathbb{C}}$

b) $z\circ 1=1\circ z=z$ for every $z\in\overline{\mathbb{C}}$

c) $z\circ(1/z)=1$ for every $z\in\overline{\mathbb{C}}$

d) $(1/z_1)\circ(1/z_2)=1/(z_1\circ z_2)$ for every $z_1,z_2\in\overline{\mathbb{C}}$

e) $z_1\circ(z_2\circ z_3)=(z_1\circ z_2)\circ z_3$ for every $z_1,z_2,z_3\in\overline{\mathbb{C}}$

f) $z_1\circ z_2=a$ if and only if $z_1=a$ or $z_2=a$ and $z_1\circ z_2=1/a$ if and only if $z_1=1/a$ or $z_2=1/a$.

It is obvious that this composition law defines a structure of Abelian group on

$\overline{\mathbb{C}}$ whose unit element is 1 and the inverse of any z is $1/z$. By removing the elements a and $1/a$ we get a subgroup G_a of this group. Since $\overline{\mathbb{C}}\setminus\{a,1/a\}$ is a differentiable manifold on which the group operations are conformal mappings the subgroup G_a is a Lie group.

Theorem 1. For every $z\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ the group $\langle z\rangle$ generated by z is a subgroup of G_a.

Proof: Let us denote $z^{(n+1)}=z\circ z^{(n)}$, every $n\in\mathbb{Z}$, where $z^{(0)}=1$ and $z^{(1)}=z$ and notice that $z\circ z^{(0)}=z\circ 1=z$. An easy induction argument shows that for every $m,n\in\mathbb{Z}$ we have $z^{(m)}\circ z^{(n)}=z^{(m+n)}$ and in particular $z^{(n)}\circ z^{(-n)}=z^{(0)}=1$, which means that indeed $\langle z\rangle$ is a subgroup of G_a. Let us notice that, for $z\neq 1$ we have $z^{(n)}=1$ if and only if $n=0$.

If $z_1\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ then $g_{z_1}(z_2)=f(z_1,z_2)=\dfrac{(Az_1-a)z_2-a(z_1-1)}{a(z_1-1)z_2+A-az_1}$ is a Möbius transformation in z_2 and if $z_2\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ then $h_{z_2}(z_1)=f(z_1,z_2)=\dfrac{(Az_2-a)z_1-a(z_2-1)}{a(z_2-1)z_1+A-az_2}$ is a Möbius transformation in z_1. Indeed, $(Az_1-a)(A-az_1)+a^2(z_1-1)^2=0$ if and only if $z_1=a$ or $z_1=1/a$, which has been excluded and similarly $(Az_2-a)(A-az_2)+a^2(z_2-1)^2=0$ if and only if $z_2=a$ or $z_2=1/a$, which again has been excluded. These properties justify the name of bi-Möbius we have given to $f(z_1,z_2)$.

Couples of bi-Möbius transformations generate mappings $M:\overline{\mathbb{C}}^2\to\overline{\mathbb{C}}^2$ of the form $M(z_1,z_2)=(f_1(z_1,z_2),f_2(z_1,z_2))$, where $f_k(z_1,z_2)=\dfrac{\omega_k z_1z_2-z_1-z_2+1}{z_1z_2-z_1-z_2+\omega_k}$, and $\omega_k=a_k+1/a_k-1$, $k=1,2$. The Proposition 1, f) shows that such a mapping has a set E of four fixed points, namely (a_1,a_2), $(1/a_1,a_2)$, $(a_1,1/a_2)$ and $(1/a_1,1/a_2)$. When restricting M to $\overline{\mathbb{C}}^2\setminus E$ its components are bijective mappings in each one of the variables. Indeed, if $z_1\in\overline{\mathbb{C}}\setminus\{a_k,1/a_k\}$, $k=1,2$, then $f_k(z_1,z_2)$ is a Möbius transformation in z_2, hence it is a bijective mapping of $\overline{\mathbb{C}}$ and since $f_k(z_1,a_k)=a_k$ and $f_k(z_1,1/a_k)=1/a_k$, it is a bijective mapping of $\overline{\mathbb{C}}\setminus\{a_k,1/a_k\}$ onto itself. Similarly, if $z_2\in\overline{\mathbb{C}}\setminus\{a_k,1/a_k\}$, $k=1,2$, then $f_k(z_1,z_2)$ is Möbius in z_1, hence it is a bijective mapping of $\overline{\mathbb{C}}\setminus\{a_k,1/a_k\}$ onto itself. Since $f_k(z_1,z_2)=f_k(z_2,z_1)$ we have $M(z_1,z_2)=M(z_2,z_1)$ hence M is not injective. However, by factorizing $\overline{\mathbb{C}}^2$ with the two elements group $\langle\sigma\rangle$ generated by the symmetry $\sigma(z_1,z_2)=(z_2,z_1)$, M induces a bijective mapping of $\tilde{M}$ of $\overline{\mathbb{C}}^2/\langle\sigma\rangle$ onto $\overline{\mathbb{C}}^2$. Indeed, an easy computation shows that for fixed ω_1 and ω_2 the equations $f_1(z_1,z_2)=b_1$ and $f_2(z_1,z_2)=b_2$ determine uniquily z_1+z_2 and z_1z_2 belonging to $\overline{\mathbb{C}}^2/\langle\sigma\rangle$. We can call this mapping Möbius transformation *of* $\overline{\mathbb{C}}^2/\langle\sigma\rangle$. This is a new concept. We are expecting Möbius transformations of $\overline{\mathbb{C}}^2/\langle\sigma\rangle$ to have similar properties with those of Möbius transformations of $\overline{\mathbb{C}}$, as well as lot of applications. Any such Möbius transformation depends on two complex parameters: ω_1 and ω_2. A composition law in the set of these transformations can be defined in the following way. Let:

$$w_1=\frac{(\omega_1 z_2-1)z_1-z_2+1}{(z_2-1)z_1-z_2+\omega_1},\ w_2=\frac{(\omega_2 z_2-1)z_1-z_2+1}{(z_2-1)z_1-z_2+\omega_2},$$

$$\zeta_1=\frac{(\omega_3 w_2-1)w_1-w_2+1}{(w_2-1)w_1-w_2+\omega_3},\ \zeta_2=\frac{(\omega_4 w_2-1)w_1-w_2+1}{(w_2-1)w_1-w_2+\omega_4}$$

Let us notice that since ζ_1 is a Möbius transformation in w_1 for every $w_2\in\overline{\mathbb{C}}\setminus\{a_3,1/a_3\}$ and w_1 is a Möbius transformation in z_1 for every $z_2\in\overline{\mathbb{C}}\setminus\{a_1,1/a_1\}$, then ζ_1 is a Möbius transformation in z_1 for every $w_2\in\overline{\mathbb{C}}\setminus\{a_3,1/a_3\}$ and $z_2\in\overline{\mathbb{C}}\setminus\{a_1,1/a_1\}$. Analogously it can be shown that ζ_1 is a Möbius transformation in z_2 and that ζ_2 is a Möbius transformation in z_1 and in z_2 when excluding some points, in other words $(\zeta_1,\zeta_2)=(\varphi_1(z_1,z_2),\varphi_2(z_1,z_2))$, where $\varphi_k(z_1,z_2)$ are Möbius transformations in z_1 when some values of z_2 are omitted and they are Möbius transformation in z_2 when some values of z_1 are omitted. Their expressions appear to be more complicated than those of $f_k(z_1,z_2)$. However, they induce Möbius transformation of $\overline{\mathbb{C}}^2/\langle\sigma\rangle$.

The study of these mappings is worthwhile, yet it exceeds the purpose of this note.

2. Multi-Möbius Transformations

The properties e) and f) from Proposition 1 show that $f(z_1,f(z_2,z_3))$ is a Möbius transformation in each one of the variables as long as the other variables belong to $\overline{\mathbb{C}}\setminus\{a,1/a\}$.

To simplify the writing, let us denote $\omega=a+1/a-1$, $s_2^{(2)}=z_1z_2$ and $s_1^{(2)}=z_1+z_2$, $s_3^{(3)}=z_1z_2z_3$, $s_2^{(3)}=z_1z_2+z_1z_3+z_2z_3$, $s_1^{(3)}=z_1+z_2+z_3$, $\cdots$, $s_n^{(n)}=z_1z_2\cdots z_n$, $\cdots$, $s_1^{(n)}=z_1+z_2+\cdots+z_n$. When no confusion is possible we can get rid of the upper subscript. Then, after a little calculation, we get:

$$f_2(z_1,z_2)=f(z_1,z_2)=\frac{\omega s_2-s_1+1}{s_2-s_1+\omega}$$

$$f_3(z_1,z_2,z_3)=f(f_2(z_1,z_2),z_3)=\frac{(1+\omega)s_3-s_2+1}{s_3-s_1+(1+\omega)}$$

$$f_4(z_1,z_2,z_3,z_4)=f(f_3(z_1,z_2,z_3),z_4)=\frac{(\omega^2+\omega-1)s_4-\omega s_3+s_2-s_1+\omega}{\omega s_4-s_3+s_2-\omega s_1+(\omega^2+\omega-1)}$$

A pattern appears regarding the coefficients of s_k in these expressions, namely in every f_m the coefficient of s_k at the numerator is the same as the coefficient of s_{m-k} at the denominator. It is reasonable to believe that this happens due to the properties a), d) and e) listed above. Indeed, we can prove:

Theorem 2. If $f_m(z_1,z_2,\cdots,z_m)=\dfrac{a_0s_n+a_1s_{n-1}+\cdots+a_m}{b_0s_n+b_1s_{n-1}+\cdots+b_m}$, then for every $k=1,2,\cdots,m$ we have $b_k=a_{m-k}$.

The function $f_m(z_1,z_2,\cdots,z_m)$ is a *m*-Möbius transformation, *i.e.* for every

$k = 1, 2, \cdots, m$ the function f_m is a Möbius transformation in z_k for any value of the other variables different of a and $1/a$.

Proof: Let us denote $\sigma_k = \frac{1}{z_1 z_2 \cdots z_k} + \cdots + \frac{1}{z_{m-k+1} z_{m-k+2} \cdots z_m}$ for every $k = 1, 2, \cdots, m$ and suppose that $f_{m-1}(z_1, z_2, \cdots, z_{m-1}) = 1 / f_{m-1}(1/z_1, 1/z_2, \cdots, 1/z_{m-1})$, which is obvious for $m = 3, 4, 5$. We have:

$$f_m(z_1, z_2, \cdots, z_m) = \frac{(\omega z_m - 1) f_{m-1}(z_1, z_2, \cdots, z_{m-1}) + (1 - z_m)}{(z_m - 1) f_{m-1}(z_1, z_2, \cdots, z_{m-1}) + (\omega - z_m)}$$

$$= \frac{(\omega - 1/z_m)\left[1/f_{m-1}(1/z_1, 1/z_2, \cdots, 1/z_{m-1})\right] + (1/z_m - 1)}{(1 - 1/z_m)\left[1/f_{m-1}(1/z_1, 1/z_2, \cdots, 1/z_{m-1})\right] + (\omega/z_m - 1)}$$

$$= \frac{(1/z_m - 1) f_{m-1}(1/z_1, 1/z_2, \cdots, 1/z_{m-1}) + (\omega - 1/z_m)}{(\omega/z_m - 1) f_{m-1}(1/z_1, 1/z_2, \cdots, 1/z_{m-1}) + (1 - 1/z_m)}$$

$$= 1/f_m(1/z_1, 1/z_2, \cdots, 1/z_m)$$

If $f_m(z_1, z_2, \cdots, z_m) = \frac{a_0 s_n + a_1 s_{n-1} + \cdots + a_m}{b_0 s_n + b_1 s_{n-1} + \cdots + b_m}$, then

$$f_m(1/z_1, 1/z_2, \cdots, 1/z_m) = \frac{a_0 \sigma_m + a_1 \sigma_{m-1} + \cdots + a_m}{b_0 \sigma_m + b_1 \sigma_{m-1} + \cdots + b_m} = \frac{a_m s_n + \cdots + a_1 s_1 + a_0}{b_m s_m + \cdots + b_1 s_1 + b_0}$$

$$= 1/f_m(z_1, z_2, \cdots, z_m) = \frac{b_0 s_m + b_1 s_{m-1} + \cdots + b_m}{a_0 s_m + a_1 s_{m-1} + \cdots + a_m}$$

These last equalities are possible if and only if $b_k = a_{m-k}$. Simplifications may occur, as in the case of f_5 below, yet they do not alter the symmetry of the coefficients.

On the other hand, if we write

$$f_m(z_1, z_2, \cdots, z_m) = f\left(f_{m-1}(z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_{m-1}), z_k\right),$$

it is obvious that f_m is a Möbius transformation in z_k as long as the other variables do not take the values a and $1/a$.

We notice that in order to find exactly what the coefficients of s_m are for a given m, we need to iteratively compute $f_j(z_1, z_2, \cdots, z_j)$ for all the values of j from 2 to m. The expressions of these coefficients as functions of ω become more and more complicated. To illustrate this affirmation as well as the Theorem 1, let us notice that an elementary computation gives:

$$f_5(z_1, z_2, \cdots, z_5) = \frac{\omega(\omega+2)s_5 - (\omega+1)s_4 + s_3 - s_1 + (\omega+1)}{(\omega+1)s_5 - s_4 + s_2 - (\omega+1)s_1 + \omega(\omega+2)}$$

$$f_6(z_1, z_2, \cdots, z_6)$$
$$= \frac{(\omega^3 + 2\omega^2 - \omega - 1)s_6 - (\omega^2 + \omega - 1)s_5 + \omega s_4 - s_3 + s_2 - \omega s_1 + (\omega^2 + \omega - 1)}{(\omega^2 + \omega - 1)s_6 - \omega s_5 + s_4 - s_3 + \omega s_2 - (\omega^2 + \omega - 1)s_1 + (\omega^3 + 2\omega^2 - \omega - 1)}$$

$$f_7(z_1, z_2, \cdots, z_7)$$
$$= \frac{(\omega^3 + 3\omega^2 + \omega - 1)s_7 - \omega(\omega+2)s_6 + (\omega+1)s_5 - s_4 + s_2 - (\omega+1)s_1 + \omega(\omega+2)}{\omega(\omega+2)s_7 - (\omega+1)s_6 + s_5 - s_3 + (\omega+1)s_2 - \omega(\omega+2)s_1 + (\omega^3 + 3\omega^2 + \omega - 1)}$$

$$f_8\left(z_1,z_2,\cdots,z_8\right) = \frac{\omega\left(\omega^3+3\omega^2-3\right)s_8-\left(\omega^3+2\omega^2-\omega-1\right)s_7+\left(\omega^2+\omega-1\right)s_6-\omega s_5+s_4-s_3+\omega s_2-\left(\omega^2+\omega-1\right)s_1+\left(\omega^3+2\omega^2-\omega-1\right)}{\left(\omega^3+2\omega^2-\omega-1\right)s_8-\left(\omega^2+\omega-1\right)s_7+\omega s_6-s_5+s_4-\omega s_3+\left(\omega^2+\omega-1\right)s_2-\left(\omega^3+2\omega^2-\omega-1\right)s_1+\omega\left(\omega^3+3\omega^2-3\right)}$$

3. Lie Groups of *m*-Möbius Transformations in $\overline{\mathbb{C}}$

For arbitrary *z*, $z_k \in \overline{\mathbb{C}}\setminus\{a,1/a\}$, $k=1,2,\cdots,m$, let us denote $g_z = f_2\left(z, f_m\left(z_1,z_2,\cdots,z_m\right)\right)$, which is a set G_m of *m*-Möbius transformations.

By Proposition 1 (see also [3]), $\overline{\mathbb{C}}\setminus\{a,1/a\}$ endowed with the composition law $z\circ w = f_2(z,w)$ is an Abelian group with the unit element 1 and for which the inverse element of *z* is z^{-1}. Moreover, an analytic atlas can be defined on $\overline{\mathbb{C}}\setminus\{a,1/a\}$ making it a differentiable manifold on which the group operations are conformal mappings and therefore this is a Lie group $\mathbf{G}_a$. Basic knowledge about Lie groups can be found in [4]. A composition law in $\mathbf{G}_m$ can be defined by $g_z\times g_w = g_{z\circ w}$. Then, for every *z*, $z_k\in\overline{\mathbb{C}}\setminus\{a,1/a\}$, $k=1,2,\cdots,m$ we have $g_z\times g_1 = g_{z\circ 1} = g_z$ and $g_z\times g_{z^{-1}} = g_{z\circ z^{-1}} = g_1$, hence g_1 is the unit element of this law and the inverse of g_z is $g_{z^{-1}}$. Moreover, $g_z\times g_w = g_w\times g_z$.

Theorem 3. The set of *m*-Möbius transformations $\mathbf{G}_m = \left\{g_z, z\in\overline{\mathbb{C}}\setminus\{a,1/a\}\right\}$ with the composition law $g_z\times g_w = g_{z\circ w}$ is a Lie group.

Proof: Indeed, the properties we listed above show that $\mathbf{G}_m$ is an Abelian group. It is isomorphic with $\mathbf{G}_a$ under the mapping $\chi(z) = g_z$ since $\chi(z\circ w) = g_{z\circ w} = g_z\times g_w$ and $\chi(1) = g_1$. A topology on G_m can be defined as the image by χ of the natural topology on $\overline{\mathbb{C}}\setminus\{a,1/a\}$. This makes $\mathbf{G}_m$ a differentiable manifold on which the composition law $g_z\times g_w = g_{z\circ w}$ defines a structure of Lie group. Different complex numbers *a* define different Lie groups of *m*-Möbius transformations, yet all of these groups are obviously isomorphic, and therefore there is no need to specify the numbers *a*, or ω when indicating such a group.

Let $\zeta\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ be arbitrary and for every $k\in\mathbb{Z}$ let us denote $\zeta^{(k+1)} = \zeta\circ\zeta^{(k)}$, where $\zeta^{(0)} = 1$. It is obvious that for every $k,l\in\mathbb{Z}$ we have $\zeta^{(k)}\circ\zeta^{(l)} = \zeta^{(k+l)}$ and then $g_{\zeta^{(k)}}\times g_{\zeta^{(l)}} = g_{\zeta^{(k+l)}}$. In particular, $g_{\zeta^{(k)}}\times g_{\zeta^{(-k)}} = g_{\zeta^{(0)}} = g_1$, hence the group $\langle g_\zeta\rangle$ generated by g_ζ is a subgroup of $\mathbf{G}_m$.

Theorem 4. For every $\zeta\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ the group $\langle g_\zeta\rangle$ is a discrete subgroup of $\mathbf{G}_m$.

Proof: Indeed, if $\zeta = 1$ then $\zeta^{(k)} = 1$ for every $k\in\mathbb{Z}$. If $\zeta\neq 1$ then we have that $\zeta^{(k+1)} = \zeta^{(k)}\circ\zeta\neq\zeta^{(k)}$. By using the expressions we have found for different f_n we can easily check that there are values of $z\neq 1$ for which $f_n(z,z,\cdots,z) = 1$. For example, if $n=3$, $z^{(3)} = f_3(z,z,z) = 1$ for every root of the equation $\omega z^2 + (\omega-3)z + \omega = 0$. Also, if $n=4$, then $z^{(4)} = 1$ for every root of the equation $(\omega+1)z^4 - 4z^3 + 4z - (\omega+1) = 0$ etc. It is obvious that for such values $\zeta = z$ the group $\langle\zeta\rangle$ is a cyclic one and so is the group $\langle g_\zeta\rangle$, hence it is a discrete subgroup of $\mathbf{G}_n$.

If $\zeta^{(k)}\neq 1$ for every $k\neq 0$, then $\langle\zeta\rangle$ is not cyclic and $\zeta^{(k+1)} = \zeta^{(k)}\circ\zeta\neq\zeta$

for every $k \neq 0$. Moreover, if $k \neq j$, then $\zeta^{(k-j)} = \zeta^{(k)} \circ \zeta^{(-j)} \neq 1$, hence $\zeta^{(k)} \neq \zeta^{(j)}$. Suppose that there is a subsequence ($\zeta^{(n_k)}$) of distinct elements such that $\lim_{n_k \to \infty} \zeta^{(n_k)} = \zeta_0$. Let us split the sequence (n_k) into two infinite subsequences (n_{k_1}) and (n_{k_2}) where $n_{k_1} + n_{k_2} = n_k$. Then $\zeta^{(n_k)} = \zeta^{(n_{k_1})} \circ \zeta^{(n_{k_2})}$ and $\zeta_0 = \lim_{n_k \to \infty} \zeta^{(n_k)} = \lim_{n_{k_1} \to \infty} \zeta^{(n_{k_1})} \circ \lim_{n_{k_2} \to \infty} \zeta^{(n_{k_2})} = \zeta_0 \circ \zeta_0$, which is possible if and only if $\zeta_0 = 1$, therefore $\lim_{n_k \to \infty} \zeta^{(n_k)} = 1$. For every $j \in \mathbb{Z}$, ($\zeta^{(n_k+j)}$) is a subsequence of ($\zeta^{(k)}$) and $\lim_{n_k \to \infty} \zeta^{(n_k+j)} = \zeta^{(j)}$, which again is possible only if $\zeta^{(j)} = 1$. Yet $\zeta^{(j)} \neq 1$ if $j \neq 0$ and this shows that there is no convergent subsequence ($\zeta^{(n_k)}$) of distinct elements. Hence the subgroup $\langle \zeta \rangle$ is discrete and so is $\langle g_\zeta \rangle$.

Corollary 1. For every $\zeta \in \overline{\mathbb{C}} \setminus \{a, 1/a\}$ the subgroup $\langle g_\zeta \rangle$ generated by ζ acts freely and properly discontinuously on G_m by left and right translations.

4. Vector Valued *m*-Möbius Transformations

We can extend the concept of *m*-Möbius transformation to $\overline{\mathbb{C}}^n$ in the following way. For $a_k \in \overline{\mathbb{C}} \setminus \{0,1\}$, let $\omega_k = a_k + 1/a_k - 1$, $k = 1, 2, \cdots, n$, and let us build the *m*-Möbius transformations $f_{m,k}(z_1, z_2, \cdots, z_m)$ as in Section 2 by using ω_k instead of ω. We will study the function $\mathbf{f} : \overline{\mathbb{C}}^m \to \overline{\mathbb{C}}^n$ defined by

$$\mathbf{f}(\mathbf{z}) = \left(f_{m,1}(\mathbf{z}), f_{m,2}(\mathbf{z}), \cdots, f_{m,n}(\mathbf{z})\right), \text{ where } \mathbf{z} = (z_1, z_2, \cdots, z_m)$$

Every $f_{m,k}$ is a *m*-Möbius transformation of the form $f_{m,k}(z_1, z_2, \cdots, z_m) = \dfrac{a_0(\omega_k)s_m + a_1(\omega_k)s_{m-1} + \cdots + a_m(\omega_k)}{a_m(\omega_k)s_m + a_{m-1}(\omega_k)s_{m-1} + \cdots + a_0(\omega_k)}$, where s_j are the symmetric functions defined in Section 2, hence $\mathbf{f}$ is a vector valued function whose every component is a *m*-Möbius transformation. For $w \in \Omega = \overline{\mathbb{C}} \setminus \{a_k, 1/a_k \mid k = 1, 2, \cdots, n\}$ let $g_w^{(k)}(\mathbf{z}) = f_2(w, f_{m,k}(\mathbf{z})) = \dfrac{\omega_k w f_{m,k}(\mathbf{z}) - w - f_{m,k}(\mathbf{z}) + 1}{w f_{m,k}(\mathbf{z}) - w - f_{m,k}(\mathbf{z}) + \omega_k}$, $k = 1, 2, \cdots, n$. Then $\mathbf{\Gamma}_\Omega = \left\{\mathbf{g}_w(\mathbf{z}) = \left(g_w^{(1)}(\mathbf{z}), g_w^{(2)}(\mathbf{z}), \cdots, g_w^{(n)}(\mathbf{z})\right)\right\}$ is a set of vector valued functions whose components are all *m*-Möbius transformations.

Theorem 5. The composition law $\mathbf{g}_z \cdot \mathbf{g}_w = g_{z \circ w}$ induces a structure of Abelian group on $\mathbf{\Gamma}_\Omega$ having the unit element $\mathbf{g}_1$ and such that the inverse element of $\mathbf{g}_z$ is $\mathbf{g}_{z^{-1}}$.

Proof: Indeed, $\mathbf{g}_z \cdot \mathbf{g}_w = \mathbf{g}_{z \circ w} = \mathbf{g}_{w \circ z} = \mathbf{g}_w \cdot \mathbf{g}_z$, for every $z, w \in \Omega$, $\mathbf{g}_z \cdot \mathbf{g}_1 = \mathbf{g}_{z \circ 1} = \mathbf{g}_z$ for every $z \in \Omega$ and $\mathbf{g}_z \cdot \mathbf{g}_{z^{-1}} = \mathbf{g}_{z \circ z^{-1}} = \mathbf{g}_1$ for every $z \in \Omega$, since the same is true for every $g_z^{(k)}$ for every *k*, by Theorem 3, hence $\mathbf{g}_{z^{-1}} = \mathbf{g}_z^{-1}$.

Theorem 6. The mapping $\mathbf{c} : \Omega \to \mathbf{\Gamma}_\Omega$ defined by $\mathbf{c}(z) = \mathbf{g}_z$ endows $\mathbf{\Gamma}_\Omega$ with a Lie group structure.

Proof: The set $\mathbf{\Gamma}_\Omega$ with the image topology induced by $\mathbf{c}$ *is* a differentiable manifold and $\mathbf{c}$ is a diffeomorphism. On the other hand, the group operations are conformal mappings and therefore of class C^∞. Therefore the mapping $\mathbf{c}$ is a *Lie group* isomorphism.

Let us notice that

$$f_{m,k}(\mathbf{1}) = f_{m,k}(1,1,\cdots,1) = \frac{a_0(\omega_k)C_m^m + a_1(\omega_k)C_m^{m-1} + \cdots + a_m(\omega_k)C_m^0}{a_m(\omega_k)C_m^m + a_{m-1}(\omega_k)C_m^{m-1} + \cdots + a_0(\omega_k)C_m^0} = 1,$$

hence $g_1^{(k)}(\mathbf{1}) = f_2(1, f_{m,k}(\mathbf{1})) = 1$, $k = 1,2,\cdots,n$, hence $\mathbf{g}_1(\mathbf{1}) = \mathbf{1}$.

When $m = n$ the function $\mathbf{f}$ is a mapping of $\overline{\mathbb{C}}^m$ onto itself. It has a set E of 2^m fixed points. Indeed, every point $(z_1, z_2, \cdots, z_m)$ where z_k is either a_k or $1/a_k$ is a fixed point of $\mathbf{f}$.

The components of $\mathbf{f}$ are *m*-Möbius transformations of $\overline{\mathbb{C}}$ in every variable z_j if the other variables belong to Ω.

Since, for fixed ω_k, every $f_{m,k}$ depends only on the symmetric sums s_j, the values of $f_{m,k}(z_1, z_2, \cdots, z_m)$ remain the same when making a permutation of the variables $z_1, z_2, \cdots, z_m$. Therefore $\mathbf{f}$ is not an injective function. Let $\wp_m$ be the group of permutations of $z_1, z_2, \cdots, z_m$ and let $\overline{\mathbb{C}}^m / \wp_m$ be the factor space of $\overline{\mathbb{C}}^m$ with respect to this group. The function $\mathbf{f}$ induces a bijective mapping $\tilde{\mathbf{f}}$ of $\overline{\mathbb{C}}^m / \wp_m$ onto $\overline{\mathbb{C}}^m$. We can call it Möbius transformation of $\overline{\mathbb{C}}^m / \wp_m$. A lot of questions remain to be answered about these transformations.

5. Conclusions

To emphasize the importance of the topic we dealt with in this paper, let us present a citation from [5]: "*Although more than* 150 *years have passed since August Ferdinand Möbius first studied the transformations that now bear his name*, *it is fair to say that the rich vein of knowledge which he hereby exposed is still far from being exhausted*".

The Möbius transformations are a chapter in any book of complex analysis. They have remarkable geometric properties and a lot of applications. The whole theory of automorphic functions is based on these transformations and they have surprising connections with the relativity theory. The concept of multi-Möbius transformation appears for the first time here and is related to the theory of Lie groups, which has itself deep connections with the Physics.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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