Advances in Pure Mathematics, 2021, 11, 785-790
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

On the Uniform and Simultaneous Approximations of Functions

Mansour Alyazidi

Department of Mathematics, King Saud University, Riyadh, Saudi Arabia
Email: yazidi@ksu.edu.sa

How to cite this paper: Alyazidi, M. (2021) On the Uniform and Simultaneous Approximations of Functions. *Advances in Pure Mathematics*, **11**, 785-790.
https://doi.org/10.4236/apm.2021.1110052

Received: September 9, 2021
Accepted: October 10, 2021
Published: October 13, 2021

Abstract

We consider the relation between the simultaneous approximation of two functions and the uniform approximation to one of these functions. In particular, F_1 and F_2 are continuous functions on a closed interval $[a,b]$, S is an n-dimensional Chebyshev subspace of $C[a,b]$ and s_1^* & s_2^* are the best uniform approximations to F_1 and F_2 from S respectively. The characterization of the best approximation solution is used to show that, under some restrictions on the point set of alternations of $F_1 - s_1^*$ and $F_2 - s_2^*$, s_1^* or s_2^* is also a best $A(1)$ simultaneous approximation to F_1 and F_2 from S with $F_1 \geq F_2$ and $n = 2$.

Keywords

Simultaneous Approximation, Uniform Approximation, Relation, Straddle Points, Alternation

1. Introduction

The interest in the simultaneous approximation started long ago [1] [2] [3] [4]. This paper concerned with the relation between the simultaneous approximation and the uniform approximation. The setting is as follows. Let $C[a,b]$ be the set of all real-valued continuous functions defined on the closed interval $[a,b]$ with the uniform norm $\|.\|$.

For $f \in C[a,b]$,

$$\|f\| = \max\{|f(x)|, x \in [a,b]\}.$$

The norms $\|F\|_{A(p)}$, $1 \leq p \leq \infty$, on $E = C[a,b] \times C[a,b]$ are defined as follows:

For $F=(F_1,F_2)\in E$

$$\|F\|_{A(\infty)}=\max\{F_1,F_2\}$$

$$\|F\|_{A(p)}=\left[\|F_1\|^p+\|F_2\|^p\right]^{\frac{1}{p}},\ 1\le p<\infty.$$

Now if S is an n-dimensional subspace of $C[a,b]$, then $U=\{(s,s):s\in S\}$ is an n-dimensional subspace of E and there exist $u^*=(s^*,s^*)$ and $v^*=(t^*,t^*)$ where $s^*,t^*\in S$ such that:

$$\begin{aligned}\left\|F-u^*\right\|_{A(\infty)}&=\inf_{u\in U}\|F-u\|_{A(\infty)}\\&=\inf_{s\in S}\max\{\|F_1-s\|,\|F_2-s\|\}\\&=\left\|F_k-s^*\right\|,\ k=1\text{ or }2.\end{aligned}$$

Such s^* is called a best $A(\infty)$ simultaneous approximation to $F=(F_1,F_2)$ from S. The set of all best $A(\infty)$ simultaneous approximations to F from S will be denoted by $P_S(F,\infty)$.

For $1\le p<\infty$,

$$\begin{aligned}\left\|F-v^*\right\|_{A(p)}&=\inf_{u\in U}\|F-u\|_{A(p)}\\&=\inf_{s\in S}\left\{\left[\|F_1-s\|^p+\|F_2-s\|^p\right]^{\frac{1}{p}}\right\}\\&=\left[\left\|F_1-t^*\right\|^p+\left\|F_2-t^*\right\|^p\right]^{\frac{1}{p}}.\end{aligned}$$

t^* is called a best $A(p)$ simultaneous approximation to $F=(F_1,F_2)$ from S. The set $P_S(F,p)$ denotes the set of all best $A(p)$ simultaneous approximation to F from S. And $P_S(F_k)$ is the set of all best uniform approximation to F_k from S, $k\in\{1,2\}$.

We are interested in the relation between the simultaneous approximation and the uniform approximation; in section two, we will show under certain conditions, that if $s_k^*\in P_S(F_k)$ then $s_k^*\in P_S(F,1)$, $k\in\{1,2\}$.

Definition 1 A point $t\in[a,b]$ is called a straddle point for two functions f and g in $C[a,b]$ if there exists $\sigma=\pm1$ such that

$$\|f\|=\sigma f(t),\ \|g\|=-\sigma g(t).$$

Definition 2 The functions f and $g\in C[a,b]$ are said to have d alternations on $[a,b]$ if there exists $d+1$ distinct points $x_1<\cdots<x_{d+1}$ in $[a,b]$ such that for some $\sigma=\pm1$,

$$f(x_i)=\sigma\|f\|,\ \text{if } i \text{ is odd}$$

$$g(x_i)=-\sigma\|g\|,\ \text{if } i \text{ is even}$$

or

$$g(x_i)=\sigma\|g\|,\ \text{if } i \text{ is odd}$$

$$f(x_i)=-\sigma\|f\|,\ \text{if } i \text{ is even}$$

We follow [5] [6] [7] [8] for the notations and the terminology of this section which will be used throughout this paper. The uniform approximation theory can be found in [9] [10]. Theorems 1 and 2 of this section and the remark thereafter which are needed for our analysis are direct consequences of theorems 1 and 3 of [6].

Theorem 1 Let S be an n-dimensional subspace of $C[a,b]$ which contains a nonzero constant, $F=(F_1,F_2)\in E$ then:

(a) $s^*\in P_S(F,1)$ if and only if there exists subsets $X_1=\{x_i, i\in I_1\}$, $X_2=\{x_i, i\in I_2\}$ of $[a,b]$ and positive numbers $\lambda_i, i\in I_1, \mu_i\in I_2$ with

$$\sum_{i\in I_1}\lambda_i=\sum_{i\in I_2}\mu_i=1$$

such that

$$\theta_i\left(F_1(x_i)-s^*(x_i)\right)=\left\|F_1-s^*\right\|,\ i\in I_1,$$

$$\theta_i\left(F_2(x_i)-s^*(x_i)\right)=\left\|F_2-s^*\right\|,\ i\in I_2,$$

$$\sum_{i\in I_1}\theta_i\lambda_i s(x_i)+\sum_{i\in I_2}\theta_i\mu_i s(x_i)=0\quad\forall s\in S,$$

$$\theta_i=\pm1.$$

(b) If $s^*\in P_s(F,1)$ with $\left\|F_1-s^*\right\|=\left\|F_2-s^*\right\|$ then $s^*\in P_S(F,p)$ for all p, $1<p\le\infty$.

Theorem 2 Let S be an n-dimensional Haar subspace of $C[a,b]$, if $F_1\ge F_2$ on $[a,b]$ then $s^*\in P_s(F,\infty)$ if and only if F_1-s^* & F_2-s^* have a straddle point or n alternations on $[a,b]$ with $\left\|F_1-s^*\right\|=\left\|F_2-s^*\right\|$. Furthermore, if F_1-s^* & F_2-s^* have n alternations on $[a,b]$ then s^* is unique.

Remark If $t\in[a,b]$ is a straddle point for F_1-s^* & F_2-s^*, $F_1\ge F_2$ on $[a,b]$ then

$$(F_1-F_2)(t)=(F_1-s^*)(t)+(F_2-s^*)(t)=\left\|F_1-s^*\right\|+\left\|F_2-s^*\right\|\ge\left\|F_1-F_2\right\|.$$

This implies that $(F_1-F_2)(t)=\left\|F_1-F_2\right\|$ and

$$\left\|F_1-s^*\right\|+\left\|F_2-s^*\right\|=\left\|F_1-F_2\right\|\le\left\|F_1-s\right\|+\left\|F_2-s\right\|\quad\forall s\in S.$$

Hence $s^*\in P_S(F,1)$.

2. The Main Result

Theorem 3 Let $s_k^*\in P_S(F_k)$, where $F_k\in C[a,b]$, $k\in\{1,2\}$, $F_1\ge F_2$ on $[a,b]$, $F=(F_1,F_2)$ and S is a 2-dimensional Chebyshev subspace of $C[a,b]$ containing a nonzero constant function. And let $X=\{a=x_1<x_2<x_3=b\}$ be the alternating set for $F_1-s_1^*$, $Y=\{a=y_1<y_2<y_3=b\}$ be the alternating set for $F_2-s_2^*$.

(i) If $\left(F_1(x_1)-s_1^*(x_1)\right)=\left\|F_1-s_1^*\right\|$ and $\left(F_2(y_1)-s_2^*(y_1)\right)=\left\|F_2-s_2^*\right\|$, then $s_1^*\in P_S(F,1)$.

(ii) If $\left(F_1(x_1)-s_1^*(x_1)\right)=-\left\|F_1-s_1^*\right\|$ and $\left(F_2(y_1)-s_2^*(y_1)\right)=-\left\|F_2-s_2^*\right\|$, then $s_2^*\in P_S(F,1)$.

Proof

(i) suppose that $\left(F_1\left(x_1\right)-s_1^*\left(x_1\right)\right)=\left\|F_1-s_1^*\right\|$ and $\left(F_2\left(y_1\right)-s_2^*\left(y_1\right)\right)=\left\|F_2-s_2^*\right\|$, since $-F_2 \geq -F_1$ then

$$\left(s_1^*\left(x_2\right)-F_2\left(x_2\right)\right) \geq \left(s_1^*\left(x_2\right)-F_1\left(x_2\right)\right)=\left\|F_1-s_1^*\right\|$$

and if $x \in [a,b]$ is such that $\left(F_2-s_1^*\right)(x) \geq 0$, then

$$\left\|F_1-s_1^*\right\| \geq \left(F_1-s_1^*\right)(x) \geq \left(F_2-s_1^*\right)(x) \geq 0 .$$

Hence there exists a $\gamma \in [a,b]$ such that

$$\left\|F_2-s_1^*\right\|=-\left(F_2-s_1^*\right)(\gamma).$$

If $\gamma=a$ or $\gamma=b$ then γ is a straddle point for $F_1-s_1^*$ & $F_2-s_1^*$ which implies that $s_1^* \in P_S(F,1)$.

If $a<\gamma<b$ then taking $x_1=z_1, \gamma=z_2, x_3=z_3$ we have:

$$\left(F_1-s_1^*\right)\left(z_1\right)=\left(F_1-s_1^*\right)\left(z_3\right)=\left\|F_1-s_1^*\right\|,$$

$$-\left(F_2-s_1^*\right)\left(z_2\right)=\left\|F_2-s_1^*\right\|,$$

$$a \leq z_1<z_2<z_3 \leq b .$$

Now, since S is a Chebyshev subspace of dimension 2, there exists $\mu_i>0, i \in \{1,2,3\}$ such that

$$\mu_1 s\left(z_1\right)-\mu_2 s\left(z_2\right)+\mu_3 s\left(z_3\right)=0 \quad \forall s \in S$$

because $1 \in S$, $\mu_2=\mu_1+\mu_3$ and setting $\omega_i=\dfrac{\mu_i}{\mu_2}, i \in \{1,2,3\}$ we have $\omega_1 s\left(z_1\right)-\omega_2 s\left(z_2\right)+\omega_3 s\left(z_3\right)=0 \quad \forall s \in S$ where $\omega_2=\omega_1+\omega_3=1$ and from theorem 1 $s_1^* \in P_S(F,1)$.

ii) If $\left(F_1\left(x_1\right)-s_1^*\left(x_1\right)\right)=-\left\|F_1-s_1^*\right\|$ and $\left(F_2\left(y_1\right)-s_2^*\left(y_1\right)\right)=-\left\|F_2-s_2^*\right\|$, since $F_1 \geq F_2$ then

$$\left(F_1\left(y_2\right)-s_2^*\left(y_2\right)\right) \geq \left(F_2\left(y_2\right)-s_2^*\left(y_2\right)\right)=\left\|F_2-s_2^*\right\|$$

and if $x \in [a,b]$ is such that $\left(s_2^*-F_1\right)(x) \geq 0$, then

$$\left\|F_2-s_2^*\right\| \geq \left(s_2^*-F_2\right)(x) \geq \left(s_2^*-F_1\right)(x) \geq 0 .$$

Hence there exists a $\gamma \in [a,b]$ such that

$$\left\|F_1-s_2^*\right\|=\left(F_1-s_2^*\right)(\gamma).$$

If $\gamma=a$ or $\gamma=b$ then γ is a straddle point for $F_1-s_2^*$ & $F_2-s_2^*$ which implies that $s_2^* \in P_S(F,1)$.

If $a<\gamma<b$ then taking $y_1=z_1, \gamma=z_2, y_3=z_3$ we have:

$$\left(F_2-s_2^*\right)\left(z_1\right)=\left(F_2-s_2^*\right)\left(z_3\right)=-\left\|F_1-s_1^*\right\|,$$

$$\left(F_1-s_2^*\right)\left(z_2\right)=\left\|F_1-s_2^*\right\|,$$

$$a \leq z_1<z_2<z_3 \leq b .$$

Now, since S is a Chebyshev subspace of dimension 2, there exists

$\mu_i > 0, i \in \{1,2,3\}$ such that

$$-\mu_1 s(z_1) + \mu_2 s(z_2) - \mu_3 s(z_3) = 0 \quad \forall s \in S$$

because $1 \in S$, $\mu_2 = \mu_1 + \mu_3$ and setting $\omega_i = \frac{\mu_i}{\mu_2}, i \in \{1,2,3\}$ we have $-\omega_1 s(z_1) + \omega_2 s(z_2) - \omega_3 s(z_3) = 0 \quad \forall s \in S$ where $\omega_2 = \omega_1 + \omega_3 = 1$ and from theorem 1 $s_2^* \in P_S(F,1)$ and the theorem is proved.

The following example shows that conditions (i) & (ii) in theorem 3 are necessary conditions.

Example 1 $S = \operatorname{span}\{1,x\}$ is a Chebyshev subspace of $C[0,1]$ and $s_1^* = \frac{1}{8} + x$ is the best uniform approximation to $F_1 = \sqrt{x}$, $s_2^* = \frac{-1}{8} + x$ is the best uniform approximation to $F_2 = x^2$, $F_1 \geq F_2$ on $[0,1]$, $s_1^* \notin P_S(F,1)$ and $s_2^* \notin P_S(F,1)$.

It is possible, under the assumptions of theorem 3 that both s_1^* and s_2^* belong to the set of best $A(1)$ simultaneous approximation as illustrated in the following example

Example 2 $S = \operatorname{span}\{1,x\}$ is a Chebyshev subspace of $C[0,1]$ and $s_1^* = \frac{-1}{8} + x$ is the best uniform approximation to $F_1 = x^2$, $s_2^* = \frac{-1}{3\sqrt{3}} + x$ is the best uniform approximation to $F_2 = x^3$, $F_1 \geq F_2$ on $[0,1]$.

$s_1^*, s_2^* \in P_S(F,1)$. Furthermore $s_2^* = \frac{-1}{3\sqrt{3}} + x$ is the unique best $A(\infty)$ simultaneous approximation to $F = (F_1, F_2)$ from *S*.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Dunham, C.B. (1967) Simultaneous Chebyshev Approximation of Two Functions on an Interval. *Proceedings of the AMS*, **18**, 472-477. https://doi.org/10.1090/S0002-9939-1967-0212463-6

[2] Watson, G.A. (1993) A Characterization of Best Simultaneous Approximation. *Journal of Approximation Theory*, **75**, 175-182. https://doi.org/10.1006/jath.1993.1097

[3] Huotari, R. and Shi, J. (1995) Simultaneous Approximation from Convex Sets. *Computers & Mathematics with Applications*, **30**, 197-206. https://doi.org/10.1016/0898-1221(95)00097-6

[4] Pinkus, A. (1997) Uniqueness in Vector-Valued Approximation. *Journal of Approximation Theory*, **91**, 17-92. https://doi.org/10.1006/jath.1993.1030

[5] Asiry, M. and Watson, G.A. (1999) On Solution of a Class of Best Simultaneous Approximation Problems. *International Journal of Computer Mathematics*, **75**, 413-425. https://doi.org/10.1080/00207169908804818

[6] Asiry, M. and Watson, G.A. (2000) Simultaneous Approximation from Chebyshev and Weak Chebyshev Spaces. *Communications in Applied Analysis*, **4**, No. 3.

[7] Alyazidi-Asiry, M. (2016) Adjoining a Constant Function to N-Dimensional Chebyshev Space. *Journal of Function Spaces*, **2016**, Article ID: 4813979. https://doi.org/10.1155/2016/4813979

[8] Alyazidi-Asiry, M. (2017) Extending a Chebyshev Subspace to a Weak Chebyshe Subspace of Higher Dimension and Related Results. *Journal of Applied and Computational Mathematics*, **6**, Article ID: 1000347. https://doi.org/10.4172/2168-9679.1000347

[9] Cheney, E.W. (1966) Introduction to Approximation Theory. McGraw-Hill, New York, London.

[10] Watson, G.A. (1980) Approximation Theory and Numerical Methods. John Wiley & Sons, Chichester.