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Non-Spectral Problem of Self-Affine Measures in $\mathbb{R}^3$

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Abstract

The problems of determining the spectrality or non-spectrality of a measure have been received much attention in recent years. One of the non-spectral problems on $\mu_{M,D}$ is to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$. In the present paper, we establish some relations inside the zero set $Z(\hat{\mu}_{M,D})$ by the Fourier transform of the self-affine measure $\mu_{M,D}$. Based on these facts, we show that $\mu_{M,D}$ is a non-spectral measure and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best possible. This extends several known conclusions.

Keywords

Iterated Function System (*IFS*), Self-Affine Measure, Orthogonal Exponentials

1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, that is to let one with all eigenvalues $|\lambda_i(M)| > 1$ and $D \subset \mathbb{Z}^n$ be a finite subset of cardinality $|D|$. The probability measure $\mu_{M,D}$ associated with an iterated function system

$$\left\{\phi_d(x) = M^{-1}(x+d)\right\}_{d\in D}$$

is uniquely determined. It only depends upon an expanding matrix M and a finite digit set D. The attractor $T := T(M,D)$ is unique non-empty compact set satisfying

$$MT = \bigcup_{d\in D}(T+d),$$

and the measure $\mu := \mu_{R,D}$ is a unique probability measure satisfying the self-affine identity

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}. \tag{1.1}$$

Such $\mu_{M,D}$ is supported on $T(M,D)$ and is called self-affine measure (see [1]).

Recall that for a probability measure $\mu_{M,D}$ of compact support on $\mathbb{R}^n$, we call $\mu_{M,D}$ a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda := \{ e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \}$ forms an orthogonal basis for $L^2(\mu_{M,D})$. The set Λ is then called a spectrum for $\mu_{M,D}$.

Spectral measure is a natural generalization of spectral set introduced by Fuglede [2] whose famous conjecture and its related problems have received much attention in recent years (see [3] [4] [5] [6]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied in the papers [5] [7] [8] [9] [10] (see also [11] [12] for the main goal). Probably the most interesting question is the spectrality or non-spectrality of self-affine measure $\mu_{M,D}$. We will focus our attention on the following question on the generalized three-dimensional Sierpinski gasket: Under what conditions on M and D is $\mu_{M,D}$ a non-spectral measure? It is known that the non-spectral problem on self-affine measures consists of the following two classes:

1) There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contains at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (see [7] [8] [13] [14] [15] [16] [17]).

2) There are natural infinite families of orthogonal exponentials, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The main question is whether some of these families can be combined to form larger collections of orthogonal exponentials. The other questions concerning this class can be found in [18].

In the present paper, we will consider the questions of the class (I) for the generalized three-dimensional Sierpinski gasket. A fractal F is a set that admits a system of scale transformations; intuitively they have the property that F looks the same as the scaling is varied. Typically a fractal comes equipped with an invariant measure. However as is illustrated by such familiar cases as the Cantor set and its invariant measure, or one of the Sierpinski examples, one must pass to a limit, and the limit typically allows intricate non-linearities. A popular representation of a class of fractals is realized with a finite set of affine transformations in Euclidean space, and this is the setting for the present paper. Now classical Fourier series relies on linearity, and so asking for Fourier series in the context of fractals is a new framework. The result below indicates the limits one encounters in such an endeavor. The main theorem improves what was previously known, *i.e.*, papers by Dutkay and Jorgensen and by Li.

Recall that the self-affine measure $\mu_{M,D}$ corresponding to

$$M=\begin{bmatrix} p_1 & 0 & 0 \\ p_2 & p_4 & 0 \\ p_3 & 0 & p_5 \end{bmatrix} \text{ and } D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} \tag{1.2}$$

is supported on the three-dimensional Sierpinski gasket $T(M,D)$. In the recent paper, Dutkay and Jorgensen [[9]: Theorem 5.1 (iii)] proved that if $p_2=p_3=0$, $p_1=p_4=p_5=p$ is odd, then $\mu_{M,D}$-orthogonal exponentials contain at most 256 elements. The general case for the non-spectrality of the self-affine measure $\mu_{M,D}$ is not known. This leaves the following open problem.

Question. How about the non-spectrality of $\mu_{M,D}$ if $p_1,p_4,p_5\in 2\mathbb{Z}+1$ and one or two of the two numbers p_2,p_3 can take any integer value?

Motivated by the previous research, in the present paper we answer the above question mostly. The main result of the paper is the following.

Theorem 1.1. *For self-affine measure* $\mu_{M,D}$ *corresponding to* (1.2), *if* $p_4=p_5=p$ *and* $p_1,p\in 2\mathbb{Z}+1\setminus\{0,-1\}$, *then* $\mu_{M,D}$ *is a non-spectral measure and there exist at most* 4 *mutually orthogonal exponential functions in* $L^2(\mu_{M,D})$, *where the number* 4 *is the best possible.*

In Section 2, we establish some relations inside the zero set of the Fourier transform of the self-affine measure $\mu_{M,D}$. Based on these established facts, we prove Theorem 1.1 in Section 3.

2. Relations Inside the Zero Set $Z(\hat{\mu}_{M,D})$

For any $\lambda_1,\lambda_2\in\Lambda$, $\lambda_1\neq\lambda_2$, the orthogonality condition

$$\left\langle e^{2\pi i\langle\lambda_1,x\rangle},e^{2\pi i\langle\lambda_2,x\rangle}\right\rangle_{L^2(\mu_{M,D})}=\int e^{2\pi i\langle\lambda_1-\lambda_2,x\rangle}d\mu_{M,D}(x)=\hat{\mu}_{M,D}(\lambda_1-\lambda_2)=0,$$

where $\hat{\mu}_{M,D}$ denotes the Fourier transform of $\mu_{M,D}$. From (1.1), we have

$$\hat{\mu}_{M,D}(\xi)=m_D\left(M^{*-1}\xi\right)\hat{\mu}_{M,D}\left(M^{*-1}\xi\right)\left(\xi\in\mathbb{R}^n\right),$$

which yields

$$\hat{\mu}_{M,D}(\xi)=\prod_{j=1}^{\infty}m_D\left(M^{*-j}\xi\right), \tag{2.1}$$

by iteration, where

$$m_D(t):=\frac{1}{|D|}\sum_{d\in D}e^{2\pi i\langle d,t\rangle} \tag{2.2}$$

and M^* denotes the conjugate transpose of *M*, in fact $M^*=M^{\mathrm{T}}$.

From (2.1), we have

$$Z(\hat{\mu}_{M,D})=\left\{\xi\in\mathbb{R}^n:\exists\, j\in\mathbb{N} \text{ such that } m_D\left(M^{*-j}\xi\right)=0\right\}. \tag{2.3}$$

Furthermore, we have the following.

Proposition 2.1. *Let* $\Theta_0:=\left\{\xi\in\mathbb{R}^n:m_D(\xi)=0\right\}$. *Then*

1) $Z(\hat{\mu}_{M,D})=\bigcup_{j=1}^{\infty}M^{*j}(\Theta_0)$;

2) $\xi_0\in Z(\hat{\mu}_{M,D})$ *if and only if* $-\xi_0\in Z(\hat{\mu}_{M,D})$.

In the following, we will restrict our discussion on the special M and D given by (1.2). Then

$$\Theta_0 = B_1 \cup B_2 \cup B_3,$$

where

$$B_1 = \left\{ \begin{pmatrix} \frac{1}{2}+k_1 \\ a+k_2 \\ \frac{1}{2}+a+k_3 \end{pmatrix} : k_1, k_2, k_3 \in \mathbb{Z},\ a \in \mathbb{R} \right\} \subset \mathbb{R}^3,$$

$$B_2 = \left\{ \begin{pmatrix} a+k_2 \\ \frac{1}{2}+k_1 \\ \frac{1}{2}+a+k_3 \end{pmatrix} : k_1, k_2, k_3 \in \mathbb{Z},\ a \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

and

$$B_3 = \left\{ \begin{pmatrix} a+k_2 \\ \frac{1}{2}+a+k_3 \\ \frac{1}{2}+k_1 \end{pmatrix} : k_1, k_2, k_3 \in \mathbb{Z},\ a \in \mathbb{R} \right\} \subset \mathbb{R}^3. \tag{2.4}$$

From Proposition 2.1, the zero set $Z(\hat{\mu}_{M,D})$ can be represented as

$$Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}(\Theta_0) = \bigcup_{j=1}^{\infty} \left(M^{*j}(B_1) \cup M^{*j}(B_2) \cup M^{*j}(B_3) \right). \tag{2.5}$$

Let $Z_1 := M^{*j}(B_1)$, $Z_2 := M^{*j}(B_2)$ and $Z_3 := M^{*j}(B_3)$ $(j = 1, 2, \cdots)$. From (2.4) and (2.5), we further have the following.

Proposition 2.2. *The sets* $Z_i, i = 1,2,3$ *satisfy the following properties*:

i) $t \in Z_i$ *if and only if* $-t \in Z_i, i = 1,2,3$;

ii) $Z(\hat{\mu}_{M,D}) \cap \mathbb{Z}^3 = \varnothing$;

iii) *if* $t = (t_1, t_2, t_3)^{\mathrm{T}} \in Z_1 \pm Z_1$, *then* $t_3 - t_2 \in \mathbb{Z}$;

iv) *if* $t = (t_1, t_2, t_3)^{\mathrm{T}} \in Z_2 \pm Z_2$, *then* $t_2 \in \mathbb{Z}$;

v) *if* $t = (t_1, t_2, t_3)^{\mathrm{T}} \in Z_3 \pm Z_3$, *then* $t_3 \in \mathbb{Z}$.

Proposition 2.3. *Let* $t = (t_1, t_2, t_3)^{\mathrm{T}} \in Z_1 \cup Z_2 \cup Z_3$. *Then the following statements hold*:

a) *if* $t \in Z_1$, *then* $t_3 - t_2 = \frac{1}{2} + \mathbb{Z}$;

b) *if* $t \in Z_2$, *then* $t_2 = \frac{1}{2} + \mathbb{Z}$;

c) *if* $t \in Z_3$, *then* $t_3 = \frac{1}{2} + \mathbb{Z}$;

d) *if* $t_2 \in \mathbb{Z}$, *then* $t_3 \notin \mathbb{Z}$ *and* $t \in Z_1 \cup Z_3$;

e) *if* $t_3 \in \mathbb{Z}$, *then* $t_2 \notin \mathbb{Z}$ *and* $t \in Z_1 \cup Z_2$.

Proposition 2.4. *If* $\xi,\eta\in Z_1$ *and* $\xi\pm\eta=(\tau_1,\tau_2,\tau_3)^{\mathrm{T}}\in Z(\hat{\mu}_{M,D})$, *then* $\xi\pm\eta\in Z_2\bigcup Z_3$, $\tau_3-\tau_2\in\mathbb{Z}$.

3. Proof of Theorem 1.1

If $\lambda_j\,(j=1,2,3,4,5)\in\mathbb{R}^3$ are such that the exponential functions

$$\mathrm{e}^{2\pi i\langle\lambda_1,x\rangle},\,\mathrm{e}^{2\pi i\langle\lambda_2,x\rangle},\,\mathrm{e}^{2\pi i\langle\lambda_3,x\rangle},\,\mathrm{e}^{2\pi i\langle\lambda_4,x\rangle},\,\mathrm{e}^{2\pi i\langle\lambda_5,x\rangle}$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then the differences $\lambda_j-\lambda_i\,(1\le i\ne j\le 5)$ are in the zero set $Z(\hat{\mu}_{M,D})$. That is, we have

$$\lambda_j-\lambda_i\in Z(\hat{\mu}_{M,D})=Z_1\bigcup Z_2\bigcup Z_3,1\le i\ne j\le 5. \tag{3.1}$$

Define $\lambda_j-\lambda_k$ by

$$\lambda_j-\lambda_i=(x_{j,i},y_{j,i},z_{j,i})^{\mathrm{T}}\in\mathbb{R}^3,1\le i\ne j\le 5.$$

We shall apply the above three propositions to deduce a contradiction below.

Observe that the following 10 differences:

$$\lambda_2-\lambda_1,\,\lambda_3-\lambda_1,\,\lambda_4-\lambda_1,\,\lambda_5-\lambda_1,$$
$$\lambda_3-\lambda_2,\,\lambda_4-\lambda_2,\,\lambda_5-\lambda_2,$$
$$\lambda_4-\lambda_3,\,\lambda_5-\lambda_3,$$
$$\lambda_5-\lambda_4,$$

belong to the union of the three sets Z_1,Z_2,Z_3. The well-known *pigeon hole principle*, there is at least one set which contain at least four elements. When Z_1 contain at least four elements, we can deduce a contradiction in the following. The other cases which Z_2 or Z_3 contain at least four elements can be discussed in the similar manner. We divided the proof into the following five typical cases:

Typical case 1. $\lambda_2-\lambda_1,\,\lambda_3-\lambda_1,\,\lambda_4-\lambda_1,\,\lambda_5-\lambda_1\in Z_1$;

Typical case 2. $\lambda_2-\lambda_1,\,\lambda_3-\lambda_1,\,\lambda_4-\lambda_1\in Z_1$;

Typical case 3. $\lambda_2-\lambda_1,\,\lambda_3-\lambda_1\in Z_1$;

Typical case 4. $\lambda_2-\lambda_1\in Z_1$;

Typical case 5. $\lambda_j-\lambda_1\notin Z_1,\ j=2,3,4,5$.

We shall give a method to deal with each typical case by considering the above remainder differences. The other cases (by applying Proposition 2.2 and Proposition 2.3) can be discussed in the same manner. Note that each typical case is concluded with a contradiction.

3.1. Typical Case 1

In the case, we have

$$\lambda_2-\lambda_1,\,\lambda_3-\lambda_1,\,\lambda_4-\lambda_1,\,\lambda_5-\lambda_1\in Z_1. \tag{3.2}$$

Since

$$\lambda_j-\lambda_i=(\lambda_j-\lambda_1)-(\lambda_i-\lambda_1)\,(2\le i<j\le 5), \tag{3.3}$$

hence, by applying Proposition 2.4, this shows that

$$\lambda_5-\lambda_2,\lambda_5-\lambda_3,\lambda_5-\lambda_4,\lambda_4-\lambda_2,\lambda_4-\lambda_3,\lambda_3-\lambda_2\in Z_2\cup Z_3. \tag{3.4}$$

The well-known *pigeon hole principle*, there are three differences in (3.4) that belong to the same set Z_2 or Z_3. Without loss of generality, we assume that

$$\lambda_5-\lambda_2,\lambda_5-\lambda_3,\lambda_5-\lambda_4\in Z_2. \tag{3.5}$$

The other cases (by applying Proposition 2.2 and Proposition 2.3) can be discussed in the same manner.

Since

$$\lambda_3-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_3)\in Z_2-Z_2$$

and

$$\lambda_4-\lambda_3=(\lambda_5-\lambda_3)-(\lambda_5-\lambda_4)\in Z_2-Z_2,$$

then, by applying Proposition 2.2(iv) and Proposition 2.3(d), we have

$$\lambda_3-\lambda_2,\lambda_4-\lambda_3\in Z_3 \tag{3.6}$$

and

$$y_{3,2},y_{4,3}\in\mathbb{Z}. \tag{3.7}$$

From

$$\lambda_4-\lambda_2=(\lambda_4-\lambda_3)-(\lambda_3-\lambda_2)\in Z_3+Z_3,$$

(3.7) and Proposition 2.2(v), we have

$$y_{4,2},z_{4,2}\in\mathbb{Z}, \tag{3.8}$$

which contradicts Proposition 2.3(d) (e). This completes the proof of Typical case 1.

3.2. Typical Case 2

In the case, we have

$$\lambda_2-\lambda_1,\lambda_3-\lambda_1,\lambda_4-\lambda_1\in Z_1. \tag{3.9}$$

Since

$$\lambda_j-\lambda_i=(\lambda_j-\lambda_1)-(\lambda_i-\lambda_1)\ (2\le i<j\le 4),$$

hence, by applying Proposition 2.4, this shows that

$$\lambda_4-\lambda_2,\lambda_4-\lambda_3,\lambda_3-\lambda_2\in Z_2\cup Z_3 \tag{3.10}$$

and

$$z_{4,2}-y_{4,2}\in\mathbb{Z},\ z_{4,3}-y_{4,3}\in\mathbb{Z},\ z_{3,2}-y_{3,2}\in\mathbb{Z}. \tag{3.11}$$

From (3.10), we know that at least one of the two sets Z_2 and Z_3, say Z_3, contains two differences, say $\lambda_4-\lambda_2$ and $\lambda_4-\lambda_3$. The other cases can be discussed in the same manner.

Since

$$\lambda_3-\lambda_2=(\lambda_4-\lambda_2)-(\lambda_4-\lambda_3)\in Z_3-Z_3,$$

then, by applying Proposition 2.2(v), we have

$$z_{3,2} \in \mathbb{Z}. \tag{3.12}$$

From (3.11), (3.12) and Proposition 2.3(d) (e), we immediately deduce a contradiction. This completes the proof of Typical case 2.

3.3. Typical Case 3

In the case, we have $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1 \in Z_1$. Since $\lambda_3 - \lambda_2 = (\lambda_3 - \lambda_1) - (\lambda_2 - \lambda_1)$, then, by applying Proposition 2.4, this shows that

$$\lambda_3 - \lambda_2 \in Z_2 \cup Z_3$$

and

$$z_{3,2} - y_{3,2} \in \mathbb{Z}. \tag{3.13}$$

We only discuss the case $\lambda_3 - \lambda_2 \in Z_2$, the other case that $\lambda_3 - \lambda_2 \in Z_3$ can be discussed in the similar manner.

Now, consider the following differences

$$\lambda_5 - \lambda_2, \lambda_5 - \lambda_3, \lambda_5 - \lambda_4, \lambda_4 - \lambda_2, \lambda_4 - \lambda_3, \tag{3.14}$$

then there are at least two differences in (3.14) that belong to the set Z_1. In fact, any two differences of (3.14) cannot belong to the same set Z_1. We only discuss the following two cases.

Case 3.1. $\lambda_4 - \lambda_2, \lambda_5 - \lambda_4 \in Z_1$;

Case 3.2. $\lambda_5 - \lambda_2, \lambda_5 - \lambda_3 \in Z_1$.

The other cases can be discussed in the similar manner.

1) In Case 3.1, Since

$$\lambda_4 - \lambda_1 = (\lambda_2 - \lambda_1) + (\lambda_4 - \lambda_2) \in Z_1 + Z_1,$$
$$\lambda_5 - \lambda_4 = (\lambda_5 - \lambda_2) - (\lambda_4 - \lambda_2) \in Z_1 - Z_1$$

and

$$\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_2) + (\lambda_2 - \lambda_1) \in Z_1 + Z_1,$$

then, by applying Proposition 2.4, we have

$$\lambda_4 - \lambda_1, \lambda_5 - \lambda_4, \lambda_5 - \lambda_1 \in Z_2 \cup Z_3 \tag{3.15}$$

and

$$z_{4,1} - y_{4,1}, z_{5,4} - y_{5,4}, z_{5,1} - y_{5,1} \in \mathbb{Z}. \tag{3.16}$$

Further, at least two differences of (3.14) belong to the same set Z_2 or Z_3. Without loss of generality, we assume that

$$\lambda_4 - \lambda_1, \lambda_5 - \lambda_4 \in Z_3. \tag{3.17}$$

The other cases can be discussed in the following same manner.

Since $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_1) \in Z_3 + Z_3$, hence, from (3.16) and Proposition 2.2(v), we have $y_{5,1}, z_{5,1} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e). This completes the proof of Case 3.1.

2) In Case 3.2, Since $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_2) + (\lambda_2 - \lambda_1) \in Z_1 + Z_1$, therefore, by ap-

plying Proposition 2.4, we have

$$\lambda_5-\lambda_1\in Z_2\bigcup Z_3 \text{ and } z_{5,1}-y_{5,1}\in\mathbb{Z}. \tag{3.18}$$

We divide the discussion of case 3.2 into the following two cases according to (3.18).

Case 3.2.1. $\lambda_5-\lambda_1\in Z_2$;

Case 3.2.2. $\lambda_5-\lambda_1\in Z_3$.

i) In Case 3.2.1, we consider the above remainder differences $\lambda_4-\lambda_1,\lambda_5-\lambda_4,\lambda_4-\lambda_2,\lambda_4-\lambda_3$. Firstly,

$$\lambda_5-\lambda_4 \text{ cannot belong to the set (or small box) } Z_1. \tag{3.19}$$

The reason is as follows.

If $\lambda_5-\lambda_4\in Z_1$, then, it follows that

$$\lambda_4-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_4)\in Z_1-Z_1,$$
$$\lambda_4-\lambda_3=(\lambda_5-\lambda_3)-(\lambda_5-\lambda_4)\in Z_1-Z_1$$

and Proposition 2.4, we have

$$\lambda_4-\lambda_2,\lambda_4-\lambda_3\in Z_2\bigcup Z_3 \tag{3.20}$$

and

$$z_{4,2}-y_{4,2},z_{4,3}-y_{4,3}\in\mathbb{Z}. \tag{3.21}$$

This is impossible. Otherwise, If $\lambda_4-\lambda_2\in Z_2$, then, from $\lambda_4-\lambda_3=(\lambda_4-\lambda_2)-(\lambda_3-\lambda_2)\in Z_2-Z_2$, (3.21) and Proposition 2.2(iv), we have $y_{4,3},z_{4,3}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

Similarly, we have $\lambda_4-\lambda_3\notin Z_2,\lambda_4-\lambda_2,\lambda_4-\lambda_3\notin Z_3$.

Secondly, for the similar discussion, we have the following facts that:

$$\lambda_4-\lambda_3 \text{ cannot belong to the set (or small box) } Z_1; \tag{3.22}$$

$$\lambda_4-\lambda_2 \text{ cannot belong to the set (or small box) } Z_1. \tag{3.23}$$

Hence

$$\lambda_4-\lambda_1,\lambda_5-\lambda_4,\lambda_4-\lambda_2,\lambda_4-\lambda_3\in Z_2\bigcup Z_3. \tag{3.24}$$

The well-known *pigeon hole principle*, there are at least two differences in (3.24) that belong to the same set Z_2 or Z_3. This is impossible.

Claim 3.1 Any three differences of (3.24) can not belong to the same set Z_3.

Claim 3.1 can be checked directly. For example, if $\lambda_4-\lambda_1,\lambda_5-\lambda_4,\lambda_4-\lambda_2\in Z_3$, then, from

$$\lambda_2-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_2)\in Z_3-Z_3,$$
$$\lambda_5-\lambda_2=(\lambda_5-\lambda_4)+(\lambda_4-\lambda_2)\in Z_3+Z_3,$$

$\lambda_5-\lambda_1=(\lambda_5-\lambda_4)+(\lambda_4-\lambda_1)\in Z_3+Z_3$ and Proposition 2.2(v), we have

$$z_{2,1},z_{5,2},z_{5,1}\in\mathbb{Z}. \tag{3.25}$$

Since $\lambda_5-\lambda_1=(\lambda_5-\lambda_2)+(\lambda_2-\lambda_1)\in Z_1+Z_1$, hence, by applying Proposition 2.4, this shows that

$$z_{5,1}-y_{5,1}\in\mathbb{Z}. \tag{3.26}$$

From (3.25) and (3.26), we have $y_{5,1}, z_{5,1} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

Claim 3.2 Any two differences of (3.24) can not belong to the same set Z_2.

Claim 3.2 can be checked directly. For example, if $\lambda_4-\lambda_3, \lambda_4-\lambda_2 \in Z_2$, then, from

$$\lambda_3-\lambda_2=(\lambda_4-\lambda_2)-(\lambda_4-\lambda_3) \in Z_2-Z_2,$$
$$\lambda_4-\lambda_2=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_2) \in Z_2+Z_2,$$

$\lambda_4-\lambda_3=(\lambda_4-\lambda_2)-(\lambda_3-\lambda_2) \in Z_2-Z_2$ and Proposition 2.2(iv), we have

$$y_{3,2}, y_{4,2}, y_{4,3} \in \mathbb{Z}. \tag{3.27}$$

Since $\lambda_3-\lambda_2=(\lambda_4-\lambda_2)-(\lambda_4-\lambda_3) \in Z_3+Z_3$, hence, by applying Proposition 2.2(v), this shows that

$$z_{3,2} \in \mathbb{Z}. \tag{3.28}$$

From (3.27) and (3.28), we have $y_{3,2}, z_{3,2} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

According to the above discussion, $\lambda_5-\lambda_1 \notin Z_2$, this completes the proof of Case 3.2.1.

ii) In Case 3.2.2, we consider the above remainder differences $\lambda_4-\lambda_1, \lambda_5-\lambda_4, \lambda_4-\lambda_2, \lambda_4-\lambda_3$. Firstly,

$$\lambda_5-\lambda_4 \text{ cannot belong to the set (or small box) } Z_1. \tag{3.29}$$

The reason is as follows.

If $\lambda_5-\lambda_4 \in Z_1$, then, it follows that

$$\lambda_4-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_4) \in Z_1-Z_1,$$
$$\lambda_4-\lambda_3=(\lambda_5-\lambda_3)-(\lambda_5-\lambda_4) \in Z_1-Z_1$$

and Proposition 2.4, we have

$$\lambda_4-\lambda_2, \lambda_4-\lambda_3 \in Z_2 \cup Z_3 \tag{3.30}$$

and

$$z_{4,2}-y_{4,2}, z_{4,3}-y_{4,3} \in \mathbb{Z}. \tag{3.31}$$

This is impossible. Otherwise, If $\lambda_4-\lambda_2 \in Z_2$, then, from $\lambda_4-\lambda_3=(\lambda_4-\lambda_2)-(\lambda_3-\lambda_2) \in Z_2-Z_2$, (3.31) and Proposition 2.2(iv), we have $y_{4,3}, z_{4,3} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

If $\lambda_4-\lambda_3 \in Z_2$, then, from $\lambda_4-\lambda_2=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_2) \in Z_2+Z_2$, (3.31) and Proposition 2.2(iv), we have $y_{4,2}, z_{4,2} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

If $\lambda_4-\lambda_2, \lambda_4-\lambda_3 \in Z_3$, then, from $\lambda_3-\lambda_2=(\lambda_4-\lambda_2)-(\lambda_4-\lambda_3) \in Z_3-Z_3$, (3.31) and Proposition 2.2(v), we have $y_{3,2}, z_{3,2} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

Secondly, for the similar discussion, we have the following facts that:

$$\lambda_4-\lambda_3 \text{ cannot belong to the set (or small box) } Z_1; \tag{3.32}$$

$\lambda_4 - \lambda_2$ cannot belong to the set (or small box) Z_1. (3.33)

Hence

$$\lambda_4 - \lambda_1, \lambda_5 - \lambda_4, \lambda_4 - \lambda_2, \lambda_4 - \lambda_3 \in Z_2 \cup Z_3. \quad (3.34)$$

The well-known *pigeon hole principle*, there are at least two differences in (3.34) that belong to the same set Z_2 or Z_3.

Claim 3.3 Any three differences of (3.34) can not belong to the same set Z_3.

Claim 3.3 can be checked directly. For example, if $\lambda_4 - \lambda_1, \lambda_5 - \lambda_4, \lambda_4 - \lambda_2 \in Z_3$, then, from

$$\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_2) \in Z_3 + Z_3,$$
$$\lambda_2 - \lambda_1 = (\lambda_4 - \lambda_1) - (\lambda_4 - \lambda_2) \in Z_3 - Z_3$$

and Proposition 2.2(v), we have

$$z_{2,1}, z_{5,2} \in \mathbb{Z}. \quad (3.35)$$

Since $\lambda_5 - \lambda_1 = (\lambda_5 - \lambda_2) + (\lambda_2 - \lambda_1) \in Z_1 + Z_1$, hence, by applying Proposition 2.4, this shows that

$$z_{5,1} - y_{5,1} \in \mathbb{Z}. \quad (3.36)$$

From (3.35) and (3.36), we have $y_{5,1}, z_{5,1} \in \mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

Claim 3.4 Any two differences of (3.34) can not belong to the same set Z_2.

Claim 3.4 can be checked directly. For example, if $\lambda_5 - \lambda_4, \lambda_4 - \lambda_2 \in Z_2$, then, from

$$\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_4) + (\lambda_4 - \lambda_2) \in Z_2 + Z_2,$$
$$\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_2) \in Z_2 - Z_2$$

and Proposition 2.2(iv), we have

$$y_{5,2}, y_{4,3} \in \mathbb{Z}. \quad (3.37)$$

Obviously, $\lambda_4 - \lambda_3 \in Z_3$. Let

$$(\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_3) = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}, (\lambda_3 - \lambda_1) + (\lambda_5 - \lambda_4) = (\eta_1, \eta_2, \eta_3)^{\mathrm{T}}.$$

By applying Proposition 2.2(v) and Proposition 2.3(e), we have

$$\xi_2 \notin \mathbb{Z}.$$

By applying Proposition 2.2, Proposition 2.3 and Proposition 2.4, we have

$$\eta_2 \in \mathbb{Z}.$$

Namely $(\lambda_5 - \lambda_1) - (\lambda_4 - \lambda_3) \neq (\lambda_3 - \lambda_1) + (\lambda_5 - \lambda_4)$, we immediately deduce a contradiction. This completes the proof of Case 3.2.2.

According to the above discussion, this completes the proof of Typical case 3.

3.4. Typical Case 4

In Typical case 4, we have $\lambda_2 - \lambda_1 \in Z_1$. We see that in this case, each set contains elements (or differences) in the following Box 2:

Z_1	Z_2	Z_3
$\lambda_2-\lambda_1$		

Box 1

Since $\lambda_5-\lambda_1, \lambda_4-\lambda_1, \lambda_3-\lambda_1 \in Z_2 \cup Z_3$, then, there are at least two differences that belong to the same set Z_2 or Z_3. We only discuss the following two cases:

Case 4.1. $\lambda_5-\lambda_1, \lambda_3-\lambda_1 \in Z_3$;

Case 4.2. $\lambda_5-\lambda_1, \lambda_4-\lambda_1, \lambda_3-\lambda_1 \in Z_2$.

The other cases can be discussed in the similar manner.

1) In Case 4.1, since $\lambda_5-\lambda_3=(\lambda_5-\lambda_1)-(\lambda_3-\lambda_1) \in Z_3-Z_3$, so, by applying Proposition 2.2(v) and Proposition 2.3(e), we have $\lambda_5-\lambda_3 \in Z_1 \cup Z_2$. There are the following two cases.

Case 4.1.1. $\lambda_5-\lambda_3 \in Z_1$;

Case 4.1.2. $\lambda_5-\lambda_3 \in Z_2$.

i) In Case 4.1.1, we consider the above remainder differences

$$\lambda_5-\lambda_4, \lambda_5-\lambda_2, \lambda_4-\lambda_2, \lambda_4-\lambda_3, \lambda_3-\lambda_2,$$

then, there are at least two differences that belong to the same set Z_1. This is impossible.

Claim 4.1 Any two differences of the above remainder differences can not belong to the same set Z_1.

We only check Claim 4.1 in the following two cases:

Case 4.1.1.1. $\lambda_5-\lambda_4, \lambda_3-\lambda_2 \in Z_1$;

Case 4.1.1.2. $\lambda_4-\lambda_2, \lambda_3-\lambda_2 \in Z_1$.

The other cases can be discussed in the similar manner.

i') In Case 4.1.1.1, since

$$\lambda_4-\lambda_3=(\lambda_5-\lambda_3)-(\lambda_5-\lambda_4) \in Z_1-Z_1,$$
$$\lambda_3-\lambda_1=(\lambda_3-\lambda_2)+(\lambda_2-\lambda_1) \in Z_1+Z_1$$

and $\lambda_5-\lambda_2=(\lambda_5-\lambda_3)+(\lambda_3-\lambda_2) \in Z_1+Z_1$, therefore, from Proposition 2.4, we have

$$\lambda_4-\lambda_3 \in Z_2 \cup Z_3 \tag{3.38}$$

and

$$z_{4,3}-y_{4,3}, z_{3,1}-y_{3,1} \in \mathbb{Z}. \tag{3.39}$$

If $\lambda_4-\lambda_3 \in Z_2$, from $\lambda_3-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_3) \in Z_2-Z_2$, (3.39) and Proposition 2.2(iv), we have $z_{3,1}, y_{3,1} \in \mathbb{Z}$, which contradicts Proposition 2.3(d)(e). If $\lambda_4-\lambda_3 \in Z_3$, from $\lambda_4-\lambda_1=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_1) \in Z_3+Z_3$, (3.39) and Proposition 2.2(v), we have $z_{4,1}, y_{4,1} \in \mathbb{Z}$, which contradicts Proposition 2.3(d)(e). Therefore, $\lambda_5-\lambda_4, \lambda_3-\lambda_2 \notin Z_1$.

ii') In Case 4.1.1.2, since

$$\lambda_4-\lambda_3=(\lambda_4-\lambda_2)-(\lambda_3-\lambda_2) \in Z_1-Z_1,$$
$$\lambda_5-\lambda_2=(\lambda_5-\lambda_3)+(\lambda_3-\lambda_2) \in Z_1+Z_1,$$

$$\lambda_3-\lambda_1=(\lambda_3-\lambda_2)+(\lambda_2-\lambda_1)\in Z_1+Z_1$$
$$\text{and } \lambda_4-\lambda_1=(\lambda_4-\lambda_2)+(\lambda_2-\lambda_1)\in Z_1+Z_1,$$

hence, from Proposition 2.4, we have

$$\lambda_4-\lambda_3\in Z_2\cup Z_3 \text{ and } z_{4,3}-y_{4,3},\ z_{3,1}-y_{3,1},\ z_{4,1}-y_{4,1}\in\mathbb{Z}. \qquad (3.40)$$

If $\lambda_4-\lambda_3\in Z_2$, from $\lambda_3-\lambda_1=(\lambda_4-\lambda_1)-(\lambda_4-\lambda_3)\in Z_2-Z_2$, (3.40) and Proposition 2.2(iv), we have $z_{3,1}, y_{3,1}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

If $\lambda_4-\lambda_3\in Z_3$, from $\lambda_4-\lambda_3=(\lambda_4-\lambda_1)-(\lambda_3-\lambda_1)\in Z_3-Z_3$, (3.40) and Proposition 2.2(v), we have $z_{4,3}, y_{4,3}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e). Therefore, $\lambda_4-\lambda_2, \lambda_3-\lambda_2\notin Z_1$.

ii) In Case 4.1.2, we consider the above remainder differences

$$\lambda_5-\lambda_4,\ \lambda_5-\lambda_2,\ \lambda_4-\lambda_2,\ \lambda_4-\lambda_3,\ \lambda_3-\lambda_2,$$

then, there are at least three differences that belong to the same set Z_1. This is impossible.

Claim 4.2 Any three differences of the above remainder differences can not belong to the same set Z_1.

We only check Claim 4.2 in the following.

Case 4.1.2.1. $\lambda_5-\lambda_4, \lambda_3-\lambda_2, \lambda_4-\lambda_2\in Z_1$.

The other cases can be discussed in the similar manner.

iii') In Case 4.1.2.1, since

$$\lambda_4-\lambda_3=(\lambda_4-\lambda_2)-(\lambda_3-\lambda_2)\in Z_1-Z_1,$$
$$\lambda_3-\lambda_1=(\lambda_3-\lambda_2)+(\lambda_2-\lambda_1)\in Z_1+Z_1$$

and $\lambda_4-\lambda_1=(\lambda_4-\lambda_2)+(\lambda_2-\lambda_1)\in Z_1+Z_1$, therefore, from Proposition 2.4, we have

$$\lambda_4-\lambda_3\in Z_2\cup Z_3 \text{ and } z_{4,3}-y_{4,3},\ z_{3,1}-y_{3,1},\ z_{4,1}-y_{4,1}\in\mathbb{Z}.$$

This is impossible. Otherwise, if $\lambda_4-\lambda_3\in Z_2$ or Z_3, then, from Proposition 2.2 and Proposition 2.3, we can deduce a contradiction.

Therefore, $\lambda_5-\lambda_3\notin Z_2$. This completes the proof of Case 4.1.

2) In Case 4.2, since

$$\lambda_5-\lambda_3=(\lambda_5-\lambda_1)-(\lambda_3-\lambda_1)\in Z_2-Z_2,$$
$$\lambda_5-\lambda_4=(\lambda_5-\lambda_1)-(\lambda_4-\lambda_1)\in Z_2-Z_2$$

and $\lambda_4-\lambda_3=(\lambda_4-\lambda_1)-(\lambda_3-\lambda_1)\in Z_2-Z_2$, then, from Proposition 2.2(iv), we have

$$\lambda_5-\lambda_4,\ \lambda_4-\lambda_3,\ \lambda_5-\lambda_3\in Z_1\cup Z_3 \text{ and } y_{5,3}, y_{5,4}, y_{4,3}\in\mathbb{Z}. \qquad (3.41)$$

Then there are at least two differences that belong to the same set Z_1 or Z_3.

In fact, any two differences in (3.41) can not belong to the same set Z_1 or Z_3. The reason is as follows.

Claim 4.3 Any two differences of the above remainder differences can not belong to the same set Z_1.

Claim 4.3 can be checked directly. For example, if $\lambda_5-\lambda_3, \lambda_4-\lambda_3\in Z_1$, then, from $\lambda_5-\lambda_4=(\lambda_5-\lambda_3)-(\lambda_4-\lambda_3)\in Z_1-Z_1$ Proposition 2.4 and (3.41), we have

$$y_{5,4},\, z_{5,4}\in\mathbb{Z},$$

which contradicts Proposition 2.3(d) (e).

Claim 4.4 Any two differences of the above remainder differences can not belong to the same set Z_3.

Claim 4.4 can be checked directly. For example, if $\lambda_5-\lambda_3,\,\lambda_4-\lambda_3\in Z_3$, then, from $\lambda_5-\lambda_4=(\lambda_5-\lambda_3)-(\lambda_4-\lambda_3)\in Z_3-Z_3$, Proposition 2.2(v) and (3.41), we have $y_{5,4},\, z_{5,4}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

According to the above discussion, this completes the proof of Typical case 4.

3.5. Typical Case 5

In the case, we have

$$\lambda_j-\lambda_1\notin Z_1 \quad (j=2,3,4,5).$$

Namely

$$\lambda_5-\lambda_1,\,\lambda_4-\lambda_1,\,\lambda_3-\lambda_1,\,\lambda_2-\lambda_1\in Z_2\cup Z_3.$$

then, from the well-known *pigeon hole principle*, there are at least two differences which belong to the same set Z_2 or Z_3. This is impossible.

Claim 5.1 Any three differences of the above four differences can not belong to the same set Z_2 or Z_3.

Claim 5.1 can be checked directly. For example, if $\lambda_5-\lambda_1,\,\lambda_4-\lambda_1,\,\lambda_3-\lambda_1\in Z_2$, then, from

$$\lambda_5-\lambda_4=(\lambda_5-\lambda_1)-(\lambda_4-\lambda_1)\in Z_2-Z_2,$$
$$\lambda_5-\lambda_3=(\lambda_5-\lambda_1)-(\lambda_3-\lambda_1)\in Z_2-Z_2,$$

$\lambda_4-\lambda_3=(\lambda_4-\lambda_1)-(\lambda_3-\lambda_1)\in Z_2-Z_2$ and Proposition 2.2(iv), we have

$$\lambda_5-\lambda_4,\,\lambda_4-\lambda_3,\,\lambda_5-\lambda_3\in Z_1\cup Z_3 \tag{3.42}$$

and

$$y_{5,3},\, y_{5,4},\, y_{4,3}\in\mathbb{Z}. \tag{3.43}$$

Then there are at least two differences which belong to the same set Z_1 or Z_3. We only discuss the following two cases:

Case 5.1. $\lambda_5-\lambda_3,\,\lambda_4-\lambda_3\in Z_1$;

Case 5.2. $\lambda_5-\lambda_3,\,\lambda_4-\lambda_3\in Z_3$.

The other cases can be discussed in the similar manner.

1) If $\lambda_5-\lambda_3,\,\lambda_4-\lambda_3\in Z_1$, then, from $\lambda_5-\lambda_4=(\lambda_5-\lambda_3)-(\lambda_4-\lambda_3)\in Z_1-Z_1$, Proposition 2.4 and (3.43), we have $y_{5,4},\, z_{5,4}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

2) If $\lambda_5-\lambda_3,\,\lambda_4-\lambda_3\in Z_3$, then, from $\lambda_5-\lambda_4=(\lambda_5-\lambda_3)-(\lambda_4-\lambda_3)\in Z_3-Z_3$, Proposition 2.2(v) and (3.43), we have $y_{5,4},\, z_{5,4}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

Claim 5.2 Any two differences of the above four differences can not belong to the same set Z_2 or Z_3.

Claim 5.2 can be checked directly. For example, if $\lambda_5-\lambda_1,\lambda_3-\lambda_1\in Z_2,\lambda_4-\lambda_1,\lambda_2-\lambda_1\in Z_3$, then, we see each set contains elements (or differences) in the following Box 3:

Z_1	Z_2	Z_3
	$\lambda_3-\lambda_1,\lambda_5-\lambda_1$	$\lambda_2-\lambda_1,\lambda_4-\lambda_1$

Box 2

Since $\lambda_5-\lambda_3=(\lambda_5-\lambda_1)-(\lambda_3-\lambda_1)\in Z_2-Z_2$ and $\lambda_4-\lambda_2=(\lambda_4-\lambda_1)-(\lambda_2-\lambda_1)\in Z_3-Z_3$, therefore, by applying Proposition 2.2(iv) (v) and Proposition 2.3(d) (e), we have

$$\lambda_5-\lambda_3\in Z_1\cup Z_3,\ \lambda_4-\lambda_2\in Z_1\cup Z_2 \text{ and } y_{5,3},\ z_{4,2}\in\mathbb{Z}. \tag{3.44}$$

We only discuss the following three cases and deduce a contradiction. The other cases can be proved in the same manner.

Case 5.3. $\lambda_5-\lambda_3\in Z_3,\ \lambda_4-\lambda_2\in Z_2$;

Case 5.4. $\lambda_5-\lambda_3\in Z_1,\ \lambda_4-\lambda_2\in Z_2$;

Case 5.5. $\lambda_5-\lambda_3\in Z_1,\ \lambda_4-\lambda_2\in Z_1$.

3) In Case 5.3, From $\lambda_5-\lambda_3=(\lambda_5-\lambda_2)-(\lambda_3-\lambda_2)\in Z_1-Z_1$, Proposition 2.4 and (3.44), we lead to a contradiction.

4) In Case 5.4, we consider the remainder differences

$$\lambda_5-\lambda_4,\ \lambda_5-\lambda_2,\ \lambda_4-\lambda_3,\ \lambda_3-\lambda_2.$$

Then, any three differences of the above four differences can not belong to the same set Z_1. In fact, for example, if $\lambda_5-\lambda_2,\ \lambda_4-\lambda_3,\ \lambda_3-\lambda_2\in Z_1$, then from $\lambda_4-\lambda_2=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_2)\in Z_1-Z_1$, Proposition 2.4 and (3.44), we lead to a contradiction.

5) In Case 5.5, the following consider the remainder differences

$$\lambda_5-\lambda_4,\ \lambda_5-\lambda_2,\ \lambda_4-\lambda_3,\ \lambda_3-\lambda_2.$$

Then, any two differences of the above four differences can not belong to the same set Z_1. In fact, for example, if $\lambda_3-\lambda_2,\ \lambda_5-\lambda_4\in Z_1$, then since

$$\lambda_5-\lambda_2=(\lambda_5-\lambda_3)+(\lambda_3-\lambda_2)\in Z_1+Z_1$$
$$\text{and } \lambda_4-\lambda_3=(\lambda_4-\lambda_2)-(\lambda_3-\lambda_2)\in Z_1-Z_1,$$

therefore, by applying Proposition 2.4, we have

$$\lambda_5-\lambda_2,\ \lambda_4-\lambda_3\in Z_2\cup Z_3 \text{ and } z_{5,2}-y_{5,2},\ z_{4,3}-y_{4,3}\in\mathbb{Z}. \tag{3.45}$$

i) $\lambda_5-\lambda_2,\ \lambda_4-\lambda_3$ can not belong to the same set Z_2 or Z_3. The reason is as follows.

For example, if $\lambda_5-\lambda_2,\ \lambda_4-\lambda_3\in Z_2$, then from

$$\lambda_4-\lambda_1=(\lambda_4-\lambda_3)+(\lambda_3-\lambda_1)\in Z_2+Z_2,$$
$$\lambda_2-\lambda_1=(\lambda_5-\lambda_1)-(\lambda_5-\lambda_2)\in Z_2-Z_2$$

and Proposition 2.2(iv), we have

$$y_{2,1},\ y_{4,1}\in\mathbb{Z}. \tag{3.46}$$

Further, from $\lambda_4-\lambda_2=(\lambda_4-\lambda_1)-(\lambda_2-\lambda_1)\in Z_3-Z_3$, Proposition 2.2(v) and (3.46), we have $y_{4,2}, z_{4,2}\in\mathbb{Z}$, which contradicts Proposition 2.3(d) (e).

ii) $\lambda_5-\lambda_2$ can not belong to the set Z_2 or Z_3, $\lambda_4-\lambda_3$ can not belong to the set Z_3 or Z_2. The reason is as follows. For example, if $\lambda_5-\lambda_2\in Z_2, \lambda_4-\lambda_3\in Z_3$, then there let

$$(\lambda_5-\lambda_2)+(\lambda_4-\lambda_3)=(\xi_1,\xi_2,\xi_3)^{\mathrm{T}} \text{ and } (\lambda_5-\lambda_3)+(\lambda_4-\lambda_2)=(\eta_1,\eta_2,\eta_3)^{\mathrm{T}}.$$

By applying Proposition 2.2, Proposition 2.3 and Proposition 2.4, we have

$$\xi_2,\xi_3\in\mathbb{Z}, \eta_2,\eta_3\notin\mathbb{Z}.$$

Namely

$$(\lambda_5-\lambda_2)+(\lambda_4-\lambda_3)\neq(\lambda_5-\lambda_3)+(\lambda_4-\lambda_2),$$

we immediately deduce a contradiction. This completes the proof of Typical case 5.

According to the above discussion of Typical cases 1 - 5, we proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 4 elements, and $\mu_{M,D}$ is a non-spectral measure. One can obtain many such orthogonal systems which contain four elements. For instance, the exponential function system E_Λ with Λ given by

$$\Lambda=\{0, M^*s_1, M^*s_2, M^*s_3\}, \tag{3.47}$$

where

$$s_1=\left(\frac{1}{2},\frac{1}{2},0\right)^{\mathrm{T}}, s_2=\left(\frac{1}{2},0,\frac{1}{2}\right)^{\mathrm{T}}, s_3=\left(0,\frac{1}{2},\frac{1}{2}\right)^{\mathrm{T}} \tag{3.48}$$

is a four-elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 4 is the best. This completes the proof of Theorem 1.1. □

By the proof of Theorem 1.1, we have the following corollaries:

Corollary 3.1. *For self-affine measure $\mu_{M,D}$ corresponding to the expanding integer matrix*

$$M=\begin{bmatrix} p_1 & 0 & 0\\ p_2 & p & 0\\ p_3 & 0 & p\end{bmatrix} \text{ and } D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}d\\0\\0\end{pmatrix},\begin{pmatrix}0\\d\\0\end{pmatrix},\begin{pmatrix}0\\0\\d\end{pmatrix}\right\},$$

if $p, p_1\notin 2\mathbb{Z}$ and $d\neq0$, then $\mu_{M,D}$ is a non-spectral measure and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best possible.

Corollary 3.2. *For self-affine measure $\mu_{M,D}$ corresponding to the expanding integer matrix*

$$M=\begin{bmatrix} p & 0 & p_1\\ 0 & p & p_2\\ 0 & 0 & p_3\end{bmatrix} \text{ and } D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}d\\0\\0\end{pmatrix},\begin{pmatrix}0\\d\\0\end{pmatrix},\begin{pmatrix}0\\0\\d\end{pmatrix}\right\},$$

if $p, p_3\notin 2\mathbb{Z}$ and $d\neq0$, then $\mu_{M,D}$ is a non-spectral measure and there exist at most 4 *mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the* *number* 4 *is the best possible.*

4. A Concluding Remark

In this section, we would like to point out that the above proof is based on the relations inside the zero set $Z\left(\hat{\mu}_{M,D}\right)$ of the Fourier transform $\mu_{M,D}$. Generally speaking, it is difficult to obtain certain properties on the zero set. If we choose the special four-elements digit set, for instance, the direct-sum-forms digit set

$$D = D_1 \oplus D_2 = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} \oplus \left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\},$$

it is easy to estimate the number of orthogonal exponentials.

Proposition 4.1. *For an expanding integer matrix* $M \in M_3(\mathbb{Z})$ *and the direct-sum-forms digit set D given by*

$$M = \begin{bmatrix} p_1 & 0 & p_5 \\ 0 & p_2 & p_4 \\ 0 & 0 & p_3 \end{bmatrix} \text{ and } D = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\},$$

if $p_1, p_2 \notin 2\mathbb{Z}$, *then* $\mu_{M,D}$ *is a non-spectral measure and there exist at most* 4 *mutually orthogonal exponential functions in* $L^2\left(\mu_{M,D}\right)$.

Proof. We consider the pair (M, D_1) and (M, D_2), let $\bar{\Theta}_0 := \left\{\xi \in \mathbb{R}^3 : m_{D_1}(\xi) = 0\right\}$ and let $\tilde{\Theta}_0 := \left\{\xi \in \mathbb{R}^3 : m_{D_2}(\xi) = 0\right\}$. Then

$$\bar{\Theta}_0 = \left\{\begin{pmatrix}\frac{1}{2}+\bar{k}\\ \bar{\alpha}\\ \bar{\beta}\end{pmatrix} : \bar{k} \in \mathbb{Z}, \bar{\alpha}, \bar{\beta} \in \mathbb{R}\right\},$$

$$\tilde{\Theta}_0 = \left\{\begin{pmatrix}\tilde{\alpha}\\ \frac{1}{2}+\tilde{k}\\ \tilde{\beta}\end{pmatrix} : \tilde{k} \in \mathbb{Z}, \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}\right\}.$$

In the case when $p_1, p_2 \notin 2\mathbb{Z}$, we see that

$$M^*\bar{\Theta}_0 \in \bar{\Theta}_0,\ M^*\tilde{\Theta}_0 \in \tilde{\Theta}_0,$$

which show that

$$Z\left(\hat{\mu}_{M,D_1}\right) \subseteq \bar{\Theta}_0,\ Z\left(\hat{\mu}_{M,D_2}\right) \subseteq \tilde{\Theta}_0.$$

If $\lambda_j\ (j = 1,2,3) \in \mathbb{R}^3$ are such that the exponential functions

$$e^{2\pi i\langle\lambda_1,x\rangle},\ e^{2\pi i\langle\lambda_2,x\rangle},\ e^{2\pi i\langle\lambda_3,x\rangle}$$

are mutually orthogonal in $L^2\left(\mu_{M,D_1}\right)$, then the differences $\lambda_j - \lambda_i\ (1 \le i \ne j \le 3)$ are in the zero set $Z\left(\hat{\mu}_{M,D_1}\right)$. That is, there are $\bar{k}_1, \bar{k}_2, \bar{k}_3 \in \mathbb{Z}$, $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \in \mathbb{R}$ and $\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3 \in \mathbb{R}$ such that

$$\lambda_3-\lambda_1=\begin{pmatrix}\frac{1}{2}+\overline{k}_1\\ \overline{\alpha}_1\\ \overline{\beta}_1\end{pmatrix},\ \lambda_2-\lambda_1=\begin{pmatrix}\frac{1}{2}+\overline{k}_2\\ \overline{\alpha}_2\\ \overline{\beta}_2\end{pmatrix},\ \lambda_3-\lambda_2=\begin{pmatrix}\frac{1}{2}+\overline{k}_3\\ \overline{\alpha}_3\\ \overline{\beta}_3\end{pmatrix}.$$

This shows that

$$(\lambda_3-\lambda_1)-(\lambda_2-\lambda_1)\neq\lambda_3-\lambda_2$$

a contradiction. Hence, there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D_1})$.

Similarly, there are at most 2 mutually orthogonal exponential functions in $L^2(\mu_{M,D_2})$.

From

$$\hat{\mu}_{M,D}(\xi)=\hat{\mu}_{M,D_1}(\xi)\hat{\mu}_{M,D_2}(\xi),$$

we have

$$Z(\hat{\mu}_{M,D})=Z(\hat{\mu}_{M,D_1})\cup Z(\hat{\mu}_{M,D_2}).$$

Hence $\mu_{M,D}$ is a non-spectral measure, and there are at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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