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Operations with Higher-Order Types of Asymptotic Variation: Filling Some Gaps

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Abstract

In this paper we exhibit some results concerning operations with higher-order types of asymptotic variation, results lacking in the general theory developed in previous papers, namely: 1) we show to what extent the standard elementary factorization of a regularly-varying function holds true for higher-order variation; 2) we exhibit an important class of higher-order regularly-varying functions requiring no restrictions on the indexes when performing multiplication; 3) we get non-obvious results on the types of higher-order variation for linear combinations. In addition, partial results are obtained concerning the type of higher-order variation of the inverse of a regularly-varying function whose index belongs to a set of "exceptional" values.

Keywords

Higher-Order Regularly, Smoothly or Rapidly-Varying Functions

1. Introduction

In three previous papers [1] [2] [3] we developed a general theory of higher-order types of asymptotic variation for functions differentiable a certain number of times on some interval $[T,+\infty)$, spending much effort on results about operations with such classes of functions in [2] and pointing out some elementary applications of the general theory to integrals and sums. Subsequently, in [4] [5], we applied the theory to the difficult problem of evaluating the exact asymptotic behaviors of Wronskians whose entries belong to one or more of the classes of regularly-, smoothly- or rapidly-varying functions. In turn, these results have been applied to the theory of finite asymptotic expansions in the real domain. Going further in this direction, namely trying to evaluate the principal part of certain Hankel determinants, we became aware of some gaps in the long list of results concerning operations with higher-order types of variation: products, linear combinations and inversion. In the present paper, we prove some complementary results that will provide useful in some applications to be developed elsewhere.

For the reader's convenience we report the general notations and essential facts of the theory, already listed in [[3]: pp. 435-438] and [[4]: pp. 6-8].

General notations.

- $\mathbb{N} := \{1, 2, \cdots\}$; $\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$;
- $\mathbb{R} :=$ real line; $\overline{\mathbb{R}} \equiv$ extended real line $:= \mathbb{R} \cup \{\pm\infty\}$;
- $f \in AC^0(I) \equiv AC(I) \Leftrightarrow f$ is absolutely continuous on each compact subinterval of the interval *I*;
- $f \in AC^k(I) \Leftrightarrow f^{(k)} \in AC(I)$;
- For $f \in AC^k(I)$ we write "$\lim_{x \to x_0} f^{(k+1)}(x)$" meaning that *x* runs through the points wherein $f^{(k+1)}$ exists as a finite number; $f(+\infty) := \lim_{x \to +\infty} f(x)$.
- The differentiation operators: $Df(x) := f'(x)$; $D^k f(x) := f^{(k)}(x)$.
- The logarithmic derivative: $D_\ell f := f'/f$.
- Hardy's notations:

"$f(x) \ll g(x),\ x \to x_0$" or, equivalently "$g(x) \gg f(x), x \to x_0$" stands for $f(x) = o(g(x)), x \to x_0$;

"$f(x) \preccurlyeq g(x), x \to x_0$" or, equivalently "$g(x) \succcurlyeq f(x), x \to x_0$" stands for $f(x) = O(g(x)), x \to x_0$.

- The relation of "*asymptotic similarity*", "$f(x) \asymp g(x),\ x \to x_0$" means that

$$\begin{cases} c_1 |g(x)| \le |f(x)| \le c_2 |g(x)| \ \ \forall\ x \text{ in a deleted neighborhood of } x_0, \\ (c_i = \text{constant} > 0). \end{cases} \tag{1.1}$$

- The relation of "*asymptotic equivalence*":

$f(x) \sim g(x),\ x \to x_0$ stands for $f(x) = g(x)[1 + o(1)],\ x \to x_0$.

- The non-standard notation:

$$f(x) = +\infty(g(x)),\ x \to x_0\ (x \in \mathcal{I}), \overset{\text{def}}{\Leftrightarrow} \begin{cases} f(x) = h(x) g(x) \ \ \forall\ x \text{ near } x_0, \\ \lim\limits_{x \to x_0,\ x \in \mathcal{I}} h(x) = +\infty; \end{cases} \tag{1.2}$$

and a similar definition for the notation $f(x) = -\infty(g(x)),\ x \to x_0\ (x \in \mathcal{I})$. In particular:

$$f(x) = \pm\infty(1),\ x \to x_0\ (x \in \mathcal{I}), \overset{\text{def}}{\Leftrightarrow} \lim_{x \to x_0,\ x \in \mathcal{I}} f(x) = \pm\infty. \tag{1.3}$$

- Factorial powers:

$$\alpha^{\underline{0}} := 1; \alpha^{\underline{1}} := \alpha; \alpha^{\underline{k}} := \alpha(\alpha - 1)\cdots(\alpha - k + 1);\ \ \alpha \in \mathbb{C}, k \in \mathbb{N}; \tag{1.4}$$

where $\alpha^{\underline{k}}$ is termed the "*k-th falling* ($\equiv$ *decreasing*) *factorial power of* α". Notice that we have defined $0^{\underline{0}} := 1$.

- Everywhere the symbol "$\log x$" stands for "$\log_e(x)$":= "the natural logarithm" of *x*.

Classes of functions and their main characterizations.

(I) (Index of asymptotic variation). *If* $f \in AC[T,+\infty)$, f *ultimately* > 0, *its index of asymptotic variation at* $+\infty$ *is defined as the value of the following limit* (*assumed to exist*):

$$\lim_{x\to+\infty} xf'(x)/f(x) = \begin{cases} 0 & (\text{slow variation at } +\infty), \\ \alpha \in \mathbb{R}\setminus\{0\} & (\text{regular variation at } +\infty), \\ \pm\infty & (\text{rapid variation at } +\infty). \end{cases} \quad (1.5)$$

In this case, we use the notation "$f \in \mathcal{R}_\alpha(+\infty)$" with the appropriate value of $\alpha \in \overline{\mathbb{R}}$.

(II) (Higher-order regular variation). *A function* $f \in AC^{n-1}[T,+\infty)$, $n \ge 1$ *is termed* "*regularly varying at* $+\infty$ (*in the strong sense*) *of order n*" *if each of the functions* $|f|, |f'|, \cdots, |f^{(n-1)}|$ *never vanishes on a neighborhood of* $+\infty$ *and is regularly varying at* $+\infty$ *with its own index of variation. If this is the case we use notation*

$$f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}, \quad \alpha := \text{"the index of } f\text{"} \in \mathbb{R}. \quad (1.6)$$

Notation "$f \in \{\mathcal{R}_\alpha(+\infty) \text{ of exact order } n\}$" implies that f is not regularly varying of order $\ge n+1$.

If $f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}$, $n \ge 1$, *then relations*

$$\begin{cases} f^{(k)}(x)/f(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)x^{-k} + o(x^{-k}) \\ \qquad \equiv \alpha^{\underline{k}} x^{-k} + o(x^{-k}), \; x\to+\infty, 1\le k\le n, \end{cases} \quad (1.7)$$

hold true whichever $\alpha \in \mathbb{R}$ *may be. The indexes of the derivatives are subject to the restrictions specified in* [[1]: Prop. 2.6, p. 796]; *in particular*:

$$f \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}, n \ge 2, \Rightarrow \begin{cases} f'(x) = o(x^{-1}f(x)), \\ f''(x) = \alpha_1 x^{-1} f'(x)[1+o(1)] \\ \text{with } \alpha_1 \le -1; \end{cases} \quad (1.8)$$

where α_1 *is the index of* f' *and the index of* $f^{(k)}$ *is* "$\alpha_1 - k + 1$" *for* $k \ge 2$.

Notice that the last derivative involved in (1.7), *i.e.* $f^{(n)}$, may have an arbitrary sign if $\alpha^{\underline{n}} = 0$.

The following partial converse of relations in (1.7) holds. *If* $f \in AC^{n-1}[T,+\infty)$, $n \ge 2$, *then* $f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}$ *for some real number* $\alpha \notin \{0,1,\cdots,n-2\}$ *iff the following relations hold true*:

$$f^{(k)}(x)/f(x) = \gamma_k x^{-k} + o(x^{-k}), \; x\to+\infty, 1\le k\le n, \quad (1.9)$$

with suitable constants γ_k *such that*

$$\gamma_1, \cdots, \gamma_{n-1} \ne 0; \; (\text{no restriction on } \gamma_n). \quad (1.10)$$

If this is the case then: $\gamma_k = \alpha^{\underline{k}}, 1 \le k \le n$.

(III) (Smooth variation). The preceding partial converse justifies the following concept.

A function $f \in AC^{n-1}[T,+\infty)$, $n \ge 1$, $f(x) \ne 0$ $\forall x$ *large enough, is termed "smoothly varying at* $+\infty$ *of order n and index* α" *if the relations in* (1.7), *referred to* $|f|$, *are satisfied. We denote this class by*: $\{\mathcal{SR}_\alpha(+\infty)$ of order $n\}$. *The following inclusions obtain*:

$$\begin{cases} \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\} = \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\} \\ \qquad\qquad\qquad\qquad \text{if } n=1 \text{ or } \{n \ge 2, \alpha \ne 0,1,\cdots,n-2\}; \\ \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\} \subsetneqq \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\} \text{ otherwise}; \end{cases} \tag{1.11}$$

the reason for the last strict inclusion being that some derivatives of a smoothly-varying function may vanish infinitely often or change sign infinitely often. The following sets of asymptotic relations, for a fixed $\alpha \in \mathbb{R}$, *are equivalent to each other*:

$$x^k f^{(k)}(x)/f(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)+o(1),\ x \to +\infty, 1 \le k \le n; \tag{1.12}$$

$$\begin{cases} xf'(x)/f(x) = \alpha + o(1),\ x \to +\infty; \\ \left(xf'(x)/f(x)\right)^{(k)} = o\left(x^{-k}\right),\ x \to +\infty, 1 \le k \le n. \end{cases} \tag{1.13}$$

(IV) (Rapid variation of first order). *A function* $f \in AC^1[T,+\infty)$ *is called "rapidly varying at* $+\infty$ *of order* 1 (*in the strong restricted sense*)" *if*:

$$\begin{cases} f(x), f'(x) \ne 0 & \forall\, x \text{ large enough}; \\ f(x)/f'(x) = o(x), & x \to +\infty; \\ \left(f(x)/f'(x)\right)' = o(1), & x \to +\infty; \end{cases} \tag{1.14a}$$

or, equivalently, if:

$$\begin{cases} f(x), f'(x) \ne 0 & \forall\, x \text{ large enough}; \\ f''(x)/f'(x) \sim f'(x)/f(x), & x \to +\infty; \end{cases} \tag{1.14b}$$

which imply $f''(x) \ne 0$ *for almost all x large enough. The asymptotic relation in* (1.14b) *is more conveniently written as*

$$f''(x)/f(x) \sim \left(f'(x)/f(x)\right)^2,\ x \to +\infty. \tag{1.14c}$$

(V) (Rapid variation of higher order). *A function* $f \in AC^n[T,+\infty)$ *is called "rapidly varying at* $+\infty$ *of order* $n \ge 2$ (*in the strong restricted sense*)" *if all the functions* $f, f', \cdots, f^{(n-1)}$ *are rapidly varying at* $+\infty$ *in the above-specified sense and this amounts to say that the following conditions hold true as* $x \to +\infty$:

$$f^{(k)}(x) \ne 0\ \ \forall\, x \text{ large enough and } 0 \le k \le n; \tag{1.15}$$

$$f(x)/f'(x) = o(x);\ f'(x)/f''(x) = o(x); \cdots; f^{(n-1)}(x)/f^{(n)}(x) = o(x); \tag{1.16}$$

$$\begin{cases} \left(f(x)/f'(x)\right)' = o(1); \left(f'(x)/f''(x)\right)' = o(1); \cdots; \\ \left(f^{(n-1)}(x)/f^{(n)}(x)\right)' = o(1); \end{cases} \tag{1.17}$$

wherein relations in (1.17) *obviously imply those in* (1.16).

If f *is rapidly varying at* $+\infty$ *of order* $n \ge 2$ *in the previous sense then all* *the functions* $f, f', \cdots, f^{(n-1)}$ *belong to the same class, either* $\mathcal{R}_{-\infty}(+\infty)$ *or* $\mathcal{R}_{+\infty}(+\infty)$, *hence we shall use notation* $f \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$ *to denote that* f *enjoys the properties in* (1.15)-(1.16)-(1.17) *plus the corresponding value* $\pm\infty$ *of the limit in* (1.5). *For an* $f \in AC^n[T, +\infty)$ *satisfying* (1.15) *we have the following characterizations*:

<< Relation

$$f \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\} \cup \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\}$$

holds true if and only if the following equivalent sets of conditions are satisfied:

$$\begin{cases} f'(x)/f(x) \sim f''(x)/f'(x) \sim \cdots \sim f^{(n)}(x)/f^{(n-1)}(x) \sim f^{(n+1)}(x)/f^{(n)}(x), \\ \text{i.e. } D_\ell\left(f^{(k)}(x)\right) \sim D_\ell\left(f(x)\right), x \to +\infty, 1 \le k \le n; \end{cases} \tag{1.18}$$

$$f^{(k+2)}(x) \sim \left(f^{(k+1)}(x)\right)^2 \Big/ f^{(k)}(x), x \to +\infty, 0 \le k \le n-1; \tag{1.19}$$

$$\begin{cases} f^{(k+2)}(x)/f(x) \sim \left(f'(x)/f(x)\right)^{k+2} \equiv \left(D_\ell\left(f(x)\right)\right)^{k+2}, x \to +\infty, \\ 0 \le k \le n-1. \end{cases} \tag{1.20}$$

It follows that even $f^{(n+1)}(x) \ne 0$ *for almost all x large enough*. >>

Remarks. (I) The concepts of regular or smooth variation of order *n* involve derivatives up to order *n*, whereas our restricted concept of rapid variation of order *n* involves derivatives up to order $n+1$.

(II) In the classical definitions of regular or rapid variation, even in the weak Karamata'sense, f is assumed ultimately strictly positive whereas in our definition of higher-order variation f is allowed to be either >0 or <0, the essential point being that it ultimately assumes only one strict sign. The above-added locution in parenthesis "(in the strong sense)" is meant to distinguish our theories from the classical ones wherein the limits in (1.5) are replaced by weaker asymptotic functional relations.

(III) To be consistent with the classical theory, notation "$f \in \mathcal{R}_\alpha(+\infty)$" is used for an (ultimately) strictly positive function whereas, when specifying the indexes of higher-order variation of a function f and its derivatives, we write "$\left|f^{(k)}\right| \in \mathcal{R}_\alpha(+\infty)$" with the absolute values.

With the following notation for the iterated natural logarithms:

$$\begin{cases} \ell_k(x) := \underbrace{\log\left(\log\left(\cdots\left(\log x\right)\cdots\right)\right)}_{k}, k \ge 1, \left(\text{defined for } x \text{ large enough}\right); \\ \ell_0(x) := x; \end{cases} \tag{1.21}$$

we have that:

- Typical functions in the class $\{\mathcal{R}_\alpha(+\infty) \text{ of any order } n \in \mathbb{N}\}, \alpha \in \mathbb{R}$ are:

$$\begin{cases} x^\alpha \cdot \left[\prod_{k=1}^{p_1}\left(\ell_k(x)\right)^{\beta_k}\right] \cdot \left[\prod_{k=1}^{p_2}\exp\left(c_k\left(\log x\right)^{\gamma_k}\right)\right] \cdot \left[\prod_{k=2}^{p_3}\exp\left(d_k\left(\ell_k(x)\right)^{\delta_k}\right)\right], \\ \alpha, \beta_k, c_k, d_k \in \mathbb{R}; 0 < \gamma_k < 1; 0 < \delta_k; \end{cases} \tag{1.22}$$

provided that they do not reduce to a power $x^p, p \in \mathbb{N}$, which belongs to the class $\{\mathcal{R}_p(+\infty) \text{ of exact order } p+1\}$.

- Typical functions in the classes $\{\mathcal{R}_{\pm\infty}(+\infty) \text{ of any order } n \in \mathbb{N}\}$ are:

$$\begin{cases} R_\alpha(x) \cdot \left[\prod_{k=1}^{p_2} \exp\left(c_k (\log x)^{\gamma_k}\right) \right] \cdot \left[\prod_{k=2}^{p_3} \exp\left(d_k x^{\delta_k}\right) \right], \\ R_\alpha \in \{\mathcal{R}_\alpha(+\infty) \text{ of any order } n \in \mathbb{N}\}; c_k, d_k \in \mathbb{R}; \gamma_k > 1; 0 < \delta_k; \end{cases} \tag{1.23}$$

where the pertinent class, either $\{\mathcal{R}_{-\infty}(+\infty)\}$ or $\{\mathcal{R}_{+\infty}(+\infty)\}$ is determined by the behavior of the function as $x \to +\infty$, according as it converges to zero or diverges to $\pm\infty$.

- Each of the following functions

$$x^\alpha; \; x^\alpha + e^{-x}; \; x^\alpha + e^{-x} \sin x \; (\alpha \in \mathbb{R}); \tag{1.24}$$

belongs to the class "$\{\mathcal{SR}_\alpha(+\infty) \text{ of any order } n \in \mathbb{N}\}$" for any $\alpha \in \mathbb{R}$. But, for $\alpha \in \mathbb{N} \cup \{0\}$ they belong to the class $\{\mathcal{R}_\alpha(+\infty) \text{ of exact order } n+1\}$.

Here is a brief summary of the results. In §2 we study to what extent the elementary factorization

$$f \in \mathcal{R}_\alpha(+\infty), \alpha \neq 0, \quad \Leftrightarrow \quad \{f(x) \equiv x^\alpha \mathcal{L}(x), \mathcal{L} \in \mathcal{R}_0(+\infty)\},$$

has an analogue for higher-order variation. After remarking that such an analogue does indeed exist for higher-order smoothly-varying functions and that the mere inference from right to left holds true for higher-order regularly-varying functions under certain restrictions on the values of α, we prove that no such restrictions are needed for an important class of functions including those in (1.22). This very class of higher-order regularly-varying functions does not require any restrictions on the indexes when performing the operation of multiplication. In §3 we obtain non-obvious results on the types of higher-order variation of linear combinations (of arbitrary signs) of various functions. In §4 we list a number of combinatorial identities elementarily inferred from the formulas for higher derivatives of composite or inverse functions and show how these identities may simplify some proofs of previous results about operations with higher-order types of variation. Finally, in §5 there is a discussion about the order of regular variation of the inverse of a higher-order regularly-varying function, pointing out significant difficulties for certain exceptional values of the indexes. Apart from smooth variation, results involving other types of asymptotic variations are obtained via direct analytic computations and estimates of the higher-order derivatives as in [[2]: §7].

Applications of the mentioned results to determining the asymptotic behaviors of certain Hankel determinants are currently being developed by the author, whereas future applications of the whole theory of higher-order types of asymptotic variation to ordinary or partial differential equations are hoped to be studied by the present or other authors.

2. Product of Higher-Order Regularly-Varying Functions

As we know from [[2]: §7.2] the product of two or more regularly-varying functions of order $n \geq 2$ is again of order n only under some restrictions on the indexes of variation, restrictions quite unnatural in applications; so it is useful to point out some classes of functions wherein such restrictions are not needed, above all because these classes include most of the functions of interest in asymptotic questions as those in (1.22). A related question concerns the factorization of a higher-order regularly-varying function which we are going to treat first.

2.1. Factoring Out a Power in a Higher-Order Regularly-Varying Function

An elementary basic property in regular variation states that:

$$f \in \mathcal{R}_\alpha(+\infty), \alpha \neq 0, \quad \Leftrightarrow \quad \{f(x) \equiv x^\alpha \mathcal{L}(x), \mathcal{L} \in \mathcal{R}_0(+\infty)\}, \tag{2.1}$$

and, for higher-order smooth variation, one of the properties in [[2]: formula (7.3), p. 820] implies the analogous property:

$$\begin{aligned} & f \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\}, \alpha \neq 0, n \geq 1, \\ & \Leftrightarrow \begin{cases} f(x) \equiv x^\alpha \mathcal{L}(x) \\ \mathcal{L} \in \{\mathcal{SR}_0(+\infty) \text{ of order } n\}. \end{cases} \end{aligned} \tag{2.2}$$

But, in general, the corresponding equivalence does not hold true for regular variation of order $n \geq 2$. As noticed in [[2]: Remark 3, p. 824], and consistently with [[2]: Prop. 7.3-(I), p. 821]:

$$\{f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}, \alpha \neq 0\} \not\Rightarrow x^{-\alpha} f(x) \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}; \tag{2.3}$$

$$\mathcal{L} \in \{\mathcal{R}_0(+\infty) \text{ of order } n\} \Rightarrow \begin{cases} x^\alpha \mathcal{L}(x) \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\} \\ \text{if } \alpha \neq 0, 1, \cdots, n-2. \end{cases} \tag{2.4}$$

Examples for (2.3) are provided by powers times the slowly-varying function $L(x) := 2 + \sin\left((\log x)^\beta\right), 0 < \beta < 1$:

$$\begin{cases} f(x) := x^\alpha L(x) \in \{\mathcal{R}_\alpha(+\infty) \text{ of any order } n \in \mathbb{N}\} \text{ if } \alpha \notin \mathbb{N} \cup \{0\}; \\ x^{-\alpha} f(x) \equiv L(x) \in \{\mathcal{R}_0(+\infty) \text{ of exact order } 1\}; \\ x^{n-\alpha} f(x) \in \{\mathcal{R}_0(+\infty) \text{ of exact order } n+1\}; \end{cases} \tag{2.5}$$

which show that the statement in (2.3) cannot be improved no matter what restrictions on the exponent α. On the other part a counterexample for the inference in (2.4) is provided by:

$$\begin{cases} g(x) := 1 + x^{-p} \in \{\mathcal{R}_0(+\infty) \text{ of any order } n \in \mathbb{N}\} \text{ whatever } p > 0; \\ x^p g(x) \equiv x^p + 1 \in \begin{cases} \{\mathcal{R}_p(+\infty) \text{ of exact order } p+1\} \text{ if } p \in \mathbb{N}, \\ \{\mathcal{R}_p(+\infty) \text{ of any order } n \in \mathbb{N}\} \text{ if } p \notin \mathbb{N}. \end{cases} \end{cases} \tag{2.6}$$

However, for the function $\mathcal{L}(x) := \log x$ which belongs to the class

$\{\mathcal{R}_0(+\infty) \text{ of any order } n\}$, it is easily checked that the inference in (2.4) is true with no restriction on α, and this leads to conjecture that the same happens for the powers of iterated logarithms and their products appearing in (1.22). Such a result would be quite convenient in asymptotic analysis and is easily inferred from the formulas for certain higher derivatives collected in the following

Lemma 2.1. If $\left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty), 0 \le k \le n-1, n \ge 2$, then the following relations hold true:

$$D^k\left(x^\alpha \mathcal{L}(x)\right)\begin{cases} \sim \alpha^{\underline{k}} x^{\alpha-k}\mathcal{L}(x), x \to +\infty, \text{ if } \alpha \notin \mathbb{N}\cup\{0\} \text{ and } 1 \le k \le n; \\ \sim \alpha^{\underline{k}} x^{\alpha-k}\mathcal{L}(x)\ x \to +\infty, \text{ if } \alpha \in \mathbb{N} \text{ and } 1 \le k \le \alpha; \end{cases} \tag{2.7}$$

$$D^{\alpha+1}\left(x^\alpha \mathcal{L}(x)\right) \sim \alpha!\mathcal{L}'(x) = o\left(x^{-1}\mathcal{L}(x)\right), x \to +\infty, \text{ if } \alpha \in \mathbb{N}; \tag{2.8}$$

$$\begin{cases} D^{\alpha+m}\left(x^\alpha \mathcal{L}(x)\right) \sim (-1)^{m-1}\alpha!(m-1)!\,x^{1-m}\mathcal{L}'(x), x \to +\infty, \\ \text{if } \alpha \in \mathbb{N}, m > 1, \end{cases} \tag{2.9}$$

and obviously $\alpha + m \le n+1$. For the special case of $\mathcal{L}(x) := x^p \log x, p \in \mathbb{N}$, we have the elementary formulas:

$$\begin{cases} D\left(x^p \log x\right) = px^{p-1}\log x + x^{p-1}; \\ D^2\left(x^p \log x\right) = p(p-1)x^{p-2}\log x + px^{p-2} + (p-1)x^{p-2}; \\ D^k\left(x^p \log x\right) = p^{\underline{k}} x^{p-k}\log x + (\text{constant})\cdot x^{p-k}, 1 \le k \le p-1; \\ D^p\left(x^p \log x\right) = p!\log x + \text{constant}; \\ D^{p+m}\left(x^p \log x\right) = p!\,D^m(\log x) \sim (-1)^{m-1}p!(m-1)!\,x^{-m}, m \ge 1; \end{cases} \tag{2.10}$$

the exact values of the constants being not presently needed.

Proposition 2.2. The following conditions

$$\begin{cases} \left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty), 0 \le k \le n-1, n \ge 2, \\ f(x) := x^\alpha \mathcal{L}(x), \alpha \in \mathbb{R}\setminus\{0\}, \end{cases} \tag{2.11a}$$

imply:

$$\begin{cases} f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\}, \\ \left|f^{(k)}\right| \in \mathcal{R}_{\alpha-k}(+\infty), 0 \le k \le n-1. \end{cases} \tag{2.11b}$$

We explicitly point out that the statement

$$\begin{Bmatrix} \mathcal{L} \in \{\mathcal{R}_0(+\infty) \text{ of order } n\}, \\ f(x) := x^\alpha \mathcal{L}(x), \alpha \in \mathbb{R}\setminus\{0\} \end{Bmatrix} \Rightarrow f \in \{\mathcal{R}_\alpha(+\infty) \text{ of order } n\},$$

is in general false without specifying the indexes of variation for the derivatives of $\mathcal{L}$, as shown by the above function $g(x) := 1 + x^{-p}, p \in \mathbb{N}$, which belongs to the class $\{\mathcal{R}_0(+\infty) \text{ of any order } n\}$ whereas $x^p g(x)$ belongs to the class $\{\mathcal{R}_p(+\infty) \text{ of exact order } p+1\}$.

Proofs of Lemma 2.1 *and Proposition* 2.2. *First case*: $\alpha \notin \mathbb{N}\cup\{0\}$. An indirect proof of the inference "(2.11a) $\Rightarrow$ (2.11b)" is to be found in [[2]: Prop. 7.3-(I), p. 821]. For a direct proof notice that the assumptions on $\mathcal{L}$ state exactly that, as $x \to +\infty$:

$$\begin{cases} \mathcal{L} \in \mathcal{R}_0(+\infty); \quad \mathcal{L}'(x) = o\left(x^{-1}L(x)\right),\ \mathcal{L}'(x) \neq 0 \text{ ultimately;} \\ \mathcal{L}''(x) \sim -x^{-1}\mathcal{L}'(x); \\ \mathcal{L}^{(3)}(x) \sim -2x^{-1}\mathcal{L}''(x) \sim 2!\,x^{-2}\mathcal{L}'(x); \\ \cdots \\ \mathcal{L}^{(k)}(x) \sim -(k-1)x^{-1}\mathcal{L}^{(k-1)}(x) \sim (-1)^{k-1}(k-1)!\,x^{1-k}\mathcal{L}'(x), \\ 2 \le k \le n; \end{cases} \tag{2.12}$$

so that by Leibniz's formula, and taking account that $\alpha^{\underline{i}} \neq 0\ \forall\, i$:

$$\begin{aligned} D^k\left(x^\alpha \mathcal{L}(x)\right) &= \sum_{i=0}^{k} \binom{k}{i} \alpha^{\underline{i}} x^{\alpha-i} \mathcal{L}^{(k-i)}(x) \\ &= \alpha^{\underline{k}} x^{\alpha-k}\mathcal{L}(x) + \sum_{i=0}^{k-1}\left\{(-1)^{k-i-1}\binom{k}{i}(k-i-1)!\,\alpha^{\underline{i}} x^{\alpha-i+1-(k-i)}\mathcal{L}'(x)\left[1+o(1)\right]\right\}; \end{aligned} \tag{2.13}$$

but the last sum is "$o\left(x^{\alpha-k}\mathcal{L}(x)\right)$" and $\alpha^{\underline{k}} \neq 0$, and so we get:

$$D^k\left(x^\alpha \mathcal{L}(x)\right) \sim \alpha^{\underline{k}} x^{\alpha-k}\mathcal{L}(x), 1 \le k \le n, \tag{2.14}$$

which is the first relation in (2.7). This implies:

$$D^{k+1}\left(x^\alpha \mathcal{L}(x)\right) \Big/ D^k\left(x^\alpha \mathcal{L}(x)\right) \sim (\alpha - k)x^{-1}, 0 \le k \le n-1,$$

which is our claim.

Second case: $f(x) := x^p \mathcal{L}(x),\ p \in \mathbb{N}$. For the special choice $\mathcal{L}(x) := x^p \log x$ formulas in (2.10) imply that "$x^p \log x \in \{\mathcal{R}_p(+\infty) \text{ of any order } n\}$" with $\left|D^k\left(x^p \log x\right)\right| \in \mathcal{R}_{p-k}(+\infty)$, and we shall show that the case of a generic $\mathcal{L}$ may be reduced to this special choice. If $k \le p$ (and obviously $k \le n$) then in the calculations in (2.13), with α replaced by *p*, we have $p^{\underline{k}} \neq 0$ so that:

$$D^k\left(x^p \mathcal{L}(x)\right) \sim p^{\underline{k}} x^{p-k}\mathcal{L}(x), 1 \le k \le p,$$

which is the second relation in (2.7) and implies that $\left|D^k\left(x^p \mathcal{L}(x)\right)\right| \in \mathcal{R}_{p-k}(+\infty)$, $1 \le k \le p-1$. For the principal parts of the derivatives of order higher than *p* we have $\left(D^{p+m}x^p\right) \equiv 0$ for $m \ge 1$ so that:

$$\begin{aligned} D^{p+m}\left(x^p \mathcal{L}(x)\right) &= \sum_{i=0}^{p+m}\binom{p+m}{i}\left(D^{p+m-i}x^p\right)\mathcal{L}^{(i)}(x) \\ &\equiv \sum_{i=m}^{p+m}\binom{p+m}{i}\left(D^{p+m-i}x^p\right)\mathcal{L}^{(i)}(x) \overset{(2.12)}{=} \\ &= \sum_{i=m}^{p+m}\left\{\binom{p+m}{i}\left(D^{p+m-i}x^p\right)(-1)^{i-1}(i-1)!\,x^{1-i}\mathcal{L}'(x)\left[1+o(1)\right]\right\} \\ &= \cdots \end{aligned} \tag{2.15}$$

Now, apart from the common factor $x\mathcal{L}'(x)$, each term inside the last sum is of type $c_i x^{-m}$, the power being independent of i; hence it is legitimate to factor out the expression $\left[1+o(1)\right]$ provided that the new sum equals a non-zero constant times x^{-m}. The following further steps are then correct:

$$D^{p+m}\left(x^p\mathcal{L}(x)\right)\overset{(2.10)}{=}$$
$$= x\mathcal{L}'(x)\left\{\sum_{i=m}^{p+m}\binom{p+m}{i}\left(D^{p+m-i}x^p\right)\cdot\left(D^i\log x\right)\right\}\left[1+o(1)\right]$$
$$\equiv x\mathcal{L}'(x)\left\{\sum_{i=0}^{p+m}\binom{p+m}{i}\left(D^{p+m-i}x^p\right)\cdot\left(D^i\log x\right)\right\}\left[1+o(1)\right] \tag{2.16}$$
$$= D^{p+m}\left(x^p\log x\right)\cdot x\mathcal{L}'(x)\left[1+o(1)\right],\ m\ge 1,$$

where the principal part of $D^{p+m}\left(x^p\log x\right)$ is reported in the last line in (2.10). This proves relations in (2.8), (2.9) and from these relations, the inference "(2.11a) $\Rightarrow$ (2.11b)" is straightforwardly obtained, namely:

$$D^{p+1}\left(x^p\mathcal{L}(x)\right)\sim p!\mathcal{L}'(x)=o\left(x^{-1}\mathcal{L}(x)\right)\quad \text{i.e.}\quad D^p\left(x^p\mathcal{L}(x)\right)\in\mathcal{R}_0(+\infty); \tag{2.17}$$

$$D^{p+m}\left(x^p\mathcal{L}(x)\right)/D^{p+m-1}\left(x^p\mathcal{L}(x)\right)\sim D^{p+m}\left(x^p\log x\right)/D^{p+m-1}\left(x^p\log x\right)$$
$$\sim -(m-1)x^{-1},\ x\to+\infty, \tag{2.18}$$

for all the values of $m>1$ admitted by the assumptions. □

Proposition 2.3. If $\mathcal{L}$ satisfies the assumptions in (2.11a) and if P is a linear combination of real powers,

$$P(x):=a_1x^{\alpha_1}+\cdots+a_mx^{\alpha_m},\ \alpha_1>\cdots>\alpha_m,\ a_i\ne 0\ \ \forall\, i, \tag{2.19}$$

then

$$\left|(P\cdot\mathcal{L})^{(k)}\right|\in\mathcal{R}_{\alpha_1-k}(+\infty),\ 0\le k\le n-1. \tag{2.20}$$

Proof. By Proposition 2.2, and whatever the α_i's, we have the relations analogous to those in (2.12):

$$\begin{cases}\left(x^{\alpha_i}\mathcal{L}(x)\right)'=x^{-1}\left(x^{\alpha_i}\mathcal{L}(x)\right)\left[\alpha_i+o(1)\right]=x^{\alpha_i-1}\mathcal{L}(x)\left[\alpha_i+o(1)\right];\\ \left(x^{\alpha_i}\mathcal{L}(x)\right)''=x^{-1}\left(x^{\alpha_i}\mathcal{L}(x)\right)'\left[(\alpha_i-1)+o(1)\right]=x^{\alpha_i-2}\mathcal{L}(x)\left[\alpha_i(\alpha_i-1)+o(1)\right];\\ \cdots\\ \left(x^{\alpha_i}\mathcal{L}(x)\right)^{(k)}=x^{\alpha_i-k}\mathcal{L}(x)\left[(\alpha_i)^{\underline{k}}+o(1)\right],\ 2\le k\le n.\end{cases} \tag{2.21}$$

Now, if $(\alpha_1)^{\underline{j}}\ne 0$ for some j we also have $(\alpha_1)^{\underline{k}}\ne 0$ for $0\le k\le j$, and:

$$\frac{\left(\sum_{i=1}^m a_ix^{\alpha_i}\mathcal{L}(x)\right)^{(k+1)}}{\left(\sum_{i=1}^m a_ix^{\alpha_i}\mathcal{L}(x)\right)^{(k)}}=\frac{\sum_{i=1}^m a_ix^{\alpha_i-k-1}\mathcal{L}(x)\left[(\alpha_i)^{\underline{k+1}}+o(1)\right]}{\sum_{i=1}^m a_ix^{\alpha_i-k}\mathcal{L}(x)\left[(\alpha_i)^{\underline{k}}+o(1)\right]}$$
$$=\frac{a_1(\alpha_1)^{\underline{k+1}}x^{\alpha_1-k-1}\mathcal{L}(x)\left[1+o(1)\right]}{a_1(\alpha_1)^{\underline{k}}x^{\alpha_1-k}\mathcal{L}(x)\left[1+o(1)\right]}$$
$$\sim(\alpha_1-k)x^{-1}, \tag{2.22}$$

proving relations in (2.20) for $0\le k\le j-1$. If $(\alpha_1)^{\underline{j-1}}\ne 0$ and $(\alpha_1)^{\underline{j}}=0$ we are just in the situation of the second case in the proof of Proposition 2.2 with $j=\alpha_1\in\mathbb{N}$ and we need the principal parts of the derivatives of order $>\alpha_1$ of the terms in the above sums. Using the last equality in (2.16) with p replaced by α_1, we get:

$$\begin{aligned}\left(\sum_{i=1}^{m} a_i x^{\alpha_i}\mathcal{L}(x)\right)^{(\alpha_1+q)} &= \sum_{i=1}^{m}\left\{a_i D^{\alpha_1+q}\left(x^{\alpha_i}\log x\right)\cdot x\mathcal{L}'(x)\left[1+o(1)\right]\right\} \\ &\equiv \sum_{i=1}^{m}\left\{a_i D^{\alpha_i+(\alpha_1-\alpha_i)+q}\left(x^{\alpha_i}\log x\right)\cdot x\mathcal{L}'(x)\left[1+o(1)\right]\right\} \\ &\sim a_1 D^{\alpha_1+q}\left(x^{\alpha_1}\log x\right)\cdot x\mathcal{L}'(x),\end{aligned} \tag{2.23}$$

having used the following estimates inferred from the last line in (2.10):

$$\begin{aligned} D^{\alpha_i+(\alpha_1-\alpha_i)+q}\left(x^{\alpha_i}\log x\right) &\sim c_{i,q} x^{-(\alpha_1-\alpha_i)-q} \\ &= o\left(D^{\alpha_1+q}\left(x^{\alpha_1}\log x\right)\right) \quad \text{for } i \ge 2. \end{aligned} \tag{2.24}$$

The conclusion straightforwardly follows as in the last lines of the proof of Proposition 2.2. □

2.2. Product of Higher-Order Regularly-Varying Functions

We know that in general, under the assumptions

$$f \in \left\{\mathcal{R}_\alpha(+\infty) \text{ of order } n\right\}, \quad g \in \left\{\mathcal{R}_\beta(+\infty) \text{ of order } n\right\}, \tag{2.25}$$

no claim may be inferred concerning the order of higher variation of the product fg without the restriction "$\alpha+\beta \ne 0,1,\cdots,n-2$". Apart from the counterexamples in [[2]: p.824], a most simple counterexample (playing a role in the next proposition) is the product $f\cdot(1/f)$ where:

$$\begin{cases} f \in \left\{\mathcal{R}_\alpha(+\infty) \text{ of any order } n\in\mathbb{N}\right\}; \\ 1/f \in \left\{\mathcal{R}_{-\alpha}(+\infty) \text{ of any order } n\in\mathbb{N}\right\}; \\ f\cdot(1/f) \in \left\{\mathcal{R}_0(+\infty) \text{ of exact order } 1\right\}, \end{cases} \tag{2.26}$$

as, e.g., $f(x) := x^\alpha$ with $\alpha \notin \mathbb{N}$, or $f(x) := \log x$. We point out two additional conditions each of which grants a precise statement concerning the higher-order variation of the product with no a-priori restriction on the indexes.

Proposition 2.4. (I) (Product of higher-order slowly-varying functions.) The inference

$$\begin{aligned} &\left\{\left|f^{(k)}\right|,\left|g^{(k)}\right| \in \mathcal{R}_{-k}(+\infty),\ 0\le k\le n-1\right\} \\ &\Rightarrow \begin{cases} f\cdot g \in \left\{\mathcal{R}_0(+\infty) \text{ of order } n\right\}, \\ \left|(f\cdot g)^{(k)}\right| \in \mathcal{R}_{-k}(+\infty),\ 0\le k\le n-1, \end{cases} \end{aligned} \tag{2.27}$$

holds true under any of the following two additional conditions:

$$\begin{cases} \text{either} \quad f'(x)g(x) \gg f(x)g'(x),\ x\to+\infty, \ \text{ or} \\ \operatorname{sign}\left(f'(x)g(x)\right) = \operatorname{sign}\left(f(x)g'(x)\right) \ \text{ ultimately.} \end{cases} \tag{2.28}$$

(II) (Product of higher-order regularly-varying functions.) If

$$\begin{cases} f_1(x) := x^{\alpha} f(x) \in \{\mathcal{R}_{\alpha}(+\infty) \text{ of order } n\}, \\ f_2(x) := x^{\beta} g(x) \in \{\mathcal{R}_{\beta}(+\infty) \text{ of order } n\}, \end{cases} \quad \alpha, \beta \in \mathbb{R}, \tag{2.29}$$

with f, g satisfying the assumptions in (2.27), then:

$$\begin{cases} f_1 \cdot f_2 \in \{\mathcal{R}_{\alpha+\beta}(+\infty) \text{ of order } n\}, \\ \left|(f_1 \cdot f_2)^{(k)}\right| \in \mathcal{R}_{\alpha+\beta-k}(+\infty), \, 0 \le k \le n-1. \end{cases} \tag{2.30}$$

The previously-mentioned case $f \cdot g$, with $g = 1/f$, is a good counterexample if both conditions in (2.28) are lacking because of the identity $f' \cdot g \equiv -f \cdot g'$.

Proof. Part (II) follows at once from part (I) and Proposition 2.2. We have to prove the property in (2.27) concerning $(f \cdot g)^{(k)}$ for $k \ge 1$, for $k = 0$ being trivial. The assumptions imply the relations, like those in (2.12):

$$\begin{cases} f'(x) = o\left(x^{-1} f(x)\right), \quad g'(x) = o\left(x^{-1} g(x)\right), \\ f^{(k)}(x) \sim (-1)^{\underline{k-1}} (k-1)! \, x^{1-k} f'(x), \ 2 \le k \le n, \\ g^{(k)}(x) \sim (-1)^{\underline{k-1}} (k-1)! \, x^{1-k} g'(x), \ 2 \le k \le n; \end{cases} \tag{2.31}$$

and, sometimes omitting the argument of the functions, we write:

$$(fg)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(i)} g^{(k-i)} = \underbrace{fg^{(k)} + f^{(k)} g}_{R_k} + \sum_{i=1}^{k-1} \binom{k}{i} f^{(i)} g^{(k-i)}, \tag{2.32}$$

where:

$$R_k(x) \sim (-1)^{\underline{k-1}} (k-1)! \, x^{1-k} \left[f'g + fg' \right] \sim (-1)^{\underline{k-1}} (k-1)! \, x^{1-k} (fg)'. \tag{2.33}$$

For each term into the sum we have:

$$f^{(i)}(x) g^{(k-i)}(x) = O\left(x^{1-i} f'(x) \cdot x^{1-k+i} g'(x)\right) = \begin{cases} o\left(x^{1-k} f'(x) g(x)\right), \\ o\left(x^{1-k} f(x) g'(x)\right), \end{cases} \tag{2.34}$$

and it follows that for each $i \in \{1, \cdots, k-1\}$ either:

$$f'g \gg fg' \ (\text{or viceversa}) \Rightarrow f^{(i)}(x) g^{(k-i)}(x) = o\left(x^{1-k} \left(f(x) g(x)\right)'\right); \tag{2.35}$$

or:

$$\begin{aligned} &\operatorname{sign}\left(f'(x) g(x)\right) = \operatorname{sign}\left(f(x) g'(x)\right) \\ &\Rightarrow \ f^{(i)}(x) g^{(k-i)}(x) = o\left(x^{1-k} \left(\left|f'(x) g(x)\right| + \left|f(x) g'(x)\right|\right)\right) \\ &\quad = o\left(x^{1-k} \left(\left|f'(x) g(x) + f(x) g'(x)\right|\right)\right) = o\left(x^{1-k} \left(f(x) g(x)\right)'\right). \end{aligned} \tag{2.36}$$

In any case, from (2.32)-(2.33) we get the relations

$$(fg)^{(k)} \sim (-1)^{\underline{k-1}} (k-1)! \, x^{1-k} (fg)', \, x \to +\infty, \tag{2.37}$$

which implies the thesis. □

3. Index of Higher-Order Variation for Linear Combinations

The exact evaluation of the index of variation of a linear combination of two functions is in general possible only under some restrictions: see [[1]: Prop. 2.1-(iii), p. 784, claims in (2.27) and in the subsequent line] for regular variation, and [[2]: relations in (7.4), p. 820] for higher-order smooth variation. For linear combinations of more than two functions we gave a result in [[1]: Prop. 2.3-(II), p. 789], under "*the least possible restrictions*", including regular or rapid variation. Extensions to more than two functions with higher-order regular or rapid variation require specifications in the proofs or in the statement. We give some extensions needed in applications.

3.1. Linear Combinations of Higher-Order Smoothly-Varying Functions

We first rewrite the mentioned results in [[2]: relations in (7.4)] as we must refer to them several times.

Lemma 3.1. If $f \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\}$ and $g \in \{\mathcal{SR}_\beta(+\infty) \text{ of order } n\}$ then:

$$\begin{cases} c_1 f + c_2 g \in \{\mathcal{SR}_{\max(\alpha,\beta)}(+\infty) \text{ of order } n\} \\ \forall\, \alpha, \beta, c_1, c_2 \in \mathbb{R},\ \alpha \neq \beta,\ c_i \neq 0; \end{cases} \tag{3.1a}$$

$$\begin{cases} c_1 f + c_2 g \in \{\mathcal{SR}_\alpha(+\infty) \text{ of order } n\} \text{ if } \alpha = \beta \text{ and} \\ \text{either } \{c_i > 0;\ f, g > 0\} \text{ or } \{c_i \neq 0;\ f(x) \gg g(x),\ x \to +\infty\}. \end{cases} \tag{3.1b}$$

And the following is the extension to more than two functions.

Proposition 3.2. Let

$$f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n\},\ 1 \le i \le p. \tag{3.2}$$

(I) If "$c_i > 0,\ f_i(x) > 0$ ultimately", and "$\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \cdots \ge \alpha_p$" then:

$$\sum_{i=1}^{p} c_i f_i \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}. \tag{3.3}$$

(II) If "$f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p$" a condition granted by the restriction "$\alpha_1 > \alpha_2 \ge \cdots \ge \alpha_p$", then:

$$\sum_{i=1}^{p} c_i f_i \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\} \quad \forall\, c_i = \text{constant} \neq 0. \tag{3.4}$$

The import of this last statement is that there is *one* function, namely f_1, with the maximal growth-order and, though we cannot be sure that the linear combination of $f_2, \cdots, f_p$ is smoothly varying of order n (whatever the index may be) we have the desired conclusion.

(III) In particular, in either case and for $n \ge 2$:

$$\alpha_1 \neq 0, 1, \cdots, n-2, \Rightarrow \sum_{i=1}^{p} c_i f_i \in \{\mathcal{R}_{\alpha_1}(+\infty) \text{ of order } n\}. \tag{3.5}$$

Proof. Both parts (I), (II) may be proved repeatedly applying Lemma 3.1 so inferring, step by step, that:

$$\begin{cases} c_1 f_1 + c_2 f_2 \equiv g_2 \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}; \\ (g_2 \sim c_1 f_1 \gg f_i, 2 \le i \le p, \text{ for part (II)}); \end{cases}$$

$$\begin{cases} g_2 + c_3 f_3 \equiv g_3 \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}; \\ (g_3 \sim g_2, \text{ for part (II)}); \end{cases}$$

and so on, arriving at the conclusion in (3.3) or (3.4). Part (III) follows from the first relation in (1.11). □

Remarks 1) Condition " $f_1 \gg f_i, 2 \le i \le p$ " in part (II) is essential in some applications wherein the more stringent condition " $f_1 \gg f_2 \gg \cdots \gg f_p$ " may not be satisfied.

2) For the conclusion about regular variation in (3.5) the restrictions on α_1 are necessary as shown by the following simple counterexamples, see [[1]: Remark 2, p. 798]:

$$\begin{cases} f_1 := x^m + x^{-1} \in \{\mathcal{R}_m(+\infty) \text{ of any order } n\}, m \in \mathbb{N} \cup \{0\}; \\ f_2 := x^m - x^{-1} \in \{\mathcal{R}_m(+\infty) \text{ of any order } n\}; \\ f_1 + f_2 = 2x^m \in \{\mathcal{R}_m(+\infty) \text{ of exact order } m+1\}; \end{cases}$$

which is a counterexample for part (I) and $n \ge 2$;

$$\begin{cases} f_1 := x^m + x^{-1} \in \{\mathcal{R}_m(+\infty) \text{ of any order } n\}, m \in \mathbb{N} \cup \{0\}; \\ f_2 := -x^{-1} \in \{\mathcal{R}_{-1}(+\infty) \text{ of any order } n\}; \\ f_1 + f_2 = x^m \in \{\mathcal{R}_m(+\infty) \text{ of exact order } m+1\}; \end{cases}$$

which is a counterexample for part (II) and $n \ge 2$.

3) If in (3.2) we assume regular (instead of smooth) variation, *i.e.* $f_i \in \{\mathcal{R}_{\alpha_i}(+\infty) \text{ of order } n\}$, and also $f_1 \gg f_i, 2 \le i \le p$, then a direct proof of the conclusion in (3.5) can be given using the results in [[1]: Prop. 3.1, p. 799] highlighting once again the necessity of the restrictions on α_1 for $n \ge 2$. In fact, by part (I) of this cited proposition the assumptions in (3.2) imply the relations:

$$f_i^{(k)}(x) = f_i(x)x^{-k}\left[(\alpha_i)^{\underline{k}} + o(1)\right], x \to +\infty, (1 \le k \le n; 1 \le i \le p), \tag{3.6}$$

whence, using relation " $f_2(x) \ll f_1(x)$ ":

$$\begin{aligned} &\frac{c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x)}{c_1 f_1(x) + c_2 f_2(x)} \\ &= \frac{c_1 f_1(x)x^{-k}\left[(\alpha_1)^{\underline{k}} + o(1)\right] + c_2 f_2(x)x^{-k}\left[(\alpha_2)^{\underline{k}} + o(1)\right]}{c_1 f_1(x) + c_2 f_2(x)} \\ &= \frac{c_1 f_1(x)x^{-k}\left[(\alpha_1)^{\underline{k}} + o(1)\right]}{c_1 f_1(x)\left[1 + o(1)\right]} = x^{-k}\left[(\alpha_1)^{\underline{k}} + o(1)\right]. \end{aligned} \tag{3.7}$$

The restrictions on α_1 imply "$(\alpha_1)^{\underline{k}} \ne 0$ for $1 \le k \le n-1$ so that from part

(II) of the mentioned proposition in [1] we infer that "$c_1 f_1 + c_2 f_2 \in \{\mathcal{R}_{\alpha_1}(+\infty) \text{ of order } n\}$". Repeating the procedure used in the proof of Proposition 3.2 we get the assertion for $\sum_{i=1}^{p} c_i f_i$.

3.2. Linear Combinations of Various Types of Asymptotic Variations

If some of the involved functions are rapidly varying some caution is required because our adopted concept of "*n*th-order rapid variation" does not simply means the validity of the limits

$$\lim_{x\to+\infty} x f^{(k+1)}(x) / f^{(k)}(x) = \pm\infty,\ 0 \le k \le n, \quad \text{with the suitable sign},$$

but requires the additional conditions in (1.17) or the equivalent formulations in (1.18)-(1.20). To make clear this point notice that repeated applications of the three results in [[1]: Prop. 2.3-(I), formula (2.52)] easily yield the following

Proposition 3.3. (*Positive linear combinations*). *If*

$$f_i(x) > 0 \ \text{ ultimately}, \ c_i > 0 \ \text{ for all the involved } i, \tag{3.8}$$

then:

$$\left\{ f_i^{(k)} \in \mathcal{R}_{-\infty}(+\infty), 1 \le i \le p, 0 \le k \le n \right\} \Rightarrow (-1)^k \sum_{i=1}^{p} c_i f_i^{(k)} \in \mathcal{R}_{-\infty}(+\infty), 0 \le k \le n; \tag{3.9}$$

$$\left\{ f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), 1 \le i \le p, 0 \le k \le n \right\} \Rightarrow \sum_{i=1}^{p} c_i f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), 0 \le k \le n; \tag{3.10}$$

$$\left\{ \begin{array}{l} f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), 1 \le i \le p, 0 \le k \le n, \\ f_i^{(k)} \in \mathcal{R}_{\alpha_i}(+\infty), \ p+1 \le i \le q, 0 \le k \le n, (\alpha_i \in \mathbb{R}), \end{array} \right\} \Rightarrow \sum_{i=1}^{q} c_i f_i^{(k)} \in \mathcal{R}_{+\infty}(+\infty), 0 \le k \le n. \tag{3.11}$$

However, the analogous inferences wherein the notation "$\mathcal{R}_{\pm\infty}(+\infty), 0 \le k \le n$" is replaced by "$\{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$", both in the hypotheses and in the theses, are not automatic facts; a counterexample will be given at the end of this section confirming the specificity of our restricted concept of rapid variation.

Proof. For $n = 0$ the three inferences above follow from direct iterations of the results in [[1]: Prop. 2.3-(I), inferences in (2.52), p. 789]; only for the inference in (3.11) one must notice that:

$$\sum_{i=1}^{p} c_i f_i \in \mathcal{R}_{+\infty}(+\infty); \ \sum_{i=p+1}^{q} c_i f_i \in \mathcal{R}_{\max \alpha_i}(+\infty) \ \text{ by [[1]: formula (2.27)]}. \tag{3.12}$$

For $n \ge 1$ only simple remarks on the derivatives are needed. The assumptions in (3.9) imply that "$(-1)^k f_i^{(k)}(x) > 0$ ultimately" so that the result for $n = 0$ applied to the derivatives gives the conclusion. Analogously for (3.10), and also for (3.11) using (3.12) referred to $f_i^{(k)}$ for each fixed *k*. □

Much more useful than Proposition 3.3 is a result on arbitrary linear combinations under certain asymptotic restrictions so extending [[1]: Prop. 2.3-(II), p. 789].

Proposition 3.4. (Arbitrary linear combinations).

(*Warning*. The notation "$f_i \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}$" in the next statement means that each f_i belongs to its own class, not necessarily the same for all of the f_i's.)

(I) Let

$$\begin{cases} f_i \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n\}, 1 \le i \le p; \\ f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p; \\ g(x) := \sum_{i=1}^{p} c_i f_i(x);\ \ c_i \in \mathbb{R} \setminus \{0\}. \end{cases} \tag{3.13}$$

If anyone of the following additional conditions is satisfied, either

$$f_1^{(k)}(x) \gg f_i^{(k)}(x),\ x \to +\infty, 1 \le k \le n+1,\ 2 \le i \le p, \tag{3.14}$$

or

$$f_1'(x)/f_1(x) \succcurlyeq f_i'(x)/f_i(x),\ 2 \le i \le p, \tag{3.15}$$

then:

$$g^{(k)}(x)/g(x) \sim \left(f_1'(x)/f_1(x)\right)^k,\ 2 \le k \le n+1, \tag{3.16}$$

which, by (1.20), implies that "g belongs to the same class of f_1".

(II) If

$$\begin{cases} f_i \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\}, 1 \le i \le p; \\ f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n+1\},\ p+1 \le i \le q;\ \alpha_i \in \mathbb{R}; \\ f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p; \\ \text{one of the conditions in (3.14)-(3.15) for the indexes } 2 \le i \le p; \\ f_{p+1}(x) \gg f_i(x),\ x \to +\infty,\ p+1 \le i \le q; \\ h(x) := \sum_{i=1}^{q} c_i f_i(x);\ \ c_i \in \mathbb{R} \setminus \{0\}, \end{cases} \tag{3.17}$$

then

$$h^{(k)}(x)/h(x) \sim \left(f_1'(x)/f_1(x)\right)^k,\ 2 \le k \le n+1, \tag{3.18}$$

which, by (1.20), implies that "$h \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\}$".

(III) If

$$\begin{cases} f_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n\}, 1 \le i \le p;\ \alpha_i \in \mathbb{R}; \\ f_i \in \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\},\ p+1 \le i \le q; \\ f_1(x) \gg f_i(x),\ x \to +\infty,\ 2 \le i \le p; \\ f_{p+1}(x) \gg f_i(x),\ x \to +\infty,\ p+1 \le i \le q; \\ \text{one of the conditions in (3.14)-(3.15) referred to the indexes } p+1 \le i \le q; \\ h(x) := \sum_{i=1}^{q} c_i f_i(x);\ \ c_i \in \mathbb{R} \setminus \{0\}, \end{cases} \tag{3.19}$$

then

$$h^{(k)}(x) = (\alpha_1)^{\underline{k}} x^{-k} + o(x^{-k}), 1 \le k \le n, \tag{3.20}$$

which, by definition, means that "$h \in \{\mathcal{SR}_{\alpha_1}(+\infty)$ of order $n\}$".

Proof. When working with the classes "$\{\mathcal{R}_{\pm\infty}(+\infty)$ of order $n\}$" we shall use the relations in (1.20) both in the assumptions and in the thesis. (I) In (3.13) we are assuming the relations

$$f_i^{(k)}(x)/f_i(x) \sim (f_i'(x)/f_i(x))^k, 2 \le k \le n+1, 1 \le i \le p.$$

Now, (3.14) imply:

$$\frac{g^{(k)}(x)}{g(x)} = \frac{c_1 f_1^{(k)}(x) + \sum_{i=2}^p c_i f_i^{(k)}(x)}{c_1 f_1(x) + \sum_{i=2}^p c_i f_i(x)} \sim \frac{c_1 f_1^{(k)}(x)}{c_1 f_1(x)}$$
$$\sim (f_1'(x)/f_1(x))^k, 2 \le k \le n+1;$$

whereas (3.15) imply:

$$\sum_{i=2}^p c_i f_i^{(k)}(x) = \sum_{i=2}^p c_i \frac{f_i^{(k)}(x)}{f_i(x)} \cdot f_i(x) = \sum_{i=2}^p O\left(\left(\frac{f_i'(x)}{f_i(x)}\right)^k \cdot f_i(x)\right)$$
$$= O\left(\left(\frac{f_1'(x)}{f_1(x)}\right)^k\right) \cdot \sum_{i=2}^p |f_i(x)| = o\left(\left(\frac{f_1'(x)}{f_1(x)}\right)^k \cdot f_1(x)\right),$$

whence:

$$\frac{g^{(k)}(x)}{g(x)} = \frac{c_1 f_1^{(k)}(x) + \sum_{i=2}^p c_i f_i^{(k)}(x)}{c_1 f_1(x) + \sum_{i=2}^p c_i f_i(x)}$$
$$= \frac{c_1 f_1^{(k)}(x) + \sum_{i=2}^p c_i f_i^{(k)}(x)}{c_1 f_1(x)}[1 + o(1)]$$
$$= \frac{f_1^{(k)}(x)}{f_1(x)} + o\left(\left(\frac{f_1'(x)}{f_1(x)}\right)^k\right) \sim (f_1'(x)/f_1(x))^k, 2 \le k \le n+1;$$

and (3.16) are proved in both cases. For parts (II) and (III) we put:

$$P(x) := \sum_{i=1}^p c_i f_i(x), \quad Q(x) := \sum_{i=p+1}^q c_i f_i(x). \tag{3.21}$$

To prove part (II) we observe that the results in part (I) and in Proposition 3.2-(II) imply:

$$P(x) \in \{\mathcal{R}_{+\infty}(+\infty) \text{ of order } n\}, \quad Q(x) \in \{\mathcal{SR}_{\alpha_{p+1}}(+\infty) \text{ of order } n+1\}, \tag{3.22}$$

whence, by the known elementary growth-order estimates in [[1]: relations in (2.19) and (2.41)] and by (1.7) we get:

$$\begin{cases} P^{(k)}(x) = +\infty(x^m) \ \forall m \in \mathbb{R}; \\ Q^{(k)}(x) = O(Q(x) x^{-k}) = O(x^m) \ \forall m > \alpha_{p+1} - k; \end{cases} \tag{3.23}$$

whence

$$\frac{h^{(k)}(x)}{h(x)} = \frac{P^{(k)}(x)+Q^{(k)}(x)}{P(x)+Q(x)} \sim \frac{P^{(k)}(x)}{P(x)}$$
$$\sim \left(\frac{P'(x)}{P(x)}\right)^k \overset{\text{by (3.16)}}{\sim} \left(\frac{f_1'(x)}{f_1(x)}\right)^k, \; 2 \le k \le n+1. \tag{3.24}$$

In the situation of part (III) we have:

$$P(x) \in \{\mathcal{SR}_{\alpha_1}(+\infty) \text{ of order } n\}, \quad Q(x) \in \{\mathcal{R}_{-\infty}(+\infty) \text{ of order } n\}, \tag{3.25}$$

whence: □

$$P^{(k)}(x) + Q^{(k)}(x) = \alpha_1^k x^{-k} + o\left(x^{-k}\right) + o\left(x^{-m}\right) \quad \forall\, m \in \mathbb{R}, 1 \le k \le n,$$

which is (3.20).

In [[1]: p. 719] there are some counterexamples showing that the imposed assumptions in the pertinent proposition practically are *the least possible ones* and such counterexamples involve regularly-varying functions. Here is a counterexample with a pair of functions rapidly varying of order 1 and satisfying neither of the two conditions (3.14)-(3.15). This non-trivial counterexample also shows that "*the elementary results in Proposition* 3.3, *concerning positive linear combinations*, *cannot be extended to higher-order rapid variation in our restricted sense*."

Consider the following two functions, both belonging to the class $\{\mathcal{R}_{+\infty}(+\infty) \text{ of order } 1\}$:

$$\begin{cases} f_1(x) := \exp\left[x^\alpha\left(2+\sin(\log x)\right)\right], (\alpha > 1), \\ f_1'(x)/f_1(x) = x^{\alpha-1}\left[2\alpha + \alpha\sin(\log x) + \cos(\log x)\right], \\ f_1''(x)/f_1(x) = x^{2\alpha-2}\left[2\alpha + \alpha\sin(\log x) + \cos(\log x)\right]^2 \\ +(\alpha-1)x^{\alpha-2}\left[2\alpha + \alpha\sin(\log x) + \cos(\log x) + \alpha\cos(\log x) - \sin(\log x)\right] \\ \sim x^{2\alpha-2}\left[2\alpha + \alpha\sin(\log x) + \cos(\log x)\right]^2, \; x \to +\infty; \end{cases} \tag{3.26}$$

$$\begin{cases} f_2(x) := \exp\left[x^\alpha\left(2+\cos(\log x)\right)\right], (\alpha > 1), \\ f_2'(x)/f_2(x) = x^{\alpha-1}\left[2\alpha + \alpha\cos(\log x) - \sin(\log x)\right], \\ f_2''(x)/f_2(x) = x^{2\alpha-2}\left[2\alpha + \alpha\cos(\log x) - \sin(\log x)\right]^2 \\ +(\alpha-1)x^{\alpha-2}\left[2\alpha + \alpha\cos(\log x) - \sin(\log x) - \alpha\sin(\log x) - \cos(\log x)\right] \\ \sim x^{2\alpha-2}\left[2\alpha + \alpha\cos(\log x) - \sin(\log x)\right]^2, \; x \to +\infty; \end{cases} \tag{3.27}$$

noticing that:

$$\left.\begin{aligned} S(x) &:= 2\alpha + \alpha\sin(\log x) + \cos(\log x) \\ C(x) &:= 2\alpha + \alpha\cos(\log x) - \sin(\log x) \end{aligned}\right\} \ge 2\alpha - \alpha - 1 = \alpha - 1 > 0 \quad \forall\, x > 0. \tag{3.28}$$

For the function $g(x) := f_1(x) + f_2(x)$ we have:

$$\begin{cases} \dfrac{g'(x)}{g(x)} = \dfrac{x^{\alpha-1}\{S(x)f_1(x)+C(x)f_2(x)\}}{f_1(x)+f_2(x)}, \\ \dfrac{g''(x)}{g(x)} \sim \dfrac{x^{2\alpha-2}\left\{[S(x)]^2 f_1(x)+[C(x)]^2 f_2(x)\right\}}{f_1(x)+f_2(x)}, \\ G(x) := \dfrac{g''(x)/g(x)}{(g'(x)/g(x))^2} \sim [f_1(x)+f_2(x)]\dfrac{[S(x)]^2 f_1(x)+[C(x)]^2 f_2(x)}{[S(x)f_1(x)+C(x)f_2(x)]^2}, \end{cases} \tag{3.29}$$

and one may check that the asymptotic relation "$G(x)\sim 1$" is not satisfied by taking, for instance, two divergent sequences $\{x_n\}_n, \{y_n\}_n$ such that

$$\sin(\log x_n)=1, \cos(\log x_n)=0; \ \sin(\log y_n)=\cos(\log y_n)=1/\sqrt{2}.$$

As a matter of fact, for any divergent sequence $\{z_n\}_n$ we have:

$$\sin(\log z_n)-\cos(\log z_n)\ge\varepsilon>0\ \forall n \Rightarrow f_1(z_n)\gg f_2(z_n)\Rightarrow \lim_n G(z_n)=1,$$

with the same conclusion if "$\cos(\log z_n)-\sin(\log z_n)\ge\varepsilon>0\ \forall n$". On the contrary:

$$\sin(\log z_n)=\cos(\log z_n)\ \forall n \Rightarrow f_1(z_n)=f_2(z_n)$$

$$\Rightarrow G(z_n)\sim 2\frac{[S(z_n)]^2+[C(z_n)]^2}{[S(z_n)+C(z_n)]^2},$$

and $\lim_n G(z_n)\ne 1$ if, for instance, $S(z_n)\equiv s\ne C(z_n)\equiv c$, as in the case of the above-chosen sequence $\{y_n\}_n$. Hence $g\notin\{\mathcal{R}_{+\infty}(+\infty)\text{ of order }1\}$.

Analogous conclusion with the pair of reciprocals $(f_1(x))^{-1}, (f_2(x))^{-1}\in\{\mathcal{R}_{-\infty}(+\infty)\text{ of order }1\}$.

4. Simplification of Previous Proofs Using Combinatorial Identities

In three different places in [2], we needed to show that certain constants were non-zero and either hinted at or provided indirect proofs. As a matter of fact, the exact values of the constants are special elementary cases of the classical formulas for the higher derivatives of composition and inversion, formulas extensively used in [2] and reported here from [[2]: §6] together with some special cases. The proofs are so brief and elementary that we report them in full.

- **Faà Di Bruno's formula for higher derivatives of a composition:**

$$\begin{aligned}(f(g(x)))^{(k)} &\equiv \frac{\mathrm{d}^k}{\mathrm{d}x^k}(f(g(x))) \\ &= \sum_{\substack{i_1+2i_2+\cdots+ki_k=k \\ 0\le i_j\le k}} \frac{k!}{i_1!\cdots i_k!(1!)^{i_1}(2!)^{i_2}\cdots(k!)^{i_k}} \\ &\quad\times f^{(i_1+\cdots+i_k)}(g(x))\cdot(g'(x))^{i_1}\cdot(g''(x))^{i_2}\cdots\left(g^{(k)}(x)\right)^{i_k}, k\ge 1,\end{aligned} \tag{4.1}$$

where the summation is taken over all possible ordered k-tuples of non-negative integers i_j such that

$$i_1 + 2i_2 + \cdots + ki_k = k \ \left(\text{hence } 1 \le i_1 + i_2 + \cdots + i_k \le k\right). \tag{4.2}$$

In the preceding sum there is only one term containing $f^{(k)}$ and only one term containing $g^{(k)}$, both with coefficient 1, namely:

$$\begin{cases} f^{(k)}\left(g(x)\right)\cdot\left(g'(x)\right)^k & \text{corresponding to } \left(i_1, i_2, \cdots, i_k\right) = \left(k, 0, \cdots, 0\right); \\ f'\left(g(x)\right)\cdot g^{(k)}(x) & \text{corresponding to } \left(i_1, i_2, \cdots, i_k\right) = \left(0, \cdots, 0, 1\right). \end{cases} \tag{4.3}$$

For convenience the coefficients into the sum in (4.1) will be denoted in the sequel by the symbol $a_{i_1, i_2, \cdots, i_k}$.

Lemma 4.1. (Special cases of Di Bruno's formula).

(I) Choosing "$f(y) := y^\beta$, $g(x) := x^\alpha$" yields the identity:

$$\begin{aligned} &\sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot \alpha^{\underline{1}} \cdot \alpha^{\underline{2}} \cdots \alpha^{\underline{k}} \cdot \beta^{\underline{i_1 + \cdots + i_k}} \\ &= (\alpha\beta)^{\underline{k}} \quad \forall\, \alpha, \beta \ne 0,\, k \in \mathbb{N}, \end{aligned} \tag{4.4}$$

which follows from:

$$\begin{aligned} D^k x^{\alpha\beta} &= (\alpha\beta)^{\underline{k}} \cdot x^{\alpha\beta - k} = D^k\left(\left(x^\alpha\right)^\beta\right) \\ &= \sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot \beta^{\underline{i_1 + \cdots + i_k}} \cdot \left(x^\alpha\right)^{\beta - (i_1 + \cdots + i_k)} \\ &\quad \times \alpha^{\underline{1}} \cdot \alpha^{\underline{2}} \cdots \alpha^{\underline{k}} \cdot x^{(\alpha-1)i_1 + (\alpha-2)i_2 + \cdots + (\alpha-k)i_k} \\ &= x^{\alpha\beta - k} \cdot \sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot \alpha^{\underline{1}} \cdot \alpha^{\underline{2}} \cdots \alpha^{\underline{k}} \cdot \beta^{\underline{i_1 + \cdots + i_k}}. \end{aligned} \tag{4.5}$$

(II) Choosing "$f(y) := y^\alpha$, $g(x) := \mathrm{e}^x$" yields the identity:

$$\begin{aligned} &\sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot \alpha^{\underline{i_1 + \cdots + i_k}} \\ &= \alpha^k \quad \forall\, \alpha \ne 0,\, k \in \mathbb{N}, \end{aligned} \tag{4.6}$$

which follows from:

$$\begin{aligned} D^k \mathrm{e}^{\alpha x} &= \alpha^k \cdot \mathrm{e}^{\alpha x} = D^k\left(\mathrm{e}^x\right)^\alpha \\ &= \sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot \alpha^{\underline{i_1 + \cdots + i_k}} \cdot \left(\mathrm{e}^x\right)^{\alpha - (i_1 + \cdots + i_k)} \cdot \mathrm{e}^{(i_1 + \cdots + i_k)x} \\ &= \mathrm{e}^{\alpha x} \cdot \sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot \alpha^{\underline{i_1 + \cdots + i_k}}. \end{aligned} \tag{4.7}$$

(III) Choosing "$f(y) := \mathrm{e}^y$, $g(x) := \log x$" yields the identity:

$$\begin{cases} \displaystyle\sum_{0 \le i_j \le k}^{i_1 + 2i_2 + \cdots + ki_k = k} a_{i_1, i_2, \cdots, i_k} \cdot (-1)^{k - i_1 - \cdots - i_k} \\ \times (0!)^{i_1} (1!)^{i_2} (2!)^{i_3} \cdots \left((k-1)!\right)^{i_k} = 0 \ \forall\, k \in \mathbb{N},\, k \ge 2, \end{cases} \tag{4.8}$$

which follows from:

$$D^k \exp(\log x) \overset{\text{for } k\ge 2}{\equiv} 0$$
$$= \sum_{\substack{i_1+2i_2+\cdots+ki_k=k \\ 0\le i_j\le k}} a_{i_1,i_2,\cdots,i_k} \cdot x^{1-i_1-2i_2-\cdots-ki_k}$$
$$\times(-1)^{(2-1)i_2+(3-1)i_3+\cdots+(k-1)i_k} \cdot +(0!)^{i_1}(1!)^{i_2}(2!)^{i_3}\cdots((k-1)!)^{i_k} \tag{4.9}$$
$$= x^{1-k} \cdot \sum_{\substack{i_1+2i_2+\cdots+ki_k=k \\ 0\le i_j\le k}} a_{i_1,i_2,\cdots,i_k} \cdot (-1)^{k-i_1-\cdots-i_k}$$
$$\times(0!)^{i_1}(1!)^{i_2}(2!)^{i_3}\cdots((k-1)!)^{i_k}.$$

(IV) And choosing " $f(y) := \log y$, $g(x) := \exp x$ " yields the identity:

$$\sum_{\substack{i_1+2i_2+\cdots+ki_k=k \\ 0\le i_j\le k}} a_{i_1,i_2,\cdots,i_k} \cdot (-1)^{i_1+\cdots+i_k-1} \cdot (i_1+\cdots+i_k-1)! = 0 \quad \forall\, k\in\mathbb{N},\, k\ge 2. \tag{4.10}$$

- **Ostrowski's formula for higher derivatives of an inverse function.**

For the inverse function of a *k*-time differentiable $f(x)$ with $f'(x)\ne 0$, the formula holds true:

$$\frac{\mathrm{d}^k}{\mathrm{d}y^k}\left(f^{-1}(y)\right)$$
$$=\left[f'\left(f^{-1}(y)\right)\right]^{1-2k} \cdot \sum_{0\le i_j\le k-1}^{\cdots\cdots} \frac{(-1)^{k-i_1-1}(2k-i_1-2)!}{i_2!\,i_3!\cdots i_k!\,(2!)^{i_2}(3!)^{i_3}\cdots(k!)^{i_k}} \tag{4.11}$$
$$\times\left[f'\left(f^{-1}(y)\right)\right]^{i_1} \cdot \left[f''\left(f^{-1}(y)\right)\right]^{i_2}\cdots\left[f^{(k)}\left(f^{-1}(y)\right)\right]^{i_k},\ k\ge 1,$$

where the summation is taken over all ordered *k*-tuples of non-negative integers i_j such that

$$i_1+\cdots+i_k = k-1; \quad i_1+2i_2+\cdots+ki_k = 2k-2. \tag{4.12}$$

For convenience the coefficients into the sum in (4.11) will be denoted in the sequel by the symbol $c_{i_1,i_2,\cdots,i_k}$.

Lemma 4.2. (Special cases of Ostrowski's formula). The non-negative indexes i_j appearing in the various sums below are subject to the restrictions in (4.12).

(I) For " $f(x) := \mathrm{e}^x$ " we trivially get the identity:

$$\sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k} = (-1)^{k-1}(k-1)! \quad \forall\, k\in\mathbb{N}. \tag{4.13}$$

(II) For " $f(x) := \log x$ " we have:

$$\begin{cases} f^{(j)}(x) = (-1)^{j-1}(j-1)!\,x^{-j};\ f^{-1}(y)\equiv \mathrm{e}^y; \\ f^{(j)}\left(f^{-1}(y)\right) = (-1)^{j-1}(j-1)!\,\mathrm{e}^{-jy},\ j\ge 1; \end{cases} \tag{4.14}$$

and:

$$D^k\mathrm{e}^y \equiv \mathrm{e}^y = \mathrm{e}^{(2k-1)y} \cdot \sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k} \cdot (-1)^{(2-1)i_2+\cdots+(k-1)i_k}$$
$$\times((2-1)!)^{i_2}((3-1)!)^{i_3}\cdots((k-1)!)^{i_k}\,\mathrm{e}^{(-i_1-2i_2-\cdots-ki_k)y} \tag{4.15}$$
$$= \mathrm{e}^y \cdot \sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k} \cdot (-1)^{k-1}(1!)^{i_2}\cdots((k-1)!)^{i_k};$$

so that we get the identity:

$$\sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k}\cdot(-1)^{k-1}(1!)^{i_2}(2!)^{i_3}\cdots((k-1)!)^{i_k}=1\quad\forall\, k\in\mathbb{N}. \tag{4.16}$$

(III) For “ $f(x):=x^{\alpha}$, $\alpha\ne 0$ ” we get the identity:

$$\sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k}\cdot\left(\alpha^{\underline{1}}\right)^{i_1}\cdot\left(\alpha^{\underline{2}}\right)^{i_2}\cdots\left(\alpha^{\underline{k}}\right)^{i_k}=\alpha^{2k-1}\cdot(1/\alpha)^{\underline{k}}\quad\forall\, k\in\mathbb{N}, \tag{4.17}$$

which follows from:

$$\begin{aligned} D^k y^{1/\alpha} &= (1/\alpha)^{\underline{k}}\cdot y^{(1/\alpha)-k}=\left(\alpha y^{1-(1/\alpha)}\right)^{1-2k}\cdot\sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k} \\ &\quad\times\left(\alpha^{\underline{1}}\right)^{i_1}\cdot\left(\alpha^{\underline{2}}\right)^{i_2}\cdots\left(\alpha^{\underline{k}}\right)^{i_k}\cdot y^{(1-(1/\alpha))i_1+(1-(2/\alpha))i_2+\cdots+(1-(k/\alpha))i_k} \\ &= \alpha^{1-2k}y^{(1/\alpha)-k}\cdot\sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k}\cdot\left(\alpha^{\underline{1}}\right)^{i_1}\cdot\left(\alpha^{\underline{2}}\right)^{i_2}\cdots\left(\alpha^{\underline{k}}\right)^{i_k}. \end{aligned} \tag{4.18}$$

We shall now use some of the foregoing identities to shorten or clarify some proofs in [2].

Proofs of Proposition 7.6-(II) in [[2]: pp. 827-829] *and of Proposition* (9.4)-(I) in [[2]: p. 850]. Relation (7.65) in [[2]: p. 829] must read:

$$h^{(k)}(x)=\frac{\left(h'(x)\right)^k}{\left(h(x)\right)^{k-1}}\cdot\left[c_{k,\alpha}+o(1)\right], \tag{4.19}$$

with suitable constants $c_{k,\alpha}$ different from the A_k ’s erroneously written therein, and provided that the $c_{k,\alpha}$ ’s are non-zero. Looking at the preceding formula (7.64) in [2] we see that

$$c_{k,\alpha}=\alpha^{-k}\cdot\sum_{0\le i_j\le k}^{i_1+2i_2+\cdots+ki_k=k} a_{i_1,i_2,\cdots,i_k}\cdot\alpha^{\underline{i_1+\cdots+i_k}}\overset{\text{by (4.6)}}{=}1, \tag{4.20}$$

and the proof of Proposition 7.6-(II) is over. In [[2]: p. 829] we hinted at an indirect proof of “ $c_{k,\alpha}=1$ ” based on the last remark in [[1]: pp. 810-811] but, admittedly, the pertinent argument is not so immediate as the reader might be led to think by the mere mention in [1]. Out of fairness, we give here the whole reasoning. The full statement of the remark in question is:

$$\left\{\begin{array}{l} f\in AC^n[T,+\infty);\ \ f^{(k)}(x)\ne 0\ \ \forall\, x \text{ large enough and } 0\le k\le n; \\ f^{(j)}\in\mathcal{R}_{\pm\infty}(+\infty)\ \text{ for some } j,\, 0\le j\le k; \\ f^{(k)}(x)\cdot f^{(k+2)}(x)\cdot\left(f^{(k+1)}(x)\right)^{-2}=c_k+o(1),\to+\infty, \\ \text{for some } k\in\{0,1,\cdots,n-1\}; \end{array}\right\} \tag{4.21}$$

$$\Rightarrow c_k=1.$$

Now, in the context of Proposition 7.6-(II) we may argue that if the constants $c_{k,\alpha}$ are known to be non-zero for all values of k and a fixed value of α, then relation (7.65) in [2] implies (with the pertinent notations):

$$h^{(k)}(x)\cdot h^{(k+2)}(x)\cdot\left(h^{(k+1)}(x)\right)^{-2}$$
$$=\left[c_{k,\alpha}+o(1)\right]\left[c_{k+2,\alpha}+o(1)\right]\left[c_{k+1,\alpha}+o(1)\right]^{-2}\equiv d_{k,\alpha}+o(1), \tag{4.22}$$

with a suitable constant $d_{k,\alpha}\neq 0$. Already knowing that "$h\in\mathcal{R}_{\pm\infty}(+\infty)$", we get $d_{k,\alpha}=1$ from (4.21). After the easy direct checking that "$c_{k,\alpha}=1$ for $k=2,3,4$" we infer that:

$$1=d_{3,\alpha}=c_{3,\alpha}\cdot c_{5,\alpha}\cdot\left(c_{4,\alpha}\right)^{-2}\ \text{and}\ c_{5,\alpha}=1;$$

and so on we get $c_{k,\alpha}=1$ for all values of *k*. Now we are in a position to prove that no $c_{k,\alpha}$ is zero assuming, if possible, that:

$$c_{i,\alpha}=1\ \text{for}\ i=2,\cdots,k+1\ \text{and}\ c_{k+2,\alpha}=0.$$

Then (4.22) would imply

$$h^{(k)}(x)\cdot h^{(k+2)}(x)\cdot\left(h^{(k+1)}(x)\right)^{-2}=o(1),$$

which is equivalent to

$$\left(h^{(k)}(x)/h^{(k+1)}(x)\right)'=1+o(1),$$

and which, in turn, implies

$$h^{(k)}(x)/h^{(k+1)}(x)\sim x,\ x\to+\infty,\ \text{i.e.}\ h^{(k)}\in\mathcal{R}_1(+\infty),$$

inconsistently with the property "$h\in\mathcal{R}_{\pm\infty}(+\infty)$".

As concerns the proof of Proposition (9.4)-(I) in [[2]: p. 850], in the final part we came across the sum now reported on the right-hand side in (4.20) and we gave another indirect proof of the identity in (4.6) intermixing algebraic arguments and facts from the theory of higher-order exponential variation.

□

Proofs of Propositions 7.7-(II) and 7.7-(III) in [[2]: pp. 830-831]. In the concluding part of Proposition 7.7-(II) in [[2]: p. 831] relation (7.79) contains a constant C_k which is defined by

$$C_k:=\sum_{0\le i_j\le k-1}^{\cdots\cdots} c_{i_1,\cdots,i_k} \tag{4.23}$$

which is non-zero by (4.13), and this fact concludes the proof.

Similarly, in the proof of Proposition 7.7-(III) in [[2]: p. 831], relation (7.82) contains a constant again denoted by C_k. This constant is obtained by replacing relations (7.80) in [2], for the derivatives $f^{(i)}$, into (7.76) in [2] which is exactly the operation performed in the claims in Lemma 4.2. This new constant is the sum appearing in our relation (4.16) which equals 1, and this concludes the proof.

□

We present a last instance wherein a combinatorial identity plays a role, namely a direct proof of a part of Proposition 7.7 in [2].

Proposition 4.3. (Former Proposition 7.7-(I) in [2]). If

$$f\in\left\{\mathcal{R}_\alpha(+\infty)\ \text{of order}\ n\right\},\ \alpha>0,\ (1/\alpha)\neq 1,2,\cdots,n-2, \tag{4.24}$$

then the inverse function

$$f^{-1} \in \left\{\mathcal{R}_{1/\alpha}(+\infty) \text{ of order } n\right\}. \tag{4.25}$$

Proof. To avoid a mix-up over the exponents we put $\tilde{f} := f^{-1}$ and $\tilde{f}^{(k)} := D^k \tilde{f}$. Relations in (1.7) imply:

$$f^{(i)}\left(\tilde{f}(y)\right) = \left(\tilde{f}(y)\right)^{-i} \cdot y \cdot \left[\alpha^{\underline{i}} + o(1)\right], 1 \le i \le n, \tag{4.26}$$

which, when replaced into (4.11), yield:

$$\begin{aligned} \tilde{f}^{(k)}(y) &= \left[f'\left(\tilde{f}(y)\right)\right]^{1-2k} \\ &\times \sum_{0 \le i_j \le k-1}^{\cdots} c_{i_1,\cdots,i_k} \left[f'\left(\tilde{f}(y)\right)\right]^{i_1} \cdot \left[f''\left(\tilde{f}(y)\right)\right]^{i_2} \cdots \left[f^{(k)}\left(\tilde{f}(y)\right)\right]^{i_k} \\ &= \left(\tilde{f}(y)\right)^{2k-1} \cdot y^{1-2k} \cdot \left[\alpha + o(1)\right] \\ &\times \sum_{0 \le i_j \le k-1}^{\cdots} \left\{c_{i_1,\cdots,i_k} \left(\tilde{f}(y)\right)^{-i_1-2i_2-\cdots-ki_k} \cdot y^{i_1+\cdots+i_k} \cdot \prod_{j=1}^{k} \left[\alpha^{\underline{i}} + o(1)\right]^{i_j}\right\} \\ &= \left(\tilde{f}(y)\right)^{2k-1+2-2k} \cdot y^{1-2k+k-1} \cdot \left[\alpha + o(1)\right] \cdot \left[d_{k,\alpha} + o(1)\right] \\ &= \tilde{f}(y) \cdot y^{-k} \cdot \left[\alpha \cdot d_{k,\alpha} + o(1)\right], \end{aligned} \tag{4.27}$$

provided that the constant

$$\alpha \cdot d_{k,\alpha} \equiv \alpha \cdot \sum_{0 \le i_j \le k-1}^{\cdots} \left\{c_{i_1,\cdots,i_k} \cdot \prod_{j=1}^{k} \left(\alpha^{\underline{i}}\right)^{i_j}\right\}$$

is non-zero as granted by (4.17) and the restrictions on α. □

5. On the Inverse of a Higher-Order Regularly-Varying Function

Referring to Proposition 4.3 the trivial counterexample of $f(x) := x^{1/p}$, $p \in \mathbb{N}$ shows the necessity of the restrictions on α. Condition "$\alpha > 0$" only grants that the inverse function is defined on some neighborhood of $+\infty$. For order $n = 1$ any α works well. Spurred on by the results in Proposition 2.4 we tried to suppress the above restrictions on α for the subclass of regularly-varying functions involved in Propositon 2.2 but the situation presents inherent difficulties and only a complete result for $\alpha = 1$ is given here together with a partial result for the remaining exceptional values of α. The difficulties for a general result are outlined in §6.

Theorem 5.1. (I) A special case of Proposition 2.2 states that:

$$\begin{aligned} &\left\{\begin{array}{l} \left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty), 0 \le k \le n-1, n \ge 2, \\ f(x) := x\mathcal{L}(x), \end{array}\right\} \\ &\Rightarrow \left\{\begin{array}{l} f \in \left\{\mathcal{R}_1(+\infty) \text{ of order } n\right\}, \\ \left|f^{(k)}\right| \in \mathcal{R}_{1-k}(+\infty), 0 \le k \le n-1. \end{array}\right. \end{aligned} \tag{5.1}$$

Here we are obviously assuming $n \ge 2$ and the assumptions in (5.1) imply that the derivatives $\mathcal{L}^{(k)}, k \le n-1$, never vanish on a neighborhood of $+\infty$ hence $\mathcal{L}$ is ultimately strictly monotonic. Analogously, the conclusions in (5.1) imply that f is ultimately strictly monotonic. If $\mathcal{L}$ is ultimately positive then the inverse function $\tilde{f}$ is defined on a neighborhood of $+\infty$. It will be presently shown that $\tilde{f}$ belongs to the same class of f and its derivatives satisfy the same relations as those of $\left|f^{(k)}\right|$ in (5.1).

(II) The following is a partial result for the other exceptional values of α:

$$\begin{cases} \left|\mathcal{L}^{(k)}\right| \in \mathcal{R}_{-k}(+\infty),\ 0 \le k \le n-1,\ n \ge 2, \\ f(x) := x^{1/p}\mathcal{L}(x),\ p \in \mathbb{N},\ p \ge 2, \end{cases} \Rightarrow \left|\tilde{f}^{(k)}\right| \in \mathcal{R}_{p-k}(+\infty),\ 0 \le k \le \min\{n-1, p\}. \tag{5.2}$$

Proof. We are using the notation $\tilde{f} := f^{-1}$ as in the proof of Proposition 4.3. (I) The function $\mathcal{L}$ satisfies relations in (2.12) whereas relations in (2.7)-(2.9) take the form:

$$\begin{cases} f'(x) = \mathcal{L}(x) + x\mathcal{L}'(x) \sim \mathcal{L}(x) \equiv x^{-1}f(x); \\ f''(x) \sim \mathcal{L}'(x) = o\left(x^{-1}\mathcal{L}(x)\right) = o\left(x^{-1}f'(x)\right); \\ f^{(k)}(x) \sim (-1)^k (k-2)!\, x^{2-k}\mathcal{L}'(x),\ k \ge 2; \end{cases} \tag{5.3}$$

whence, with the natural substitution $x = \tilde{f}(y)$:

$$\tilde{f}'(y) \equiv \left[f'\left(\tilde{f}(y)\right)\right]^{-1} \sim \tilde{f}(y)\cdot\left[f\left(\tilde{f}(y)\right)\right]^{-1} \equiv y^{-1}\tilde{f}(y) \quad \text{i.e. } \tilde{f} \in \mathcal{R}_1(+\infty); \tag{5.4}$$

$$\begin{cases} \tilde{f}''(y) \equiv -\left(\tilde{f}'(y)\right)^3 \cdot f''\left(\tilde{f}(y)\right); \\ f''\left(\tilde{f}(y)\right) \overset{(5.3)}{=} o\left(\left(\tilde{f}(y)\right)^{-1}\cdot f'\left(\tilde{f}(y)\right)\right) \overset{(5.4)}{\equiv} o\left(\left[\tilde{f}(y)\cdot\tilde{f}'(y)\right]^{-1}\right); \end{cases} \tag{5.5}$$

$$\begin{aligned} \tilde{f}''(y)/\tilde{f}'(y) &\equiv -\left(\tilde{f}'(y)\right)^2 \cdot f''\left(\tilde{f}(y)\right) \overset{(5.5)}{=} o\left(\tilde{f}'(y)/\tilde{f}(y)\right) \overset{(5.4)}{=} \\ &= o\left(y^{-1}\right),\ y \to +\infty, \quad \text{i.e. } \tilde{f}' \in \mathcal{R}_0(+\infty). \end{aligned} \tag{5.6}$$

For $k \ge 3$ we use Ostrowski's formula (4.11) highlighting the signs of the coefficients and we get as $y \to +\infty$:

$$\begin{aligned} \tilde{f}^{(k)}(y) &= \left(\tilde{f}'(y)\right)^{2k-1} \cdot \sum_{0 \le i_j \le k-1}^{\cdots} \Big\{\left|c_{i_1,\cdots,i_k}\right| (-1)^{k-i_1-1}(-1)^{2i_2+\cdots+ki_k}(0!)^{i_2}(1!)^{i_3}\cdots\left((k-2)!\right)^{i_k} \\ &\quad \times \left(\tilde{f}(y)\right)^{(2-2)i_2+(2-3)i_3+\cdots+(2-k)i_k} \cdot \left(\mathcal{L}'\left(\tilde{f}(y)\right)\right)^{i_1} \cdot \left(\mathcal{L}'\left(\tilde{f}(y)\right)\right)^{i_2} \cdots \left(\mathcal{L}'\left(\tilde{f}(y)\right)\right)^{i_k} \cdot [1+o(1)]\Big\} \\ &= \left(\tilde{f}'(y)\right)^{2k-1} \cdot \sum_{0 \le i_j \le k-1}^{\cdots} \Big\{\left|c_{i_1,\cdots,i_k}\right| (-1)^{3(k-1)} \left(\prod_{j=0}^{k-2}(j!)^{i_{j+2}}\right)\left(\tilde{f}(y)\right)^{-i_1} \\ &\quad \times \left(\mathcal{L}\left(\tilde{f}(y)\right)\right)^{i_1} \cdot \left(\mathcal{L}'\left(\tilde{f}(y)\right)\right)^{k-1-i_1} \cdot [1+o(1)]\Big\} \\ &= (-1)^{3(k-1)} \cdot \left(\tilde{f}'(y)\right)^{2k-1} \cdot \left(\mathcal{L}'\left(\tilde{f}(y)\right)\right)^{k-1} \\ &\quad \times \sum_{0 \le i_j \le k-1}^{\cdots} \Big\{\overline{c}_{i_1,\cdots,i_k} \left(\mathcal{L}\left(\tilde{f}(y)\right)\right)^{i_1} \cdot \left(\mathcal{L}'\left(\tilde{f}(y)\right)\cdot\tilde{f}(y)\right)^{-i_1} \cdot [1+o(1)]\Big\}, \end{aligned} \tag{5.7}$$

with suitable positive coefficients $\overline{c}_{i_1,\cdots,i_k}$. We now put

$$F(y) := \mathcal{L}\left(\tilde{f}(y)\right) / \left(\mathcal{L}'\left(\tilde{f}(y)\right) \cdot \tilde{f}(y)\right), \tag{5.8}$$

and notice that the assumptions in (5.1) imply

$$x\mathcal{L}'(x)/\mathcal{L}(x) = o(1),\ x \to +\infty; \quad \mathcal{L}'(x) \neq 0 \text{ ultimately}; \tag{5.9}$$

whence $F(y) \to \pm\infty,\ y \to +\infty$, and:

$$\sum_{0 \le i_j \le k-1}^{\cdots} \left\{\overline{c}_{i_1,\cdots,i_k} \left(F(y)\right)^{i_1} \cdot \left[1 + o(1)\right]\right\} \sim c_k \left(F(y)\right)^{\max i_1},\ y \to +\infty, \tag{5.10}$$

where $c_k > 0$ and the number "$\max i_1$" a priori depends only on *k*. Now, the exponents i_j satisfy the two equations in (4.12) whence $i_1 < k-1$; and for the choice $i_1 = k-2$ the system in (4.12) reduces to

$$i_2 + \cdots + i_k = 1,\ 2i_2 + \cdots + ki_k = k, \tag{5.11}$$

which has the solution "$i_2 = \cdots = i_{k-1} = 0,\ i_k = 1$". It follows that "$\max i_1 = k-2$" and:

$$\tilde{f}^{(k)}(y) \sim (-1)^{3(k-1)} c_k \cdot \left(\tilde{f}'(y)\right)^{2k-1} \cdot \left(\mathcal{L}'\left(\tilde{f}(y)\right)\right)^{k-1} \cdot \left(F(y)\right)^{k-2}; \tag{5.12}$$

$$\begin{cases} \tilde{f}^{(k+1)}(y)/\tilde{f}^{(k)}(y) \sim -\dfrac{c_{k+1}}{c_k}\left(\tilde{f}'(y)\right)^2 \cdot \mathcal{L}'\left(\tilde{f}(y)\right) \cdot F(y) \\ \qquad\qquad = -\dfrac{c_{k+1}}{c_k}\left(\tilde{f}'(y)\right)^2 \cdot \left(\mathcal{L}\left(\tilde{f}(y)\right)/\tilde{f}(y)\right). \end{cases} \tag{5.13}$$

The first relation in (5.3) gives $\mathcal{L}\left(\tilde{f}(y)\right) \sim f'\left(\tilde{f}(y)\right) = \left(\tilde{f}'(y)\right)^{-1}$ and we get:

$$\tilde{f}^{(k+1)}(y)/\tilde{f}^{(k)}(y) \sim -\frac{c_{k+1}}{c_k}\tilde{f}'(y)/\tilde{f}(y) \overset{(5.4)}{\sim} -\frac{c_{k+1}}{c_k} y^{-1},\ y \to +\infty, \tag{5.14}$$

which means that "$\tilde{f}^{(k)} \in \mathcal{R}_{\alpha_k}(+\infty)$" for suitable $\alpha_k < 0$ and $2 \le k \le n-1$. By (5.4) and (5.6) "$\tilde{f} \in \{\mathcal{R}_1(+\infty) \text{ of order } n\}$" and $\alpha_1 = 0$, and we shall now repeatedly use the general result about the index of variation of a derivative, [[1]: Prop. 2.6-(I), p. 796], to show that $\alpha_k = 1-k$. For the index of $\tilde{f}''$ we preliminarily remark that

$$\lim_{y\to+\infty} \tilde{f}'(y) \overset{(5.4)}{=} \lim_{y\to+\infty} y^{-1}\tilde{f}(y) = \lim_{x\to+\infty} \left(f(x)\right)^{-1} \cdot x \equiv \lim_{x\to+\infty} \left(\mathcal{L}(x)\right)^{-1}, \tag{5.15}$$

and that "$\lim_{x\to+\infty} \mathcal{L}(x)$" exists in $[0,+\infty]$ by the positivity and monotonicity of $\mathcal{L}(x)$.

First case. If this last limit is either zero or $+\infty$ then the same is true (with inverted values) for the "$\lim_{y\to+\infty} \tilde{f}'(y)$" and, by the mentioned general result, $\tilde{f}'' \in \mathcal{R}_{-1}(+\infty)$. This in turn implies the mentioned values of the indexes for the higher-order derivatives.

Second case. If "$\lim_{x\to+\infty} \mathcal{L}(x) = \ell \in \left]0,+\infty\right)$" we may suppose $\ell = 1$ and represent f in the form "$f(x) := x\left[1 + \overline{\mathcal{L}}(x)\right]$" where the new function $\overline{\mathcal{L}}$ satisfies the same assumptions satisfied by $\mathcal{L}$ in (5.1) and "$\overline{\mathcal{L}}(x) = o(1)$". All the previous calculations from (5.3) to (5.14) remain valid because no use was made of the value $\mathcal{L}(+\infty)$, hence "$\tilde{f}^{(k)} \in \mathcal{R}_{\alpha_k}(+\infty)$" for suitable $\alpha_k < 0$ and $2 \le k \le n-1$. In the present case we need to make explicit the asymptotic behavior of $\tilde{f}''$ and we have:

$$\begin{cases} f(x) = x\left[1+\bar{\mathcal{L}}(x)\right]; \\ f'(x) = 1+\bar{\mathcal{L}}(x)+x\bar{\mathcal{L}}'(x) = 1+\bar{\mathcal{L}}(x)\left[1+o(1)\right] = 1+o(1); \\ f''(x) = 2\bar{\mathcal{L}}'(x)+x\bar{\mathcal{L}}''(x) = 2\bar{\mathcal{L}}'(x)-\bar{\mathcal{L}}'(x)\left[1+o(1)\right] = \bar{\mathcal{L}}'(x)\left[1+o(1)\right]; \end{cases} \tag{5.16}$$

$$\begin{cases} \log y = \log x + \log\left[1+\bar{\mathcal{L}}(x)\right] = \log x + o(1),\ x \to +\infty,\ \text{where } y \equiv f(x); \\ \log x = \log y + o(1),\ y \to +\infty,\ \text{and } x \equiv \tilde{f}(y) = y\left[1+o(1)\right],\ y \to +\infty; \end{cases} \tag{5.17}$$

whence:

$$\begin{cases} \tilde{f}'(y) \equiv \left[f'\left(\tilde{f}(y)\right)\right]^{-1} = \left[1+o(1)\right]; \\ \tilde{f}''(y) \equiv -\left(\tilde{f}'(y)\right)^3 \cdot f''\left(\tilde{f}(y)\right) \sim -f''\left(\tilde{f}(y)\right) \sim -\bar{\mathcal{L}}'\left(\tilde{f}(y)\right) \\ \qquad = -\bar{\mathcal{L}}'\left(y\left[1+o(1)\right]\right) \sim -\bar{\mathcal{L}}'(y),\ y \to +\infty; \end{cases} \tag{5.18}$$

where the last expression follows from a result about asymptotic functional relations: [[1]: Prop. 5.1-(I), p. 811]. As "$\bar{\mathcal{L}}' \in \mathcal{R}_{-1}(+\infty)$" by hypothesis, the relation "$\tilde{f}''(y) \sim -\bar{\mathcal{L}}'(y)$" implies $\alpha_2 = -1$ by [[1]: first inference in (2.31), p. 785]. The values of the other α_k's follow automatically as pointed out in the first case and the proof of part (I) is over.

(II) Putting $\alpha := 1/p$, relations in (2.7) state that:

$$f^{(k)}(x) \sim \alpha^{\underline{k}} x^{\alpha-k} \mathcal{L}(x),\ x \to +\infty, 1 \le k \le n-1, \tag{5.19}$$

so that:

$$\tilde{f}'(y) \equiv \left[f'\left(\tilde{f}(y)\right)\right]^{-1} \sim p \cdot \left(\tilde{f}(y)\right)^{1-\alpha} \cdot \left(\mathcal{L}\left(\tilde{f}(y)\right)\right)^{-1} \equiv p \cdot \tilde{f}(y) \cdot y^{-1}; \tag{5.20}$$

$$\begin{aligned} \tilde{f}^{(k)}(y) &= \left(\tilde{f}'(y)\right)^{2k-1} \cdot \sum_{0 \le i_j \le k-1}^{\cdots} \Big\{ \left|c_{i_1,\cdots,i_k}\right| (-1)^{k-1-i_1} \left(\alpha^{\underline{1}}\right)^{i_1} \left(\alpha^{\underline{2}}\right)^{i_2} \cdots \left(\alpha^{\underline{k}}\right)^{i_k} \\ &\quad \times \left(\tilde{f}(y)\right)^{(\alpha-1)i_1+(\alpha-2)i_2+\cdots+(\alpha-k)i_k} \cdot \left(\mathcal{L}\left(\tilde{f}(y)\right)\right)^{i_1+i_2+\cdots+i_k} \cdot \left[1+o(1)\right] \Big\} \\ &= \left(\tilde{f}'(y)\right)^{2k-1} \cdot \left(\tilde{f}(y)\right)^{(\alpha-2)(k-1)} \left(\mathcal{L}\left(\tilde{f}(y)\right)\right)^{k-1} \\ &\quad \times (-1)^{k-1} \cdot \sum_{0 \le i_j \le k-1}^{\cdots} \Big\{ \left|c_{i_1,\cdots,i_k}\right| (-1)^{i_1} \left(\alpha^{\underline{1}}\right)^{i_1} \left(\alpha^{\underline{2}}\right)^{i_2} \cdots \left(\alpha^{\underline{k}}\right)^{i_k} \cdot \left[1+o(1)\right] \Big\}. \end{aligned} \tag{5.21}$$

At this point the proof could be completed if we only knew that the sum

$$\sum_{0 \le i_j \le k-1}^{\cdots} \left|c_{i_1,\cdots,i_k}\right| (-1)^{i_1} \left(\alpha^{\underline{1}}\right)^{i_1} \left(\alpha^{\underline{2}}\right)^{i_2} \cdots \left(\alpha^{\underline{k}}\right)^{i_k} \tag{5.22}$$

is non-zero which may not be the case. As a matter of fact, for $\mathcal{L}(x) \equiv 1$ we have $\tilde{f}(y) \equiv y^p$ and the above formula gives the expression of $\left(\mathrm{d}^k/\mathrm{d}y^k\right) y^p \equiv p^{\underline{k}} y^{p-1}$; hence the sum in (5.22) equals $(-1)^{k-1} p^{\underline{k}}$, being non-zero for $k \le p$ and zero for $k \ge p+1$. Hence the following relations hold true for the admissible values of k:

$$\tilde{f}^{(k)}(y) \sim c_k y^{1-2k}\left(\tilde{f}(y)\right)^{2k-1+(\alpha-2)(k-1)}\left(\mathcal{L}\left(\tilde{f}(y)\right)\right)^{k-1}, 1 \le k \le p; c_k \ne 0; \quad (5.23)$$

$$\tilde{f}^{(k+1)}(y)/\tilde{f}^{(k)}(y) \sim \frac{c_{k+1}}{c_k} y^{-2}\left(\tilde{f}(y)\right)^{\alpha} \cdot \mathcal{L}\left(\tilde{f}(y)\right) \equiv \frac{c_{k+1}}{c_k} y^{-1}, 1 \le k \le p-1; \quad (5.24)$$

where the last equality follows from the definition of $f:\left(\tilde{f}(y)\right)^{1/p} \cdot \mathcal{L}\left(\tilde{f}(y)\right) \equiv f\left(\tilde{f}(y)\right)$. Hence $\tilde{f} \in \mathcal{R}_{\alpha_k}(+\infty)$ for certain indexes α_k and, by [[1]: Prop. 2.6-(I), p. 796], $\alpha_k = p-k$ for $k \le p-1$. Moreover:

$$\tilde{f}^{(p+1)}(y) = o\left(y^{-1}\tilde{f}^{(p)}(y)\right) \text{ hence } \tilde{f}^{(p)} \in \mathcal{R}_0(+\infty). \quad (5.25)$$

□

6. Conclusions

I) The main facts presented in this paper are:

1) The elementary factorization of regularly-varying functions:

$$f \in \mathcal{R}_{\alpha}(+\infty), \alpha \ne 0, \quad \Leftrightarrow \quad \left\{f(x) \equiv x^{\alpha}\mathcal{L}(x), \mathcal{L} \in \mathcal{R}_0(+\infty)\right\}, \quad (6.1)$$

which holds true for higher-order smoothly-varying functions as well, cannot be extended to higher-order regularly-varying functions. It is known that the analogue of the sole inference from right to left in (6.1) holds true for higher-order regularly-varying functions under certain restrictions on α and we have shown in §2 that no such restrictions are needed for a useful class of functions including those in (1.22).

2) The foregoing result in turn implies that there is an important class of higher-order regularly-varying functions that require no impractical restrictions on their indexes of variation when performing on them the operation of multiplication, and this is proved in §2 as well.

3) Useful non-obvious results can be obtained on the types of higher-order variation for arbitrary linear combinations of various functions where "arbitrary" means "of any signs", and appropriate counterexamples can be exhibited highlighting once more the special restricted character of our concept of higher-order rapid variation defined by relations in (1.15)-(1-17). The most meaningful results are those in Proposition 3.4 concerning linear combinations of both smoothly- and rapidly-varying functions. §3 ends with a non-trivial counterexample showing that the imposed assumptions are the least possible ones.

II) In §4 there is a list of combinatorial identities, directly inferred from the formulas for higher derivatives of composite or inverse functions, which are used to simplify some proofs of previous results about operations with higher-order types of variation. The reader may notice that the whole matter of operations with higher-order types of asymptotic variations, amply developed in [2], is quite complicated and some sporadic alternative proofs as those in §4, though longer, may be considered to provide support for these so many results.

III) In contrast to the results in §2 and §3 only partial results can be given in studying the type of higher-order variation of the inverse of a regularly-varying function and trying to suppress restrictions on the indexes. But this is due to inherent difficulties. It is the author's firm conviction that the thesis in (5.2) holds true for all the admissible values of $k \ge p+1$ but, for derivatives of order greater than p it is not clear how to proceed by direct calculations in order to highlight the principal part of $\tilde{f}^{(k)}(y)$ and, consequently, of the ratio $\tilde{f}^{(k+1)}(y) / \tilde{f}^{(k)}(y)$. The difficulty is due to the factor $(-1)^{i_1}$ in (5.22) which does not automatically assure that the constant is non-zero because the global sign of the other factors in (5.22) does not depend on the various exponents i_j:

$$\begin{cases} \alpha \equiv 1/p \in]0,1[\Rightarrow \text{sign } \alpha^{\underline{j}} = (-1)^{j-1} \Rightarrow \text{sign } \left(\alpha^{\underline{j}}\right)^{i_j} = (-1)^{(j-1)i_j} \\ \Rightarrow \text{sign } \prod_{j=1}^{k} \left(\alpha^{\underline{j}}\right)^{i_j} = (-1)^{(2-1)i_2+(3-1)i_3+\cdots+(k-1)i_k} = (-1)^{k-1}. \end{cases} \tag{6.2}$$

The factor $(-1)^{i_1}$ will be always present even if a two-term expansion of $f^{(k)}(x)$, inferred from (2.13), is used, namely:

$$f^{(k)}(x) = \alpha^{\underline{k}} x^{\alpha-k} \mathcal{L}(x) + x^{\alpha-k+1} \mathcal{L}'(x) \left[A_k(\alpha) + o(1)\right], \tag{6.3}$$

noticing that even the coefficient $A_k(\alpha)$ may be zero as, e.g., $A_2(1/2) = 0$.

We tried some calculations for a function f of the type appearing in (1.22) with $\alpha > 0$ and not reduced to power. In this special case, each derivative $f^{(k)}$ is a linear combination of functions with the same algebraic structure of f (*i.e.* powers times the sort of slowly-varying functions specified therein) and, as such, it is either $\equiv 0$, a case excluded by the assumption that f is not a power, or has a principal part as $x \to +\infty$ with the same algebraic structure: hence $f^{(k)}(x)$ either converges to zero or diverges to $\pm\infty$ as $x \to +\infty$. This grants that the limit "$\lim_{x\to+\infty} x f^{(k+1)}(x) / f^{(k)}(x)$" exists as an extended real number for each $k \ge 0$ and, by the cited reference [[1]: Prop. 2.6-(I), p. 796], "$\left|f^{(k)}\right| \in \mathcal{R}_{\alpha-k}(+\infty) \ \forall k \in \mathbb{N}$".

Unfortunately, it is not obvious how to proceed to show the analogous property for the inverse function, *i.e.* that the limit "$\lim_{y\to+\infty} y \tilde{f}^{(k+1)}(y) / \tilde{f}^{(k)}(y)$" exists as an extended real number for each $k \in \mathbb{N}$ implying that "$\tilde{f} \in \{\mathcal{R}_p(+\infty) \text{ of any order } n\}$".

This remains an "Open Problem" in addition to those stated in [[2]: p. 866] save for Open Problem 4 completely solved in [3].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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