Scientific Research Publishing

Advances in Pure Mathematics, 2021, 11, 377-385
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

The Homotopical Proof of $\Pi_1(S, x_o)$ as a Fundamental Group in a General Interval

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How to cite this paper: Delali1, Z.D., William, O.-D., Lewis, B. and Kwame, A.R. (2021) The Homotopical Proof of $\Pi_1(S, x_o)$ as a Fundamental Group in a General Interval. *Advances in Pure Mathematics*, **11**, 377-385.
https://doi.org/10.4236/apm.2021.115024

Received: March 18, 2021
Accepted: May 14, 2021
Published: May 17, 2021

Abstract

The aim of this study is to establish that, the equivalent class $\Pi_1(S, x_o)$ which is made up of homotopic loops is a group with respect to "$\circ$" in the general interval $[m, n]$. The study proved from homotopical point of view that $\Pi_1(S, x_o)$ is associative, has an identity and inverse function. The study established with proof that $\Pi_1(S, x_o)$ is a fundamental group in $[m, n]$, $\forall n, m \in Z^+$.

Keywords

Homotopy, Fundamental Group, Homeomorphism, Equivalent Class, Path Concatenation

1. Introduction

The properties of a topological space that were developed so far have been depended on the choice of topology, the collection of open sets. Taking a different tack, we introduce a different structure, algebraic in nature, associated to a space together with a choice of base point (X, x_0). The structure will allow us to bring to bear the power of algebraic arguments [1] [2] [3]. The fundamental group was introduced by Poincairé in his investigations of the actions of a group on a manifold. In a long paper entitled Analysis Situs, Poincare introduced the concept of the fundamental group of a topological space. The research begun with a heuristic introduction, using functions $F_i(i = 1, \cdots, A)$ (not necessarily single valued) on a manifold defined by equations between coordinates $(x, k)(k = 1, \cdots, n)$. It assumed that these functions satisfy certain differential equations, where (x_i, k) are known single-valued differentiable functions of (x, k) and F_i which satisfy certain integrability conditions [4]. Then it considered that the transformations of F_i produces result if one traces their values along a closed loop. Dugundji, put, for the first time, a topology on fundamental groups of certain spaces and deduced a classification theorem for connected covers of a space.

Furthermore, topologists of the early 20th century dreamed of a generalisation to higher dimensions of the non-abelian fundamental group, for applications to problems in geometry and analysis for which group theory had been successful.

For some decades now, the theorem $\Pi_1(S, x_o)$ is a group, established to be a fundamental group with respect to "$\circ$" in the interval [0, 1] [5]. It has been used to prove some mathematical concepts such as connectedness, metric space, isomorphism, Cech homotopy etc. The theorem has led few researchers to work within this interval [0, 1] [6] [7]. Cannon and Conner (2005) used the fact that $\Pi_1(S, x_o)$ is a group to work in one dimensions. Their research examined the fundamental groups of complicated one-dimensional spaces. In attempt to prove some other mathematical concepts, it was established that if X is a space of dimension at most 1, then, the fundamental group is isomorphic to a subgroup of the first Cech homotopy group based on finite open covers. Consequently, for a one-dimensional continuum X, the fundamental group is isomorphic to a subgroup of the first Cech homotopy group [8]. A potentially new approach to homotopy theory derived from the expositions in Brown's (1968) and Higgins' (1971), which in effect suggested that most of 1-dimensional homotopy theory can be better expressed in terms of groupoids rather than groups [9].

This led to a search for the uses of groupoids in higher homotopy theory, and in particular for higher homotopy groupoids. The basic intuitive concept was generalising from the usual partial compositions of homotopy classes of paths to partial compositions of homotopy classes (of some form) of complexes. But a search for such constructs proved abortive for some years from 1966 [10].

Recently, the concept of fundamental group was used on the ageing process of human. The homotopy relates the topological shape of the infant to the topological shape of the adult. The compact connected human body with boundary is assumed to be topologically equivalent to a cylinder. This complex connected cylindrical shape of the body $x = S^1 \times I$, described by the homotopic functions $f, h: x \to x$ provided the ageing process in vertical interval $I = [0, \beta]$. But this work is limited to the interval $[0, \beta]$ [11].

Therefore this research intends to establish the proof that $\Pi_1(S, x_o)$ is a group in other domains other than [0, 1]. In this paper, we defined homotopy and presented some group properties. We then described the fundamental group and its related properties such as group homomorphisms. We also looked at useful definitions and theorems that play an important role in computing fundamental groups, and our main result was actually built upon these theorems. In the end, we established the proof that the equivalent class $\Pi_1(S, x_o)$ is a fundamental group in the interval $[m, n]$ $\forall n, m \in Z^+$.

Throughout this paper we assumed the knowledge of basic algebra and general topology.

2. Preliminaries

2.1. Homotopy

Definition 2.1.1

Let X be a topological space. A path in X from x_0 to x_1 is a continuous map $f: I \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We say that x_0 is the *initial* point and x_1 the final point [12].

2.2. Homotopic Path Concatenation

Definition 2.2.1

Suppose X is a space and $x_0 \in X$ is a choice of base point in X. The space of based loops in X denoted $\Omega(X, x_0)$, is the subspace of $\operatorname{map}([0,1], X)$,

$$\Omega(X, x_0) = \{\lambda \operatorname{map}([0,1], X) \mid \lambda(0) = \lambda(1) = x_0\} \tag{1}$$

Composition of loops determines a binary operation $*: \Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$. We restrict the notion of homotopy when applied to the space of based loops in X in order to stay in that space during the deformation.

Definition 2.2.2

Given two based loops λ and μ, a loop homotopy between them is a homotopy of paths $H: [0,1] \times [0,1] \to X$ with $H(t,0) = \lambda(t)$, $H(t,1) = \lambda(t)$ and $H(0,s) = H(1,s) = x_0$. That is, for each $s \in [0,1]$, the path $t \mapsto H(t,s)$ is a loop at x_0.

Proposition 2.2.3

Continuous mappings $F: W \to X$ and $G: Y \to Z$ induce well-defined functions $F^*: [X,Y] \to [W,Y]$ and $G_*: [X,Y] \to [X,Z]$ by $F^*([h]) = [h \circ F]$ and $G_*([h]) = [G \circ h]$ for $[h] \in [X,Y]$.

Proof. We need to show that if $h \simeq h'$, then $h \circ F \simeq h' \circ F$ and $G \circ h \simeq G \circ h'$. Fixing a homotopy $H: X \times [0,1] \to Y$ with $H(x,0) = h(x)$ and $H(x,1) = h'(x)$, then the desired homotopies are $H_F(w,t) = H(F(w),t)$ and $H_G(x,t) = G(H(x,t))$.

To a space X we associate a space particularly rich in structure, the *mapping space of paths in X*, $\operatorname{map}([0,1], X)$. Recall that $\operatorname{map}([0,1], X)$ is the set of continuous mappings $\operatorname{Hom}([0,1], X)$ with the compact-open topology. The space $\operatorname{map}([0,1], X)$ has the following properties:

1) X embeds into $\operatorname{map}([0,1], X)$ by associating to each point $x \in X$ to the *constant path*, $c_x(t) = x$ for all $t \in [0,1]$.

2) Given a path $\lambda: [0,1] \to X$, we can *reverse* the path by composing with $t \mapsto 1-t$. Let $\lambda^{-1}(t) = \lambda(1-t)$.

3) Given a pair of paths $\lambda, \mu: [0,1] \to X$ for which $\lambda(1) = \mu(0)$, we can *compose* paths by

$$\lambda * \mu(t) = \begin{cases} \lambda(2t), & \text{if } 0 \le t \le 1/2 \\ \mu(2t-1), & \text{if } 1/2 \le t \le 1 \end{cases} \quad (2)$$

Thus, for certain pairs of paths λ and μ, we obtain a new path $\lambda * \mu \in \text{map}([0,1], X)$. Composition of paths is always defined when we restrict to a certain subspace of $\text{map}([0,1], X)$ [13].

Definition 2.2.4

Let *X* be a topological space, and x_0 a point in *X*. The fundamental group of *X* is the set of path homotopy classes $[f]$ of loops $f: I \to X$ based at x_0, together with the operation "$\circ$". We denote it by $\Pi_1(X, x_0)$ [14].

Definition 2.2.5

Given two loop classes $[f]$ and $[g]$ we define:

1) $[f]*[g] = [f * g]$.

2) The inverse of $[f]$ is given by $[f^{-1}]$, that is $[f]^{-1}$, where $f^{-1}(t) = \overline{f}(t) = f(1-t)$.

Theorem 2.2.6

If X is a convex subset of $\Re^n$, *and if* $a, b \in X$, $\alpha_t(s) = (1-t)\alpha_0(s) + t\alpha_1(s)$. *This defines a homotopy between* α_0 *and* α_1.

Theorem 2.2.7

The relation $f \simeq g$ is an equivalence relation on the set, $\text{Hom}(X, Y)$, of continuous mappings from *X* to *Y*.

Proof. Let $f: X \to Y$ be a given mapping. The homotopy $H(x,t) = f(x)$ is a continuous mapping $H: X \times [0,1] \to Y$ and so $f \simeq f$.

If $f_0 \simeq f_1$ and $H: X \times [0,1] \to Y$ is a homotopy between f_0 and f_1, then the mapping $H': X \times [0,1] \to Y$ given by $H'(x,t) = H(x, 1-t)$ is continuous and a homotopy between f_1 and f_0 that is $f_1 \simeq f_0$.

Finally, for $f_0 \simeq f_1$ and $f_1 \simeq f_0$, suppose that $H_1: X \times [0,1] \to Y$ is a homotopy between f_0 and f_1, and $H_2: X \times [0,1] \to Y$ is a homotopy between f_1 and f_2. Define the homotopy $H: X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} H_1(x, 2t), & \text{if } 0 \le t \le 1/2 \\ H_2(x, 2t-1), & \text{if } 1/2 \le t \le 1 \end{cases}$$

Since $H_1(x,1) = f_1(x) = H_2(x,0)$, the piecewise definition of *H* gives a continuous function. By definition, $H(x,0) = f_0(x)$ and $H(x,1) = f_2(x)$ and so $f_0 \simeq f_2$ [15].

Definition 2.2.8

The equivalence classes of maps from *X* to *Y* as in the Theorem 2.2.7 are called the homotopy classes ad denoted by [*f*] as the homotopy class of the map *f*. While we use the notation $\langle \gamma \rangle$ when γ is a loop in *X* based at $x_0 \in X$.

2.3. Homotopy Classes

Definition 2.3.1

The equivalent classes $[f]$ determined by homotopy modulo x_0 on the collection $C(S, x_\circ)$ of all closed paths *f* on *S* based at $x_\circ \in S$ are called homotopy classes of $C(S,x_\circ)$. The collection of these homotopy classes is denoted by $\Pi_1(S,x_\circ)$ [12].

3. Main Results

Path Concatenation in [*m*, *n*]

Definition 3.1.1

1) If $f,g \in \Pi_1(S,x_0)$ we define the juxtaposition $f \circ g$ of *f* and *g* as follows:

$$(f \circ g)(s) = \begin{cases} f(2s) & \text{if } \frac{m}{2} \le s \le \frac{n}{2} \\ g(2s-n) & \text{if } \frac{m+n}{2} \le s \le n \end{cases}, \forall m,n \in Z^+$$

Thus $f \circ g \in \Pi_1(S,x_0)$ and "$\circ$" is a binary operation on $\Pi_1(S,x_0)$.

2) If $[f],[g] \in \Pi_1(S,x_0)$, then let $[f]\circ[g] = [f \circ g]$

Proposition 3.1.2

Let X,Y be topological spaces and let A a subset of *X*. Then $\sim_A$ is ann equivalence relation on the set $\text{Map}_A(X,Y)$ of maps from *X* to *Y* which agree with with a given map on *A*.

Proof. For notational convenience, drop the subscript *A* from the notation.

1) Reflexive property $f \sim f$: Define $H(x,t) = f(x)$. This is the composition of *f* with the projection of $X \times I$ on *X*. Since it is a composition of two continuous maps, it is continuous.

2) Symmetric property $f \sim g \Rightarrow g \sim f$: Suppose $H : X \times I \to Y$ is a homotomy (relative to *A*) of *f* to *g*. Let $H'(x,t) = H(x,1-t)$. The $H'(x,0) = H(x,1) = g(x)$ and similarly for $t = 1$. Also, if $H(a,t) = f(a) = g(a)$ for $a \in A$, the same is true for H'. H' is a composition of two continuous maps. What are they?

3) Transitive property $f \sim g, g \sim h \Rightarrow f \sim h$: This is somewhat harder. Let $H' : X \times I \to Y$ be a homotopy (relative to *A*) from *f* to *g*, and let H'' be such a homotopy of *g* to *h*. Define

$$H(x,t) = \begin{cases} H'(x,2t), & \text{for } 0 \le t \le 1/2 \\ H''(x,2t-1), & \text{for } 1/2 \le t \le 1 \end{cases}$$

Note that the definitions agree for $t = 1/2$. We need to show *H* is continuous.

Corollary 3.1.3

$$\pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(\mathbb{S}^{n-1}).$$

Theorem 3.1.4

(Eckmann-Hilton). Let *G* be a group space and $e \in G$ be the identity point. Then $pi_1(G,e)$ is abelian.

Proof. To show $pi_1(G,x)$ is abelian, we will show that for any two loops $\gamma,\delta : I \to G$ we have $\gamma * \delta \sim \delta * \gamma$. Indeed for this, we construct a homotopy between the two loops above.

Theorem 3.1.5

(Van Kampen's theorem). Suppose $X = \bigcup_{i=1}^{n} \mathcal{A}_\alpha$ where each $\mathcal{A}_\alpha$ contains a green basepoint $x_0 \in X$ so that each $\mathcal{A}_\alpha$ is path connected and each $\mathcal{A}_\alpha \cap \mathcal{A}_\beta$ is path connected. We have homomorphisms $\pi_1(\mathcal{A}_\alpha, x_0) \to \pi_1(X_0, x_0)$ induced by the inclusions $\mathcal{A}_\alpha \to X$ and homomorphisms $i_{\alpha\beta} : \pi_1(\mathcal{A}_\alpha \cap \mathcal{A}_\beta, x_0) \to \pi_1(\mathcal{A}_\alpha, x_0)$ induced by the inclusions $\mathcal{A}_\alpha \cap \mathcal{A}_\beta \to X$.

1) The homomorphism $\Phi : \pi_1(\mathcal{A}_1, x_0) * \cdots * \pi_1(\mathcal{A}_n, x_0) \to \pi_1(X, x_0)$ is surjective.

2) If further each $\mathcal{A}_\alpha \cap \mathcal{A}_\beta \cap \mathcal{A}_\gamma$ is path connected, then the kernel of Φ is the minimal normal subgroup $\mathcal{N}$ generated by all the elements of the form $i_{\alpha\beta}(\omega) i_{\alpha\beta}(\omega)^{-1}$ for $\omega \in \pi_1(\mathcal{A}_\alpha \cap \mathcal{A}_\beta, x_0)$ so Φ induces an isomorphism $\pi_1(X, x_0) \simeq \pi_1(\mathcal{A}_1, x_0) * \cdots * \pi_1(\mathcal{A}_n, x_0) / \mathcal{N}$.

Theorem 3.1.6

$\Pi_1(S, x_\circ)$ is a fundamental group with respect to "$\circ$" in the general interval $[m, n]$.

Proof

1) "$\circ$" is associative. We need to show that $(f \circ g) \circ k \,\tilde{\tilde{x}}_\circ\, f \circ (g \circ k)$ for $f, g, k \in \Pi_1(S, x_\circ)$.

$$[(f \circ g) \circ k](s) = \left\{ \begin{pmatrix} f(2s) \\ g(2s-n) \end{pmatrix} \circ k = \begin{cases} f[2(2s)] \\ g[2(2s)-n] \\ k(2s-n) \end{cases} \right.$$

$$= \begin{cases} f(4s) & \text{if } \frac{m}{4} \le s \le \frac{n}{4} \\ g(4s-n) & \text{if } \frac{m+n}{4} \le s \le \frac{n}{2}, \forall m, n \in Z^+ \\ k(2s-n) & \text{if } \frac{m+n}{2} \le s \le n \end{cases}$$

and

$$[f \circ (g \circ k)](s) = \left\{ f \circ \begin{pmatrix} g(2s) \\ k(2s-n) \end{pmatrix} = \begin{cases} f(2s) \\ g[2(2s-n)] \\ k[2(2s-n)-n] \end{cases} \right.$$

$$= \begin{cases} f(2s) & \text{if } \frac{m}{2} \le s \le \frac{n}{2} \\ g(4s-2n) & \text{if } \frac{n+2m}{4} \le s \le \frac{3n}{4}, \forall m, n \in Z^+ \\ k(4s-3n) & \text{if } \frac{m+3n}{4} \le s \le n \end{cases}$$

We define a homotopy between $(f \circ g) \circ k$ and $f \circ (g \circ k)$ as follows:

$$h(s,t) = \begin{cases} f\left(\frac{4s}{n+t}\right) & \text{if } \langle s,t \rangle \in I^2 \text{ and } t \ge \frac{4s - mn}{m} \\ g(4s-t-n) & \text{if } \langle s,t \rangle \in I^2 \text{ and } \frac{m+n}{4s} \ge t \ge 4s-2n, \forall m, n \in Z^+ \\ k\left(\frac{4s-t-2n}{2-t}\right) & \text{if } \langle s,t \rangle \in I^2 \text{ and } 4s-2n \ge t \end{cases}$$

Then the following is true:

$$h(s,m)=\begin{cases} f(4s) & \text{if } 0\ge 4s-n & \left[\text{i.e. } \frac{m}{4}\le s\le \frac{n}{4}\right] \\ g(4s-n) & \text{if } 4s-n\ge 0\ge 4s-2n & \left[\text{i.e. } \frac{m+n}{4}\le s\le \frac{n}{2}\right] \\ k(2s-n) & \text{if } 4s-2\ge 0 & \left[\text{i.e. } \frac{m+n}{2}\le s\le n\right] \end{cases}$$

$$h(s,n)=\begin{cases} f(2s) & \text{if } n\ge 4s-n & \left[\text{i.e. } 0\le s\le \frac{m}{2}\right] \\ g(4s-2n) & \text{if } 4s-n\ge n\ge 4s-2n & \left[\text{i.e. } \frac{m+n}{4}\le s\le \frac{3n}{4}\right] \\ k(4s-3n) & \text{if } 4s-2n\ge 1 & \left[\text{i.e. } \frac{m+3n}{4}\le s\le n\right] \end{cases}$$

Thus $h(s,m)=[(f\circ g)\circ k](s)$ and $h(s,n)=[f\circ(g\circ k)](s)$ for all $s\in I^n$. Also $h(m,t)=f(m)=x_0$ and $h(n,t)=k(n)=x_0$ for all $t\in I^n$.

Hence $(f\circ g)\circ k\,\tilde{\tilde{x}}_0\, f\circ(g\circ k)$.

2) We show that the constant mapping $c:I^n\to\{x_0\}$ is such that [c] is the identity element of $\Pi_1(S,x_0)$ with respect to "$\circ$". Thus we must show that $f\circ c\,\tilde{\tilde{x}}_0\, f$ for any $f\in\Pi_1(S,x_0)$. Let $h:I^2\to S$ be defined as follows:

$$h(s,t)=\begin{cases} f\left(\frac{2s}{n+t}\right) & \text{if } \langle s,t\rangle\in I^2 \text{ and } t\ge \frac{2s-mn}{m} \\ x_\circ & \text{if } \langle s,t\rangle\in I^2 \text{ and } \frac{2s-mn}{m}\ge t \end{cases},\ \forall m,n\in Z^+$$

Then

$$(f\circ c)(s)=h(s,m)=\begin{cases} f(2s) & \text{if } 0\ge 2s-n & \left[i.e.\ \frac{m}{2}\le s\le \frac{n}{2}\right] \\ x_0 & \text{if } 2s-n\ge 0 & \left[i.e.\ \frac{m+n}{2}\le s\le n\right] \end{cases}$$

and $h(s,n)=f(s)$ if $1\ge 2s-n$ [*i.e.* $m\le s\le n$]. Thus $h(s,m)=(f\circ c)(s)$ and $h(s,n)=f(s)$ for all $s\in I^n$. More so, $h(m,t)=f(m)=x_\circ$ and $h(n,t)=f(n)=x_0$ for all $t\in I^n$.

Hence $f\circ c\,\tilde{\tilde{x}}_0\, f$.

3) Finally we want to show that each homotopy class $[f]\in\Pi_1(S,x_0)$ has an inverse $[g]\in\Pi_1(S,x_0)$ such that $[f]\circ[g]=[c]$. Thus we want to show that if $f\in\Pi_1(S,x_0)$ there exist a $g\in\Pi_1(S,x_0)$ such that $f\circ g\,\tilde{\tilde{x}}_0\, c$. Let $g(s)=f(n-s)\quad\forall s\in I^n$. Since $g(m)=f(n)=x_\circ=f(m)=g(n)$, $g\in\Pi_1(S,x_\circ)$

by definition, we have

$$(f\circ g)(s)=\begin{cases} f(2s) & \text{if } \frac{m}{2}\le s\le \frac{n}{2} \\ g(2s-n)=f(2n-2s) & \text{if } \frac{m+n}{2}\le s\le n \end{cases},\forall m,n\in Z^+$$

We then define homotopy h between $f \circ g$ and c as follows:

$$h(s,t)=\begin{cases} x_0 & \text{if } 0 \le s \le \frac{nt}{2} \\ f(2s-t) & \text{if } \frac{nt}{2} \le s \le \frac{n}{2} \\ g(2s+t-n) & \text{if } \frac{n}{2} \le s \le n-\frac{nt}{2} \\ x_0 & \text{if } n-\frac{nt}{2} \le s \le n \end{cases}, \forall m,n \in Z^+$$

Since f and g are continuous, h is continuous and we have

$$h(s,m)=\begin{cases} f(2s) & \text{if } \frac{m}{2} \le s \le \frac{n}{2} \\ g(2s-n) & \text{if } \frac{m+n}{2} \le s \le n \end{cases}, \forall m,n \in Z^+$$

and

$$h(s,n)=x_0 \quad \text{if } m \le s \le n$$

Thus $h(s,m)=(f \circ g)(s)$ and $h(s,n)=c(s)$ $\forall s \in I^n$. Also $h(m,t)=h(n,t)=x_\circ$ for all $t \in I^n$. Hence $f \circ g \,\tilde{\overline{x}}_\circ\, c$.

4. Conclusions

This paper, in full accordance with the principles of homotopy, has been able to establish the proof that $\Pi_1(S,x_\circ)$ is a fundamental group in the general interval $[m,n]$, $\forall m,n \in Z^+$.

In general, $\Pi_1(S,x_\circ)$ depends upon $x_\circ$. However, in the case of an arc-wise-connected space S, we can show that $\Pi_1(S,x_\circ)$ is independent of $x_\circ$.

Acknowledgements

Sincere thanks to Mrs. Belinda Zigli and Prof. William Obeg-Denteh for their immersed supports. The authors wish to acknowledge the support of their respective Universities and the anonymous referees for their helpful comments towards the improvement of the paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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