Rational Approximation to |x| at Logarithmic Nodes

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Abstract

Based on a node group $X = \left\{ x_k = \log\left(\frac{n+k}{n} \right) \right\}_{k=1}^{n}, k = 1, 2, \cdots, n$, the Newman type rational operator is constructed in the paper. The convergence rate of approximation to a class of non-smooth functions is discussed, which is $O\left(\frac{1}{n \log(n)} \right)$ regarding to X. Moreover, if the operator is constructed based on further subdivision nodes, the convergence rate is $O\left(\frac{1}{n^2 \log(n)} \right)$. The result in this paper is superior to the approximation results based on equidistant nodes, Chebyshev nodes of the first kind and Chebyshev nodes of the second kind.

Keywords

Newman Type Rational Operator, Logarithmic, Nodes, Approximation Order

1. Introduction

In approximation theory, rational approximation to $|x|$ plays an important role. The approximation to non-smooth functions $|x|$ began with Bernstein's polynomial approximation [1]. In 1964, Newman [2] constructed rational functions $r_n(x)$ and approximated $|x|$, with the constructed node set

$$X = \left\{ -a, -a^2, \cdots, -a^{n-1}, 0, a^{n-1}, \cdots, a^2, a \right\}, a = \exp\left(-n^{-\frac{1}{2}} \right), n = 1, 2, \cdots,$$

and the correspondingly rational function to set *X*. It was found that the approximation effect is much better than polynomial approximation.

After that, scholars have studied the convergence rate of $|x|$ by operators based on different node groups. In 1997, Brutman and Passow [3] considered the zeros of Chebyshev polynomial $T_{2n}(x) = \cos(2n \arccos x)$ in interval $(0,1]$, namely, $\left\{ x_k = \cos\left(\dfrac{(2k-1)\pi}{4n} \right) \right\}_{k=1}^{n}$, or

$$T = \left\{ x_k = \sin\left(\frac{(2k-1)\pi}{4n} \right) = \cos\left(\frac{(2n-2k+1)\pi}{4n} \right) \right\}_{k=1}^{n}, \tag{1}$$

and investigated the rational interpolation problem of $|x|$. The order of approximation was $O\left(\dfrac{1}{n\log(n)} \right)$.

In 2010, Huiming Zhang [4] constructed Newman-α type rational operators and the zeros of the second kind of Chebyshev polynomials $U_{2n-1}(x) = \dfrac{\sin(2n\arccos x)}{\sqrt{1-x^2}}$ in interval $[-1,1]$ were considered on the rational interpolation of $|x|$, $X = \left\{ \cos\left(\dfrac{k\pi}{2n} \right) = \sin\left(\dfrac{(n-k)\pi}{2n} \right) \right\}_{k=1}^{n-1}$, namely,

$U(x) = \left\{ x_k = \sin\left(\dfrac{k\pi}{2n} \right) = \cos\left(\dfrac{(n-k)\pi}{2n} \right) \right\}_{k=1}^{n-1}$. The order of approximation was $O\left(\dfrac{1}{n\log(n)} \right)$.

In reference [5], a new set of interpolating nodes was constructed, and the Newman type rational interpolation operator was used to approximate the function $|x|$, and the order of approximation was $O\left(e^{-\frac{2}{1+e}\sqrt{n}} \right)$, where $\varepsilon = \sqrt{\dfrac{3}{2\sqrt{n}} + \dfrac{1}{n}}$. In reference [6], the rational operator pairs on a group of encrypted Newman nodes $N = \left\{ e^{-\frac{k}{\sqrt{n}}} \right\}_{k=1}^{2n}$ approximated $|x|$, and the order of approximation was $O\left(e^{-2\sqrt{n}} \right)$.

The approximation $|x|$ of different operators based on different nodes has a wide past. Among these node groups, the zeros of polynomials and exponential type nodes have been considered. It is natural to ask how do the logarithmic nodes perform in the approximation problem.

In this paper, we focus on the convergence rate of $|x|$ at logarithmic nodes $X := \left\{ x_k := \log\left(\dfrac{n+k}{n} \right) \right\}_{k=1}^{n}$. We prove that, with the node group X, the convergence rate of approximation is $O\left(\dfrac{1}{n\log(n)} \right)$. Furthermore, if the operator

is constructed based on further subdivision nodes, the convergence rate is improved to $O\left(\dfrac{1}{n^2 \log(n)}\right)$. This result reveals the essential to some extent: the more dense the nodes near zero, the better rate of convergence, which also explains the results in some former research, see for example, reference [3].

The Newman type rational operator is constructed in this paper. The Newman type rational operator is defined as

$$r_n(X;x) = x\frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)}, p_n(x) = \prod_{k=1}^{n}(x_k + x), \tag{2}$$

and $h_n(X;x) = \dfrac{p_n(-x)}{p_n(x)}$.

2. Auxiliarily Lemmas

To verify the desired result, we shall need the following auxiliary results.

Lemma 1. *For* $x \in N_+$, *the inequality* $\mathrm{e}^{\frac{1}{2}} \leq \left(1 + \dfrac{1}{x}\right)^{x}$ *holds true.*

Proof. Notice that the function $f(x) = \left(1 + \dfrac{1}{x}\right)^{x}$ is monotonically increasing for $x > 1$. Then, $f(1) = 2$ is the minimum value and $f(1) > \mathrm{e}^{\frac{1}{2}}$. It is easy to observe that

$$\left(1 + \frac{1}{x}\right)^{x} \geq \mathrm{e}^{\frac{1}{2}}. \tag{3}$$

Lemma 2. *For* $x, y \in (1, \mathrm{e})$, *the inequality* $\dfrac{\log(x)}{\log(y)} \leq \dfrac{x}{y}$ *holds true.*

Proof. Write $f(x) = \dfrac{\log(x)}{x}$. Take the derivative of $f(x)$, we have

$$f'(x) = \frac{1}{x^2} - \frac{\log(x)}{x^2}. \tag{4}$$

The function $f(x) = \dfrac{\log(x)}{x}$ is monotonically increasing for $x \in (1, \mathrm{e})$. So the inequality $\dfrac{\log(x)}{x} < \dfrac{\log(y)}{y} \Rightarrow \dfrac{\log(x)}{\log(y)} \leq \dfrac{x}{y}$ holds true for $x < y$.

Lemma 3. [7] *For* $0 \leq x \leq \dfrac{1}{2}$, *the inequality* $\dfrac{1-x}{1+x} > 3^{-2x}$ *holds true.*

Lemma 4. *For* $x \in [0,1]$, *the inequality* $\dfrac{x}{2} \leq \log(x+1) \leq x$ *holds true.*

Proof. Take the derivative of $f(x) = \log(x+1) - \dfrac{x}{2}$, we obtain

$$f'(x) = \frac{1}{x+1} - \frac{1}{2}. \tag{5}$$

The function $f(x)$ is monotonically increasing for $x \in [0,1]$ and

$f(0) = 0$. Thus the inequality $\frac{x}{2} \le \log(x+1)$ holds true for $x \in [0,1]$.

Similarly, for function $g(x) = x - \log(x+1)$, the inequality $\log(x+1) \le x$ holds true when $x \in [0,1]$.

Lemma 5. [8] *For* $x \in N_+$, *the inequality* $\log(n) < \sum_{k=1}^{n} \left(\frac{1}{k}\right)$ *holds true.*

3. Main Results and Proofs

Theorem 1. *Based on a node group* $X = \left\{ x_k = \log\left(\frac{n+k}{n}\right) \right\}_{k=1}^{n}$, *we have*

$$\left| E_n(X;x) \right| = \left\| x \right\| - r_{2n-1}(X;x) = O\left(\frac{1}{n \log(n)} \right). \tag{6}$$

Proof of Theorem 1. Since $r_n(X;x) = x \dfrac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)}$ and $|x|$ are even

function, we only need to consider the approximation on the interval $[0,1]$. The proof will be divided into the following cases.

Case 1. $x \in \left[0, \log\left(\frac{n+1}{n}\right) \right]$. Considering

$$A_n = \sum_{k=1}^{n} \frac{1}{\log\left(\dfrac{n+k}{n}\right)}, \tag{7}$$

and

$$A_n = \sum_{k=1}^{n} \frac{1}{\log\left(\dfrac{n+k}{n}\right)} \ge \sum_{k=1}^{n} \frac{n}{k} \ge n \log(n). \tag{8}$$

Then, we see

$$\left| h_n(X;x) \right| = \left| \frac{p_n(-x)}{p_n(x)} \right| = \left| \prod_{k=1}^{n} \frac{x_k - x}{x_k + x} \right| = \left| \prod_{k=1}^{n} \frac{1 - \dfrac{x}{x_k}}{1 + \dfrac{x}{x_k}} \right| \le e^{-2x \sum_{k=1}^{n} \frac{1}{x_k}}. \tag{9}$$

Therefore,

$$\left| E_n(X;x) \right| = \left| x - r_n(X;x) \right| = \left| 2x \frac{p_n(-x)}{p_n(-x) + p_n(x)} \right|$$

$$= \frac{|2x| \left| \dfrac{p_n(-x)}{p_n(x)} \right|}{1 + \left| \dfrac{p_n(-x)}{p_n(x)} \right|} \le 2x \frac{\left| h_n(X;x) \right|}{1 - \left| h_n(X;x) \right|}$$

$$= 2x \left| h_n(X;x) \right| = 2x \frac{1}{e^{2xA_n}} \le \frac{x}{xA_n} \le \frac{1}{n \log(n)},$$

it is clear that the function achieves its maximum value in the interval

$\left[0, \log\left(\dfrac{n+1}{n} \right) \right]$ at $x = \dfrac{1}{n \log(n)}$.

Case 2. $x \in \left[\log\left(\dfrac{n+1}{n} \right), \log 2 \right]$. We can see

$$
\begin{aligned}
\left| h_n(X;x) \right| &= \left| \frac{p_n(-x)}{p_n(x)} \right| = \left| \prod_{k=1}^{n} \frac{x_k - x}{x_k + x} \right| \\
&= \prod_{k=1}^{n} \frac{\log 2 - \log\left(\dfrac{n+k}{n} \right)}{\log 2 + \log\left(\dfrac{n+k}{n} \right)} \le \prod_{k=1}^{n} \frac{\log\left(2\dfrac{n}{n+k} \right)}{\log\left(2\dfrac{n+k}{n} \right)} \\
&\le \prod_{k=1}^{n} \left(\frac{n}{n+k} \right)^2 = \left\{ \left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right) \cdots \left(1 + \frac{n}{n} \right) \right\}^{-2}.
\end{aligned}
$$

Applying the lemma 1,

$$
\left| h_n(X;x) \right| \le \left(\mathrm{e}^{\frac{1}{2n}} \mathrm{e}^{\frac{2}{2n}} \cdots \mathrm{e}^{\frac{n}{2n}} \right)^{-2} = \mathrm{e}^{-\frac{n(n+1)}{2n}} = \mathrm{e}^{-\frac{n+1}{2}}.
$$

Therefore,

$$
\left| E_n(X;x) \right| = \left| x - r_n(X;x) \right| = \left| 2x \frac{p_n(-x)}{p_n(-x) + p_n(x)} \right| \le 2x \left| h_n(X;x) \right| \le 2\mathrm{e}^{-\frac{n+1}{2}}.
$$

Case 3. $x \in \left[\log 2, 1 \right]$. Noting that

$$
\left| h_n(X;x) \right| = \left| \frac{p_n(-x)}{p_n(x)} \right| = \left| \prod_{k=1}^{n} \frac{x_k - x}{x_k + x} \right| = \prod_{k=1}^{n} \frac{\log \mathrm{e} - \log 2}{\log \mathrm{e} + \log 2} \le \prod_{k=1}^{n} \frac{\log\left(\dfrac{\mathrm{e}}{2} \right)}{\log(2\mathrm{e})} \le 4^{-n}.
$$

Thus we have

$$
\left| E_n(X;x) \right| = \left| x - r_n(X;x) \right| = \left| 2x \frac{p_n(-x)}{p_n(-x) + p_n(x)} \right| \le 2x \left| h_n(X;x) \right| \le 4^{-n+\frac{1}{2}}.
$$

Combining three cases completes the proof of Theorem 1.

Theorem 2. *Considering* $x^* = \dfrac{1}{n \log(n)}$, *we have*

$$
\left| E_n(X;x^*) \right| \ge \frac{1}{729 n \log(n)}. \tag{10}
$$

Proof. let $n > 4$, we can see that

$$
\left| h_n(X;x^*) \right| = \prod_{k=1}^{n} \frac{\log\left(\dfrac{n+k}{n} \right) - x^*}{\log\left(\dfrac{n+k}{n} \right) + x^*} = \prod_{k=1}^{n} \frac{1 - \dfrac{x^*}{\log\left(\dfrac{n+k}{n} \right)}}{1 - \dfrac{x^*}{\log\left(\dfrac{n+k}{n} \right)}}.
$$

Applying Lemma 3,

$$\left|h_n\left(X;x^*\right)\right| \geq 3^{-2x^*\sum\limits_{k=1}^{n}\frac{1}{x_k}},$$

where

$$x^*\sum_{k=1}^{n}\frac{1}{x_k} = x^*\sum_{k=1}^{n}\frac{1}{\log\left(\dfrac{n+k}{n}\right)} \leq x^*\sum_{k=1}^{n}\frac{2n}{k} = 2nx^*\sum_{k=1}^{n}\frac{1}{k}$$

$$\leq \frac{2n}{n\log(n)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) \leq 3.$$

Then we obtain,

$$n\log(n)\left|E_n\left(X;x^*\right)\right| = \frac{2n\log(n)\cdot\dfrac{1}{n\log(n)}\cdot h_n\left(X;x^*\right)}{1+h_n\left(X;x^*\right)} \tag{11}$$

$$\geq h_n\left(X;x^*\right) \geq 3^{-6} = \frac{1}{729}.$$

This shows that the approximation order in Theorem 1 cannot be improved.

It is noticed that the density near zero is closely related to the approximation order. If the node near the zero point is further subdivided, the following theorem can be obtained.

Theorem 3. *Considering inserting n-degree nodes into the interval* $\left[0,\log\left(\dfrac{n+1}{n}\right)\right]$, *which is* $\left\{\dfrac{1}{n}\log\left(\dfrac{n+k}{n}\right)\right\}_{k=1}^{n}$. *The order of approximation on the interval* $[0,1]$ *is* $O\left(\dfrac{1}{n^2\log(n)}\right)$ *for* $k\geq 1$.

Proof. Taking interval $x\in\left[0,\dfrac{1}{n}\log\left(\dfrac{n+1}{n}\right)\right]$ as an example. Considering

$$A_n = \sum_{k=1}^{n}\left(\frac{1}{\dfrac{1}{n}\log\left(\dfrac{n+k}{n}\right)} + \frac{1}{\log\left(\dfrac{n+k}{n}\right)}\right),$$

we can see

$$A_n = \sum_{k=1}^{n}\left(\frac{1}{\dfrac{1}{n}\log\left(\dfrac{n+k}{n}\right)} + \frac{1}{\log\left(\dfrac{n+k}{n}\right)}\right) \geq \sum_{k=1}^{n}\frac{n^2}{k} + \sum_{k=1}^{n}\frac{n}{k}$$

$$\geq n^2\log(n) + n\log(n) \geq n^2\log(n).$$

Direct computation together with (9) result in

$$\left|E_n\left(X;x\right)\right| = \left|x-r_n\left(X;x\right)\right| = \left|2x\frac{p_n\left(-x\right)}{p_n\left(-x\right)+p_n\left(x\right)}\right|$$

$$\leq 2x\left|h_n\left(X;x\right)\right| = 2x\frac{1}{e^{2xA_n}} \leq \frac{1}{A_n} \leq \frac{1}{n^2\log(n)}.$$

The proof on other intervals can refer to theorem 1.

4. Conclusions

In this paper, based on the node group $X = \left\{ x_k = \log\left(\dfrac{n+k}{n}\right)\right\}_{k=1}^{n}$, we use Newman type rational operators to approximate $|x|$, and derive the approximation order $O\left(\dfrac{1}{n\log(n)}\right)$. When $r_n(x)$ is discussed on the interval $\left[0, \log\left(\dfrac{n+1}{n}\right)\right]$, the approximation effect is the worst, which is $O\left(\dfrac{1}{n\log(n)}\right)$; on the interval $\left[\log\left(\dfrac{n+1}{n}\right), \log 2\right]$, the approximation effect is better, which is $O\left(e^{-n}\right)$; on the interval $\left[\log 2, 1\right]$, the approximation effect is the best, it is $O\left(4^{-n}\right)$.

Considering that the higher the density of the node near zero, the better the approximation effect is, we further insert the nodes of degree n to make the approximation order reach $O\left(\dfrac{1}{n^2\log(n)}\right)$. This result is better than $|x|$ on equidistant nodes ([9]), Chebyshev nodes of the first kind ([3]) and Chebyshev nodes of the second kind ([4]).

Acknowledgements

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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