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Distributed Control for $n \times n$ Cooperative Systems Governed by Hyperbolic Operator of Infinite Order

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Abstract

In this study, a distributed optimal control problem for $n \times n$ cooperative hyperbolic systems with infinite order operators and Dirichlet conditions are considered. The existence and uniqueness of the state of these systems are proved. The necessary and sufficient conditions for optimality of distributed control with constraints are found, and the set of equations and inequalities that defining the optimal control of these systems is also obtained. Finally, some examples for the control problem without constraints are given.

Keywords

Cooperative Systems, Hyperbolic Systems, Optimal Control, Infinite Order, Distributed Control Problem, Dirichlet Conditions

1. Introduction

The earliest theory of optimal control was introduced by Lions [1].

Majority of the research in this field has focused on discussing the optimal control problem by using several operator types (such as elliptic, parabolic, or hyperbolic operators) [2] [3] [4].

The discussion was extended to systems involving different types of operators (such as infinite order [5]-[11] or infinite number of variables [12] [13] [14]).

In [3] [15] [16] [17], the studies continued to develop using different types of systems (cooperative or non-cooperative).

Based on the theories proposed by Lions [1] and Dubinskii [18] [19] [20], the distributed control problem with Dirichlet conditions for 2×2 non-cooperative hyperbolic systems involving infinite order operators was discussed in a previous study [17]; in this study, we extend this problem to $n \times n$ cooperative hyperbolic systems.

The system can be defined as

$$\left.\begin{array}{l}\frac{\partial^2}{\partial t^2} y_i + \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y_i = \sum_{j=1}^{n} a_{ij} y_j + f_i \text{ in } Q \\ y_i \to 0,\ |x| \to \infty \\ D^\omega y_i = 0 \text{ on } \Sigma \text{ for } |\omega| = 0,1,2,\cdots,|\omega| \le \alpha - 1,\ i = 1,2,\cdots,n \\ y_i(x,0) = y_{i,0}(x),\ \frac{\partial y_i(x,0)}{\partial t} = y_{i,1}(x),\ x \in \Omega \end{array}\right\} \quad (1)$$

with $y_i \in L^2(Q)$, $\frac{\partial y_i}{\partial t} \in L^2(Q)$.

Where

$a_{ij} > 0$ for all $i \ne j$. (This implies that the system (1) is cooperative), (2)

$$a_{ij} = a_{ji} \text{ for all } 1 \le i, j \le n, \quad (3)$$

and $Q = \Omega \times]0,T[$ with boundary $\Sigma = \Gamma \times]0,T[$.

This paper is constituted of four sections. Section 1 presents the Sobolev spaces of infinite order, which we refer to later in the paper. In section 2, the state of $n \times n$ cooperative system with Dirichlet conditions is studied. In Section 3, the formulation of the distributed control with constraints is introduced. Finally, Section 4 presents some examples for the control problem without constraints.

2. Necessary Spaces: [18] [19] [20]

The Sobolev spaces of infinite order operators, which are used in this study, have already been presented in Reference [17].

We will list them briefly below:

$$* \quad H^\infty(\Omega) = H^\infty\{a_\alpha, 2\}(\Omega) = \left\{\psi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha \psi\right\|_2^2 \le \infty\right\},$$

* The conjugate space of $H^\infty(\Omega)$ is defined as,

$$* \quad H^{-\infty}(\Omega) = H^{-\infty}\{a_\alpha, 2\}(\Omega) = \left\{\vartheta(x) : \vartheta(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \vartheta_\alpha(x)\right\},$$

where $\vartheta_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha \vartheta_\alpha\right\|_2^2 < \infty$.

Then we have the following chains:

$$* \quad H^\infty(\Omega) \subseteq L^2(\Omega) \subseteq H^{-\infty}(\Omega),$$

$$* \quad H_0^\infty(\Omega) \subseteq L^2(\Omega) \subseteq H_0^{-\infty}(\Omega),$$

where

$$H_0^\infty(\Omega) = H_0^\infty\{a_\alpha, 2\}(\Omega) = \left\{\psi(x) \in C_0^\infty(\Omega) : \|\psi\|^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha \left\|D^\alpha \psi\right\|_2^2 \prec \infty, D^\omega \psi\big|_\Gamma = 0, |\omega| \le \alpha\right\}.$$

* $L^2(Q) = L^2(0,T,L^2(\Omega))$ is a Hilbert space of measurable functions $t \to \psi(t)$, $t \in]0,T[$, that map an interval (0, *T*) in to the space $L^2(\Omega)$, such that: $\|\psi\|_{L^2(Q)} = \left(\int_0^T \|\psi(t)\|_2^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \le \infty$, and

$$(f,g)=\int_0^T \left(f(t),g(t)\right)_{L^2(\Omega)}\mathrm{d}t,$$

* In a similar manner as that of $L^2(Q)$, we obtain the constructed space $L^2\left(0,T,H_0^{\infty}(\Omega)\right)=L^2\left(H_0^{\infty}(Q)\right)$, and the following chains:

* $L^2\left(H_0^{\infty}(Q)\right)\subseteq L^2(Q)\subseteq L^2\left(H_0^{-\infty}(Q)\right)$,

* $\left(L^2\left(H_0^{\infty}(Q)\right)\right)^n\subseteq\left(L^2(Q)\right)^n\subseteq\left(L^2\left(H_0^{-\infty}(Q)\right)\right)^n$.

Finally,

* $W_0(0,T)=\left\{f\in L^2\left(H_0^{\infty}(Q)\right):\dfrac{\mathrm{d}f}{\mathrm{d}t}\in L^2\left(H_0^{-\infty}(Q)\right)\right\}$,

with the norm:

$$\|f(t)\|_{W_0(0,T)}=\left(\int_{(0,T)}\|f(t)\|^2_{H_0^{\infty}(\Omega)}\mathrm{d}t+\int_{(0,T)}\left\|\frac{\mathrm{d}f}{\mathrm{d}t}\right\|^2_{H_0^{-\infty}(\Omega)}\mathrm{d}t\right)^{1/2}$$

which is also a Hilbert space.

3. State of the System

We study the following cooperative hyperbolic systems with Dirichlet conditions:

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}y_i+Ay_i=\sum_{j=1}^{n}a_{ij}y_j+f_i \quad \text{in } Q\\
&y_i\to 0,\ |x|\to\infty\\
&D^{\omega}y_i=0 \ \text{ on }\Sigma \ \text{ for } |\omega|=0,1,2,\cdots,|\omega|\le\alpha-1,\ i=1,2,\cdots,n\\
&y_i(x,0)=y_{i,0}(x),\ \frac{\partial y_i(x,0)}{\partial t}=y_{i,1}(x),\ x\in\Omega
\end{aligned}\right\} \tag{4}$$

with $y_i\in L^2\left(H_0^{\infty}(Q)\right)$, $\dfrac{\partial y_i}{\partial t}\in L_2(Q)$.

We have the operators $A\in\mathcal{L}\left(\left(L^2\left(H_0^{\infty}(Q)\right)\right)^n,\left(L^2\left(H_0^{-\infty}(Q)\right)\right)^n\right)$ such that

$$A\left(\overline{y}=(y_1,y_2,\cdots,y_n)\right)=(Ay_1,Ay_2,\cdots,Ay_n)$$
$$=\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_{\alpha}D^{2\alpha}y_1,\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_{\alpha}D^{2\alpha}y_2,\cdots,\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_{\alpha}D^{2\alpha}y_n\right),$$

it is easy to write A as a matrix take the form:

$$A\overline{y}=\begin{bmatrix}\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_{\alpha}D^{2\alpha} & \cdots & 0\\ 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_{\alpha}D^{2\alpha}\end{bmatrix}_{n\times n}\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix},$$

i.e.

$$Ay_i = \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2\alpha} y_i,\ \ i = 1,2,\cdots,n \tag{5}$$

Let M be $(n\times n)$ square coefficients matrix such that

$$M\left(\bar{y} = (y_1, y_2, \cdots, y_n)\right) = \left(\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \cdots, \sum_{j=1}^{n} a_{nj}y_j\right)$$

$$= \left(a_{11}y_1 + \cdots + a_{1n}y_n, a_{21}y_1 + \cdots + a_{2n}y_n, a_{n1}y_1 + \cdots + a_{nn}y_n\right)$$

i.e. $M\bar{y} = \sum_{j=1}^{n} a_{ij}y_j$, $i = 1,2,\cdots,n$.

Let $S = A - M$, so that S represents $(n\times n)$ square matrix takes the form

$$S = \begin{bmatrix} \sum\limits_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2\alpha} - a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \sum\limits_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2\alpha} - a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \sum\limits_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2\alpha} - a_{nn} \end{bmatrix}_{n\times n}$$

Therefore, $S_i y_i = \sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2\alpha} y_i - \sum_{j=1}^{n} a_{ij}y_j$, $i = 1,2,\cdots,n$.

Hence, we can rewrite the first equation in system (4) as follows:

$$\frac{\partial^2}{\partial t^2} y_i + S_i y_i = f_i \ \text{ in } Q$$

Definition 1:

The bilinear form $\pi(t,\bar{y},\bar{\phi})$ is defined on $\left(L^2\left(H_0^\infty(Q)\right)\right)^n$ as follows:

$$\pi(t,\bar{y},\bar{\phi}) = \left(S\bar{y},\bar{\phi}\right)_{\left(L^2(Q)\right)^n},\ \ \bar{y} = (y_i)_{i=1}^n,\ \ \bar{\phi} = (\phi_i)_{i=1}^n \in \left(L^2\left(H_0^\infty(Q)\right)\right)^n,$$

where S maps $\left(L^2\left(H_0^\infty(Q)\right)\right)^n$ onto $\left(L^2\left(H_0^{-\infty}(Q)\right)\right)^n$, so that

$$\pi(t,\bar{y},\bar{\phi}) = \sum_{i=1}^{n}\left(S_i y_i, \phi_i\right)_{L^2(Q)},$$

$$\begin{aligned} \pi(t,\bar{y},\bar{\phi}) &= \sum_{i=1}^{n}\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} a_\alpha D^{2\alpha} y_i - \sum_{j=1}^{n} a_{ij}y_j, \phi_i\right)_{L^2(Q)} \\ &= \sum_{i=1}^{n}\int_Q \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha y_i D^\alpha \phi_i \,\mathrm{d}x\mathrm{d}t - \sum_{i=1}^{n}\int_Q \sum_{j=1}^{n} a_{ij} y_j \phi_i \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{6}$$

Lemma 1:

There exists a constant $c, c_1 \in R$, such that:

$$\pi(t,\bar{y},\bar{y}) + c_1\|y\|^2_{\left(L^2(Q)\right)^n} \ge c\|y\|^2_{\left(L^2\left(H_0^\infty(Q)\right)\right)^n},\ \ c, c_1 > 0, \tag{7}$$

that is, (6) is coercive on $\left(L^2\left(H_0^\infty(Q)\right)\right)^n$.

Proof:

We have:

$$\pi(t,\bar{y},\bar{\phi}) = \sum_{i=1}^{n}\int_Q \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha y_i D^\alpha \phi_i \,\mathrm{d}x\mathrm{d}t - \sum_{i=1}^{n}\int_Q \sum_{j=1}^{n} a_{ij} y_j \phi_i \,\mathrm{d}x\mathrm{d}t.$$

Thus,

$$\pi\left(t,\overline{y},\overline{y}\right)=\sum_{i=1}^{n}\int_{Q}\sum_{|\alpha|=0}^{\infty}a_{\alpha}\left|D^{\alpha}y_{i}\right|^{2}\mathrm{d}x\mathrm{d}t-\sum_{i=1}^{n}\int_{Q}a_{ii}\left|y_{i}\right|^{2}\mathrm{d}x\mathrm{d}t$$
$$-\sum_{i\neq j}^{n}\int_{Q}a_{ij}y_{i}y_{j}\mathrm{d}x\mathrm{d}t,$$

then we deduce

$$\pi\left(t,\overline{y},\overline{y}\right)+\sum_{i=1}^{n}\int_{Q}a_{ii}\left|y_{i}\right|^{2}\mathrm{d}x\mathrm{d}t+\sum_{i\neq j}^{n}\int_{Q}a_{ij}y_{i}y_{j}\mathrm{d}x\mathrm{d}t$$
$$=\sum_{i=1}^{n}\int_{Q}\sum_{|\alpha|=0}^{\infty}a_{\alpha}\left|D^{\alpha}y_{i}\right|^{2}\mathrm{d}x\mathrm{d}t,$$

then,

$$\pi\left(t,\overline{y},\overline{y}\right)+c_{1}\left\|y\right\|_{\left(L^{2}(Q)\right)^{n}}^{2}\geq\left\|y\right\|_{\left(L^{2}\left(H_{0}^{\infty}(Q)\right)\right)^{n}}^{2},\quad c_{1}>0$$

which proves the coerciveness condition on $\left(L^{2}\left(H_{0}^{\infty}\left(Q\right)\right)\right)^{n}$.

Lemma 2:

If (2), (3) and (7) are hold, then $\exists!\ \overline{y}=\left(y_{i}\right)_{i=1}^{n}\in\left(L^{2}\left(H_{0}^{\infty}\left(Q\right)\right)\right)^{n}$ for system (4), for $f_{i}=f_{i}\left(x,t\right)\in L^{2}\left(Q\right)$.

Proof:

Let $\overline{\psi}=\left(\psi_{i}\right)_{i=1}^{n}\rightarrow L\left(\overline{\psi}\right)$ be a continuous linear form defined on $\left(L^{2}\left(H_{0}^{\infty}\left(Q\right)\right)\right)^{n}$ by

$$\forall\,\overline{\psi}=\left(\psi_{i}\right)_{i=1}^{n}\in\left(L^{2}\left(H_{0}^{\infty}\left(Q\right)\right)\right)^{n},$$
$$L\left(\overline{\psi}\right)=\sum_{i=1}^{n}\left\{\int_{Q}f_{i}\psi_{i}\mathrm{d}x\mathrm{d}t+\int_{\Omega}y_{i,1}\psi_{i}\left(x,0\right)\mathrm{d}x\right\},\tag{8}$$

where $f_{i}\in L^{2}\left(Q\right),y_{i,0}\in L^{2}\left(\Omega\right)$ and $y_{i,1}\in L^{2}\left(\Omega\right)$.

Then, by the Lax-Milgram lemma,

$\exists!\ \overline{y}\in\left(L^{2}\left(H_{0}^{\infty}\left(Q\right)\right)\right)^{n}$ such that

$$L\left(\overline{\psi}\right)=\frac{\partial^{2}}{\partial t^{2}}\left(y_{i},\psi_{i}\right)+\pi\left(t,\overline{y},\overline{\psi}\right),\ \forall\,\overline{\psi}=\left(\psi_{i}\right)_{i=1}^{n}\in\left(L^{2}\left(H_{0}^{\infty}\left(Q\right)\right)\right)^{n}.\tag{9}$$

Now, let us multiply system (4) by ψ_{i}, and then integrate it over Q:

$$\sum_{i=1}^{n}\int_{Q}\left(\frac{\partial^{2}}{\partial t^{2}}y_{i}+Ay_{i}\right)\psi_{i}\mathrm{d}x\mathrm{d}t-\sum_{j=1}^{n}\int_{Q}a_{ij}y_{j}\psi_{i}\mathrm{d}x\mathrm{d}t=\sum_{i=1}^{n}\int_{Q}f_{i}\psi_{i}\mathrm{d}x\mathrm{d}t.$$

By using Green's formula:

$$\sum_{i=1}^{n}\left\{\int_{Q}\frac{\partial^{2}\psi_{i}}{\partial t^{2}}y_{i}\mathrm{d}x\mathrm{d}t+\int_{Q}\sum_{|\alpha|=0}^{\infty}a_{\alpha}D^{\alpha}y_{i}D^{\alpha}\psi_{i}\mathrm{d}x\mathrm{d}t-\sum_{j=1}^{n}\int_{Q}a_{ij}y_{j}\psi_{i}\mathrm{d}x\mathrm{d}t\right.$$
$$\left.+\int_{\Omega}\frac{\partial y_{i}\left(x,0\right)}{\partial t}\psi_{i}\left(x,0\right)\mathrm{d}x-\int_{\Sigma}\frac{\partial y_{i}}{\partial v_{A}}\psi_{i}\mathrm{d}\Sigma\right\}=\sum_{i=1}^{n}\int_{Q}f_{i}\psi_{i}\mathrm{d}x\mathrm{d}t$$

from (6), (8) and (9) we have

$$\sum_{i=1}^{n}\left\{\int_{\Omega}\frac{\partial y_{i}\left(x,0\right)}{\partial t}\psi_{i}\left(x,0\right)\mathrm{d}x-\int_{\Sigma}\frac{\partial y_{i}}{\partial v_{A}}\psi_{i}\mathrm{d}\Sigma\right\}=\sum_{i=1}^{n}\int_{Q}y_{i,1}\left(x\right)\psi_{i}\left(x,0\right)\mathrm{d}x.$$

Then, we deduce that

$$D^{\omega} y_i = 0 \text{ on } \Sigma \text{ for } |\omega| = 0,1,2,3,\cdots, |\omega| \le \alpha - 1,\ i = 1,2,3,\cdots,n$$

$$\frac{\partial y_i(x,0)}{\partial t} = y_{i,1}(x),\ \ x \in \Omega,$$

Thus, the proof is complete.

4. Control Problem with Constraints

The space $U = \left(L^2(Q)\right)^n$ is the space of controls $\bar{u} = (u_i)_{i=1}^n$.

The state of the system $\bar{y}(\bar{u}) = \left(y_i(\bar{u})\right)_{i=1}^n \in \left(L^2\left(H^{\infty}(Q)\right)\right)^n$ is determined by the solution of

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2} y_i(\bar{u}) + A y_i(\bar{u}) = \sum_{j=1}^n a_{ij} y_j + f_i + u_i \ \text{ in } Q,\\
&y_i \to 0,\ |x| \to \infty,\\
&D^{\omega} y_i = 0 \ \text{ on } \Sigma \ \text{ for } |\omega| = 0,1,2,3,\cdots, |\omega| \le \alpha - 1,\ i = 1,2,3,\cdots,n\\
&y_i(x,0,\bar{u}) = y_{i,0}(x),\ \frac{\partial y_i(x,0,\bar{u})}{\partial t} = y_{i,1}(x),\ \ x \in \Omega,
\end{aligned}\right\} \tag{10}$$

with $y_i \in L^2\left(H_0^{\infty}(Q)\right)$, $\frac{\partial y_i}{\partial t} \in L_2(Q)$.

The observation function is given by

$$\bar{z}(\bar{u}) = \left(z_i(\bar{u})\right)_{i=1}^n = \left(y_i(\bar{u})\right)_{i=1}^n = \bar{y}(\bar{u}).$$

The cost function $J(\bar{u})$ is given by

$$J(\bar{u}) = \sum_{i=1}^n \left\| y_i(\bar{u}) - z_{id} \right\|_{L^2(Q)}^2 + M \sum_{i=1}^n \left\| u_i \right\|_{L^2(Q)}^2 \tag{11}$$

where $\bar{z}_d = (z_{id})_{i=1}^n \in \left(L^2(Q)\right)^n$ and $M \ge 0$ is a constant.

Then, the control problem is to minimize J over U_{ad} which is a closed convex subset of $U = \left(L^2(Q)\right)^n$.

i.e. to determine $\bar{u}$ such that

$$J(\bar{u}) = \inf_{\bar{v} \in U_{ad}} J(\bar{v}),\ \ \bar{v} = (v_i)_{i=1}^n.$$

Based on the above data and previous results, we have the following theorem:

Theorem 1:

Assuming that (7),(10) and (11) hold, ∃! the optimal control $\bar{u} = (u_i)_{i=1}^n \in U_{ad}$ such that: $J(\bar{u}) \le J(\bar{v}), \forall \bar{v} = (v_i)_{i=1}^n \in U_{ad}$, and it is determined by:

$$\left.\begin{aligned}
&\frac{\partial^2 p_i(\bar{u})}{\partial t^2} + A p_i(\bar{u}) - \sum_{j=1}^n a_{ij} p_j = y_i(\bar{u}) - z_{id} \ \text{ in } Q\\
&D^{\omega} p_i(\bar{u}) = 0 \ \text{ on } \Sigma \ \text{ for } |\omega| = 0,1,2,\cdots, |\omega| \le \alpha - 1,\ i = 1,2,\cdots,n\\
&p_i(x,0,\bar{u}) = 0,\ \frac{\partial p_i(x,0,\bar{u})}{\partial t} = 0,\ \ x \in \Omega
\end{aligned}\right\} \tag{12}$$

with $y_i, p_i \in L^2\left(H_0^{\infty}(Q)\right)$, $\frac{\partial y_i}{\partial t}, \frac{\partial p_i}{\partial t} \in L_2(Q)$

and

$$\sum_{i=1}^{n}\left(p_i\left(\overline{u}\right)+Mu_i,v_i-u_i\right)_{L^2(Q)}\geq 0,\ \ \forall\overline{v}=\left(v_i\right)_{i=1}^{n}\in U_{ad} \tag{13}$$

where $p_i\left(\overline{u}\right)$ is the adjoint state.

Proof:

As in [1], $\overline{u}=\left(u_i\right)_{i=1}^{n}\in U_{ad}$ is determined by:

$$\sum_{i=1}^{n}J_i'\left(\overline{u}\right)\left(v_i-u_i\right)\geq 0,\ \ \forall\overline{v}=\left(v_i\right)_{i=1}^{n}\in U_{ad},$$

i.e.

$$\sum_{i=1}^{n}\left(y_i\left(\overline{u}\right)-z_{id},y_i\left(\overline{v}\right)-y_i\left(\overline{u}\right)\right)_{L^2(Q)}+\sum_{i=1}^{n}M\left(u_i,v_i-u_i\right)_{L^2(Q)}\geq 0$$

which is equivalent to:

$$\sum_{i=1}^{n}\int_0^T\left(y_i\left(\overline{u}\right)-z_{id},y_i\left(\overline{v}\right)-y_i\left(\overline{u}\right)\right)_{L^2(\Omega)}\mathrm{d}t+\sum_{i=1}^{n}M\left(u_i,v_i-u_i\right)_{L^2(Q)}\geq 0. \tag{14}$$

Now, let us define a hyperbolic infinite order operator B as follows:

$$B\overline{y}\left(\overline{u}\right)=By_i\left(\overline{u}\right)=\frac{\partial^2 y_i\left(\overline{u}\right)}{\partial t^2}+Ay_i\left(\overline{u}\right)-\sum_{j=1}^{n}a_{ij}y_j,\ \ \forall\overline{u}=\left(u_i\right)_{i=1}^{n}\in U_{ad},$$

Since, $\left(\overline{p},B\overline{y}\right)_{L^2(Q)}=\int_0^T\left(p_i\left(\overline{u}\right),\frac{\partial^2 y_i\left(\overline{u}\right)}{\partial t^2}+Ay_i\left(\overline{u}\right)-\sum_{j=1}^{n}a_{ij}y_j\right)_{L^2(\Omega)}\mathrm{d}t$, from (3), we obtain

$$\left(\overline{p},B\overline{y}\right)_{L^2(Q)}=\int_0^T\left(\frac{\partial^2 p_i\left(\overline{u}\right)}{\partial t^2}+Ap_i\left(\overline{u}\right)-\sum_{j=1}^{n}a_{ij}p_j,y_i\left(\overline{u}\right)\right)_{L^2(\Omega)}\mathrm{d}t$$
$$=\left(B^*\overline{p},\overline{y}\right)_{L^2(Q)},$$

then $B^*\overline{p}\left(\overline{u}\right)=B^*\left(p_i\left(\overline{u}\right)\right)=\left(\frac{\partial^2 p_i\left(\overline{u}\right)}{\partial t^2}+Ap_i\left(\overline{u}\right)-\sum_{j=1}^{n}a_{ij}p_j\right), i=1,2,\cdots,n$.

Now, let us set the following notation:

$$S^*\overline{p}\left(\overline{u}\right)=S_i^*\left(p_i\left(\overline{u}\right)\right)=Ap_i\left(\overline{u}\right)-\sum_{j=1}^{n}a_{ij}p_j,\ \ i=1,2,\cdots,n.$$

According to the form of the adjoint equation in [1]:

$$\frac{\partial^2 p\left(\overline{u}\right)}{\partial t^2}+S_i^*p\left(\overline{u}\right)=y\left(\overline{u}\right)-z_d,$$

and by Lemma 2,

$\exists!$ Solution $p_i\left(\overline{u}\right)\in L^2\left(Q\right)$ for (12).

Now, we transform (14) as follows:

we multiply (12) by $\left(y_i\left(\overline{v}\right)-y_i\left(\overline{u}\right)\right)$ and integrating between 0, T, then we obtain:

$$\int_0^T\left(y_i\left(\overline{u}\right)-z_{id},y_i\left(\overline{v}\right)-y_i\left(\overline{u}\right)\right)_{L^2(\Omega)}\mathrm{d}t$$
$$=\int_0^T\left(\left(\frac{\partial^2}{\partial t^2}+A\right)p_i\left(\overline{u}\right)-\sum_{j=1}^{n}a_{ij}p_j,\left(y_i\left(\overline{v}\right)-y_i\left(\overline{u}\right)\right)\right)_{L^2(\Omega)}\mathrm{d}t$$

$$= \int_0^T \left(p_i(\bar{u}), \left(\frac{\partial^2}{\partial t^2} + A \right) \left(y_i(\bar{v}) - y_i(\bar{u}) - \sum_{j=1}^{n} a_{ij} \left(y_i(\bar{v}) - y_i(\bar{u}) \right) \right) \right)_{L^2(\Omega)} \mathrm{d}t$$

$$= \int_0^T \left(p_i(\bar{u}), v_i - u_i \right)_{L^2(\Omega)} \mathrm{d}t,$$

hence (14) becomes

$$\sum_{i=1}^{n} \int_0^T \left(p_i(\bar{u}), v_i - u_i \right)_{L^2(\Omega)} \mathrm{d}t + \sum_{i=1}^{n} M \left(u_i, v_i - u_i \right)_{L^2(Q)} \geq 0, \quad \forall \bar{v} = (v_i)_{i=1}^{n} \in U_{ad},$$

i.e. $\sum_{i=1}^{n} \int_0^T \left(p_i(\bar{u}) + M u_i, v_i - u_i \right)_{L^2(\Omega)} \mathrm{d}t \geq 0, \forall \bar{v} = (v_i)_{i=1}^{n} \in U_{ad}$.

Thus, the proof is complete.

5. Control Problem without Constraints

1) The case if $U_{ad} = \left(L^2(Q) \right)^n$ *i.e.* (there are no constraints on the control $\bar{u}$), then (13) takes the form $p_i(\bar{u}) + N_i u_i = 0, x \in Q$, hence

$$u_i = -N_i^{-1} p_i(\bar{u}). \tag{15}$$

Example 1:

Let us consider n=2 in (1), also (2) and (3) are satisfied, the space $\left(L_2(Q) \right)^2$ is the space of controls $u = (u_1, u_2)$ and the state $y(u) = \left(y_1(u), y_2(u) \right) \in \left(L^2\left(H_0^\infty(Q) \right) \right)^2$ is determined by:

$$\begin{cases} \dfrac{\partial^2 y_1(\bar{u})}{\partial t^2} + A y_1(\bar{u}) + N_1^{-1} p_1(\bar{u}) = a_{11} y_1(\bar{u}) + a_{12} y_2(\bar{u}) + f_1 \quad \text{in } Q, \\ \dfrac{\partial^2 y_2(\bar{u})}{\partial t^2} + A y_2(\bar{u}) + N_2^{-1} p_2(\bar{u}) = a_{21} y_1(\bar{u}) + a_{22} y_2(\bar{u}) + f_2 \quad \text{in } Q, \\ y_1, y_2 \to 0, \quad |x| \to \infty, \\ y_1(\bar{u})\big|_\Sigma = 0, \quad y_2(\bar{u})\big|_\Sigma = 0, \\ y_1(x,0;\bar{u}) = y_{0,1}(x), \dfrac{\partial y_1(x,0;\bar{u})}{\partial t} = y_{1,1}(x), \quad x \in \Omega, \\ y_2(x,0;\bar{u}) = y_{0,2}(x), \dfrac{\partial y_2(x,0;\bar{u})}{\partial t} = y_{1,2}(x), \quad x \in \Omega. \end{cases} \tag{16}$$

$$y_1(\bar{u}), y_2(\bar{u}), \frac{\partial y_1(\bar{u})}{\partial t}, \frac{\partial y_2(\bar{u})}{\partial t} \in L^2(Q).$$

$$\begin{cases} \dfrac{\partial^2 p_1(\bar{u})}{\partial t^2} + A p_1(\bar{u}) - a_{11} p_1(\bar{u}) - a_{12} p_2(\bar{u}) = y_1(\bar{u}) - z_{d1} \quad \text{in } Q, \\ \dfrac{\partial^2 p_2(\bar{u})}{\partial t^2} + A p_2(\bar{u}) - a_{21} p_1(\bar{u}) - a_{22} p_2(\bar{u}) = y_2(\bar{u}) - z_{d2} \quad \text{in } Q, \\ p_1(\bar{u})\big|_\Sigma = 0, \quad p_2(\bar{u})\big|_\Sigma = 0, \\ p_1(x,T,\bar{u}) = 0, \dfrac{\partial p_1(x,T\bar{u})}{\partial t} = 0, \quad x \in \Omega, \\ p_2(x,T,\bar{u}) = 0, \dfrac{\partial p_2(x,T\bar{u})}{\partial t} = 0, \quad x \in \Omega, \end{cases} \tag{17}$$

$$y_1, y_2, \frac{\partial y_1(\bar{u})}{\partial t}, \frac{\partial y_2(\bar{u})}{\partial t} \in L^2(Q), \quad p_1, p_2, \frac{\partial p_1(\bar{u})}{\partial t}, \frac{\partial p_2(\bar{u})}{\partial t} \in L^2(Q),$$

$$u_1 = -N_1^{-1} p_1(\bar{u}), u_2 = -N_2^{-1} p_2(\bar{u}), \ \forall \bar{u} = (u_1, u_2) \in U_{ad}, \tag{18}$$

together with (16), where $p(\bar{u}) = (p_1(\bar{u}), p_2(\bar{u}))$ is the adjoint state.

2) The case if there are no constraints on u_1,

$$i.e.\ U_{ad} = \left\{\bar{u} : u_1 \text{ arbitrary in } L^2(Q), u_i \geq 0 \ a.e. \text{ in } Q, i = 2, \cdots, n\right\}, \tag{19}$$

hence, (13) takes the following form:

$$\begin{cases} p_1(\bar{u}) + N_1 u_1 = 0, \\ p_i(\bar{u}) + N_i u_i \geq 0, u_i \geq 0 \ a.e. \ i = 2, \cdots, n, \\ u_i\left(p_i(\bar{u}) + N_i u_i\right) = 0, \ \ i = 2, \cdots, n. \end{cases} \tag{20}$$

Example 2:

If we take $n = 2$,

$$\text{then } U_{ad} = \left\{\bar{u}/u_1 \text{ arbitrary in } L^2(Q), u_2 \geq 0 \ a.e. \text{ in } Q\right\}. \tag{21}$$

So, (13) is equivalent to

$$\begin{cases} p_1(\bar{u}) + N_1 u_1 = 0, \\ p_2(\bar{u}) + N_2 u_2 \geq 0, u_2 \geq 0 \ \ a.e., \\ u_2\left(p_2(\bar{u}) + N_2 u_2\right) = 0. \end{cases} \tag{22}$$

so, the optimal control is determined by:

$$\begin{cases} \dfrac{\partial^2 y_1(\bar{u})}{\partial t^2} + A y_1(\bar{u}) - a_{11} y_1(\bar{u}) - a_{12} y_2(\bar{u}) + N_1^{-1} p_1(\bar{u}) = f_1 \ \text{ in } Q, \\ \dfrac{\partial^2 y_2(\bar{u})}{\partial t^2} + A y_2(\bar{u}) - a_{21} y_1(\bar{u}) - a_{22} y_2(\bar{u}) - f_2 \geq 0 \ \text{ in } Q, \\ \dfrac{\partial^2 p_1(\bar{u})}{\partial t^2} + A p_1(\bar{u}) - a_{11} p_1(\bar{u}) - a_{12} p_2(\bar{u}) = y_1(\bar{u}) - z_{d1} \ \text{ in } Q, \\ \dfrac{\partial^2 p_2(\bar{u})}{\partial t^2} + A p_2(\bar{u}) - a_{21} p_1(\bar{u}) - a_{22} p_2(\bar{u}) = y_2(\bar{u}) - z_{d2} \ \text{ in } Q, \\ p_2(\bar{u}) + N_2\left\{\dfrac{\partial^2 y_2(\bar{u})}{\partial t^2} + A y_2(\bar{u}) - a_{21}\ y_1(\bar{u}) - a_{22} y_2(\bar{u}) - f_2\right\} \geq 0, \\ \left(\dfrac{\partial^2 y_2(\bar{u})}{\partial t^2} + A y_2(\bar{u}) - a_{21} y_1(\bar{u}) - a_{22} y_2(\bar{u}) - f_2\right) \\ \cdot\left(p_2(\bar{u}) + N_2\left(\dfrac{\partial^2 y_2(\bar{u})}{\partial t^2} + A y_2(\bar{u}) - a_{21} y_1(\bar{u}) - a_{22} y_2(u) - f_2\right)\right) = 0, \\ y_1(\bar{u})\big|_\Sigma = 0, \quad y_2(\bar{u})\big|_\Sigma = 0, \quad p_1(\bar{u})\big|_\Sigma = 0, \ \ p_2(\bar{u})\big|_\Sigma = 0, \\ y_1(x, 0; \bar{u}) = y_{0,1}(x), \dfrac{\partial y_1(x, 0; \bar{u})}{\partial t} = y_{1,1}(x), \ \ x \in \Omega, \\ y_2(x, 0; \bar{u}) = y_{0,2}(x), \dfrac{\partial y_2(x, 0; \bar{u})}{\partial t} = y_{1,2}(x), \ \ x \in \Omega, \\ p_1(x, T, \bar{u}) = p_2(x, T, \bar{u}) = 0, \ \ x \in \Omega, \\ \dfrac{\partial p_1(x, T\bar{u})}{\partial t} = \dfrac{\partial p_2(x, T\bar{u})}{\partial t} = 0, \ \ x \in \Omega. \end{cases} \tag{23}$$

Further

$$\begin{cases} u_1 = N_1^{-1} p_1(\bar{u}), \\ u_2 = \dfrac{\partial^2 y_2(\bar{u})}{\partial t^2} + A y_2(\bar{u}) - a_{21} y_1(\bar{u}) - a_{22} y_2(\bar{u}) - f_2. \end{cases} \tag{24}$$

6. Conclusion

In this paper, we have some important results. First of all we proved the existence and uniqueness of the state for system (4), which is (2 × 2) cooperative hyperbolic systems involving infinite order operators (Lemma 2). Then we found the necessary and sufficient conditions of optimality for system (10), that give the characterization of optimal control (Theorem 1).

Finally, we derived the necessary and sufficient conditions of optimality for some cases without control constraints.

Also it is evident that by modifying:

- the nature of the control (distributed, boundary),
- the nature of the observation (distributed, boundary),
- the initial differential system,
- the type of equation (elliptic, parabolic and hyperbolic),
- the type of system (non-cooperative, cooperative),
- the order of equation, many of variations on the above problem are possible to study with the help of Lions formalism.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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