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Continuity of Solution Mappings for Parametric Set Optimization Problems under Partial Order Relations

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Abstract

This paper mainly investigates the semicontinuity of solution mappings for set optimization problems under a partial order set relation instead of upper and lower set less order relations. To this end, we propose two types of monotonicity definition for the set-valued mapping introduced by two nonlinear scalarization functions which are presented by these partial order relations. Then, we give some sufficient conditions for the semicontinuity and closedness of solution mappings for parametric set optimization problems. The results presented in this paper are new and extend the main results given by some authors in the literature.

Keywords

Parametric Set Optimization Problem, Nonlinear Scalarization Function, Semicontinuity, Partial Order Relation

1. Introduction

Set-valued optimization which is a generalization of vector optimization has been studied and applied in many fields, such as engineering, mathematical finance, medicine, robust and fuzzy optimization; see [1] [2] [3] [4] and the references therein.

As we know, for set-valued optimization problems, there are two types of criteria which are vector optimization criterion and set optimization criterion. Based on the vector optimization criterion, discussing the continuity of solution set mapping for set-valued vector optimization problems is very similar to discuss vector variational inequalities or vector equilibrium problems [5]. However, this criterion is not always suitable for all types of set-valued optimization problems. To surmount the shortcoming, Kuroiwa [6] [7] introduced and discussed the set optimization criterion which is based on a comparison among the values of objective set-valued mapping. The method seems to be more natural than classical methods, whenever one needs to consider preferences over sets. Many research results have been studied for parametric set optimization problems under different kinds of set order relations, such as the optimality conditions, convexity, well-posedness, existence, duality theory and algorithms; see [8]-[15] and the references therein.

It is well-known that stability analysis plays a key role in optimization theories and related applications [16] [17] [18] [19]. Especially, the continuity of solution mappings to perturbed optimization problem is one of the important contents. The semicontinuity of solution mappings for parametric set optimization problems was firstly investigated by Xu and Li in [20]. They obtained the continuity of the weak minimal solution set mapping and the minimal solution set mapping to a parametric u-set optimization problem under strong assumption conditions. Subsequently, Xu and Li [21] studied the continuity of the minimal solution set mappings to parametric set optimization problems by the semicontinuity of the u-lower level mappings. In terms of the continuity of the level mappings, Khoshkhabar-amiranloo [16] discussed the stability of the minimal solutions of set optimization problems by using the C-Hausdorff continuity instead of the continuity in the sense of Berge. Recently, Chen *et al.* [22] used another proof method to get the semicontinuity of parametric set optimization problems with lower set less order relation under some strong assumption conditions. Through the above papers, we can find that they all need the continuity assumptions of set-valued objective mappings when they proved the semicontinuity of solution mappings for parametric set optimization problems.

Recently, Karaman [23] introduced two new order relations on family of sets called m_1 and m_2 set order relations which are partial order relations and are smaller than the vector order relations introduced in [24]. Preechasilp and Wangkeeree [25] gave some sufficient conditions for the semicontinuity of the solution mapping to a parametric set optimization problem with (converse) m_1-property assumptions under m_1 set order relation. Since it is difficult to check the (converse) m_1-property assumptions, this limits the applications of stability of set optimization problems (See Example 5.1 and 5.2). To the best of our knowledge, there are few stability results on the solution mapping to parametric set optimization problems in terms of nonlinear scalarization methods.

From what has been mentioned above, the first aim of our paper is to get the semicontinuity of solution mappings to parametric set optimization problems when the set-valued objective mapping is not continuous. The second aim of the paper is to obtain the semicontinuity of solution mappings to parametric set optimization problems via nonlinear scalarization methods. Hence, we define two types of new monotonicity of the set-valued mapping by using two nonlinear scalarization functions under partial order relation. The third aim of this paper is to investigate the semicontinuity of solution mapping of parametric set optimization problems under weaker and simpler assumptions. We give some sufficient conditions for the semicontinuity and closedness of solution mappings for parametric set optimization problems. In our results, we have no (converse) m_1-property assumptions for objective mappings and the assumption (iv) of Theorem 4.2 in [21] and [26]. Moreover, the continuity of the objective mapping is replaced by some monotonicity assumptions. So, our results do not need any additional assumptions to get the lower semicontinuity of solution mapping of parametric set optimization problems. Our results extend and improve the corresponding ones of [21] [25] [26].

The rest of this paper is organized as follows. In Section 2, some basic concepts and preliminary results about the semicontinuity of set-valued mappings are introduced. In Section 3, by using the nonlinear scalarization functions, two types of new monotonicity of the set-valued mapping are defined. In Section 4, the semicontinuity and closedness of the solution mappings for the parametric set optimization problems are discussed without any continuities of the set-valued objective mapping. In Section 5, we give the comparisons between our results in Section 4 and ones in [16] [21] [22] [25] [26]. In Section 6, we give some concluding remarks.

2. Preliminaries

Throughout this paper, let *X*, *Y* and *Z* be real normed linear spaces, and let Λ be a nonempty subset of *Z*, $C \subseteq Y$ and $K \subseteq X$ are closed, convex and pointed cones. Let $\wp_0(Y)$ be the family of all nonempty subsets of *Y*, and $\wp_0^*(Y)$ be the family of all nonempty bounded subsets of *Y*. Let X^* be the topological dual space of *X* and K^* be the dual cone of *K*, defined by

$$K^* := \{k \in X^* : k(x) \geq 0, \forall x \in K\}.$$

Denote the quasi-interior of K^* by $K^\#$, *i.e.*,

$$K^\# := \{k \in X^* : k(x) > 0, \forall x \in K \setminus \{0\}\}.$$

Proposition 2.1. [24]

(i) If *K* is a closed convex cone in a real locally convex linear space *X*, then $K = \{x \in K : k(x) \geq 0 \text{ for all } k \in K^*\}$.

(ii) Let $k \in K^\#$. If $k(z) \geq 0$, then $z \notin -K \setminus \{0\}$.

We now recall some order relations on $\wp_0(Y)$. The first one is lower set less ($\preceq_C^l$) and upper set less ($\preceq_C^u$) order relations on $\wp_0(Y)$.

Definition 2.1. [24] Let $A, B \in \wp_0(Y)$ be arbitrarily closen sets.

(i) The upper set less order relations ($\preceq_C^u$) is defined by $A \preceq_C^u B \Leftrightarrow A \subseteq B - C$,

(ii) The lower set less order relations ($\preceq_C^l$) is defined by $A \preceq_C^l B \Leftrightarrow B \subseteq A + C$.

($\preceq_C^u$) and ($\preceq_C^l$) order relations have been extensively studied in many literature. Recently, by using properties of Minkowski difference, Karaman *et al.* established the following new order relations on a family of sets.

Definition 2.2. [23] Let $A, B, C \in \wp_0(Y)$.

$$A \preceq_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap C \neq \emptyset,$$

$$A \preceq_C^{m_2} B \Leftrightarrow (A \dot{-} B) \cap (-C) \neq \emptyset.$$

The set $B \dot{-} A = \{x \in X : x + A \subseteq B\} = \bigcap_{a \in A} (B - a)$ is called Minkowski (Pontryagin) difference of B and A.

Moreover, when $\operatorname{int} C \neq \emptyset$, the strictly version of $A \preceq_C^{m_1} B$ and $A \preceq_C^{m_2} B$ are defined by

$$A \prec_C^{m_1} B \Leftrightarrow (B \dot{-} A) \cap \operatorname{int} C \neq \emptyset,$$

$$A \prec_C^{m_2} B \Leftrightarrow (A \dot{-} B) \cap \operatorname{int}(-C) \neq \emptyset.$$

Remark 2.1. It is easily seen that, for $A, B \in \wp_0(Y)$, $A \preceq_C^{m_1} B$ implies $A \preceq_C^{u} B$ and $A \preceq_C^{m_2} B$ implies $A \preceq_C^{l} B$. However, the converse is not true. The detailed counter-example sees in [23].

Let M be a nonempty set of X and $F : M \rightrightarrows Y$ be a set-valued mapping. The set optimization problem is defined as follows:

$$(\text{SOP}) \quad \min F(x) \text{ subject to } x \in M.$$

Definition 2.3. [23] An element $x_0 \in M$ is said to be

(i) a m_1-minimal solution of (SOP) if there does not exist any $x \in M$ with $F(x) \neq F(x_0)$ such that $F(x) \preceq_C^{m_1} F(x_0)$, that is, either $F(x) \npreceq_C^{m_1} F(x_0)$ or $F(x) = F(x_0)$ for any $x \in M$;

(ii) a m_2-minimal solution of (SOP) if there does not exist any $x \in M$ with $F(x) \neq F(x_0)$ such that $F(x) \preceq_C^{m_2} F(x_0)$, that is, either $F(x) \npreceq_C^{m_2} F(x_0)$ or $F(x) = F(x_0)$ for any $x \in M$;

If the set M and the mapping F are perturbed by a parameter λ which varies over a set $\Lambda \subseteq Z$, we definite the following parametric set optimization problem (PSOP):

$$(\text{PSOP}) \quad \min F(x, \lambda) \text{ subject to } x \in M(\lambda),$$

where $F : B \times \Lambda \subseteq X \times Z \rightrightarrows Y$, $M : \Lambda \rightrightarrows X$, F and M are set-valued mappings, $M(\Lambda) = \bigcup_{\lambda \in \Lambda} M(\lambda) \subseteq B$. For each $\lambda \in \Lambda$, we denote $S_{m_1}(\lambda)$ and $S_{m_2}(\lambda)$ by the m_1-minimal solution mapping and m_2-minimal solution mapping to (PSOP), respectively. That is, $S_{m_1}(S_{m_2}) : \Lambda \rightrightarrows X$ is defined as follows:

$$S_{m_1}(\lambda) := \{\bar{x} \in M(\lambda) : F(x, \lambda) \npreceq_C^{m_1} F(\bar{x}, \lambda) \text{ for all } x \in M(\lambda) \setminus \{\bar{x}\} \text{ and } F(x, \lambda) \neq F(\bar{x}, \lambda)\};$$

$$S_{m_2}(\lambda) := \{\bar{x} \in M(\lambda) : F(x, \lambda) \npreceq_C^{m_2} F(\bar{x}, \lambda) \text{ for all } x \in M(\lambda) \setminus \{\bar{x}\} \text{ and } F(x, \lambda) \neq F(\bar{x}, \lambda)\}.$$

In this paper, we assume that $S_{m_i}(\lambda) \neq \emptyset$ for each $\lambda \in \Lambda$, $i = \{1, 2\}$. Next, we will discuss the upper and lower semicontinuities of $S_{m_i}(\cdot)$ at a point $\lambda_0 \in \Lambda$. Now, let us recall some basic definitions and their properties.

Definition 2.4. [27] Suppose that $\varphi : \Lambda \rightrightarrows X$ is a set-valued mapping, and $\bar{\lambda} \in \Lambda$ is given.

(i) φ is called lower semicontinuous (l.s.c) at $\bar{\lambda}$ iff for any open set $V \subseteq X$ with $V \cap \varphi(\bar{\lambda}) \neq \varnothing$, there exists a neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$ such that $\varphi(\lambda) \cap V \neq \varnothing$, for all $\lambda \in N(\bar{\lambda})$.

(ii) φ is called upper semicontinuous (u.s.c) at $\bar{\lambda}$ iff for any open set $V \subseteq X$ with $\varphi(\bar{\lambda}) \subseteq V$, there exists a neighborhood $N(\bar{\lambda})$ of $\bar{\lambda}$ such that $\varphi(\lambda) \subseteq V$, for all $\lambda \in N(\bar{\lambda})$.

(iii) φ is called closed at $\bar{\lambda}$ iff for each sequence $(\lambda_n, x_n) \in \text{graph}\,\varphi := \{(\lambda, x) : x \in \varphi(\lambda)\}$, $(\lambda_n, x_n) \to (\bar{\lambda}, \bar{x})$, it follows that $(\bar{\lambda}, \bar{x}) \in \text{graph}\,\varphi$.

$\varphi(\cdot)$ is said to be l.s.c (resp. u.s.c) on Λ, if and only if it is l.s.c (resp. u.s.c) at each $\bar{\lambda} \in \Lambda$. $\varphi(\cdot)$ is said to be continuous on Λ if and only if it is both l.s.c and u.s.c on Λ.

Proposition 2.2. [27] [28]

(i) φ is l.s.c at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \to \bar{\lambda}$ and any $\bar{x} \in \varphi(\bar{\lambda})$, there exists $x_n \in \varphi(\lambda_n)$ such that $x_n \to \bar{x}$.

(ii) If φ has compact values at $\bar{\lambda}$ (*i.e.*, $\varphi(\bar{\lambda})$ is a compact set), then φ is u.s.c at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subseteq \Lambda$ with $\lambda_n \to \bar{\lambda}$ and any $x_n \in \varphi(\lambda_n)$, there exist $\bar{x} \in \varphi(\bar{\lambda})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \to \bar{x}$.

3. Monotonicity via Scalarization

In this section, new monotonicity definitions will be introduced by the scalarization method. When we want to obtain the semicontinuity results of solution mappings of (SOP), we always use some assumptions that include the continuity of the objective mapping F (see [16] [21] [22]). In this paper, we will get the semicontinuity results of solution mappings of (PSOP) without the continuity of the objective mapping F by using a new monotonicity condition. Firstly, we recall two nonlinear scalarization functions and some properties of these functions.

Definition 3.1. [23] Let the functions $I_e^{m_1} : \wp_0(Y) \times \wp_0(Y) \to \mathbb{R}$ and $I_e^{m_2} : \wp_0(Y) \times \wp_0(Y) \to \mathbb{R}$ as

$$I_e^{m_1}(A, B) := \inf\{t \in \mathbb{R} : A \preceq_C^{m_1} te + B\},$$

$$I_e^{m_2}(A, B) := \inf\{t \in \mathbb{R} : A \preceq_C^{m_2} te + B\}.$$

for all $A, B \in \wp_0(Y)$, where $e \in \text{int}\, C$.

Proposition 3.1. [23] Let $A, B \in \wp_0(Y)$ and $I_e^{m_1}(A, B)$ be finite. Then, the following conditions are satisfied:

(i) when $B \dot{-} A$ is compact and C is closed,

$$A \preceq_C^{m_1} B \Leftrightarrow I_e^{m_1}(A, B) \leq 0,$$

(ii) $A \prec_C^{m_1} B \Leftrightarrow I_e^{m_1}(A, B) < 0.$

Next, two types of new monotonicity of the set-valued mapping will be defined by the above nonlinear scalarization functions.

Definition 3.2. Let $\Omega \subseteq X, e \in \text{int}\, C$. A set-valued mapping $G : \Omega \rightrightarrows Y$ is said to be

(i) m_1-monotonically increasing on Ω iff $\forall k \in K^* \setminus \{0_{X^*}\}$, one has that

$$\left\langle I_e^{m_1}\left(G(y), G(x)\right), k(y)-k(x)\right\rangle > 0, \forall x, y \in \Omega \text{ with } x \neq y.$$

(ii) m_2-monotonically increasing on Ω iff $\forall k \in K^* \setminus \{0_{X^*}\}$, one has that

$$\left\langle I_e^{m_2}\left(G(y), G(x)\right), k(y)-k(x)\right\rangle > 0, \forall x, y \in \Omega \text{ with } x \neq y.$$

Remark 3.1. If G is a single-valued mapping and $X = Y = \mathbb{R}$, $K = C = \mathbb{R}_+$. It is easy to see that $K^* = \mathbb{R}_+$. Therefore, k and e are all positive real numbers. Obviously, $k(y) = ky$ and $k(x) = kx$, thus

$$\begin{aligned} I_e^{m_1}\left(G(y), G(x)\right) &= \inf\left\{t \in \mathbb{R} : G(y) \preceq_C^{m_1} te + G(x)\right\} \\ &= \inf\left\{t \in \mathbb{R} : \left(G(x) + te \dot{-} G(y)\right) \cap C \neq \varnothing\right\} \\ &= \inf\left\{t \in \mathbb{R} : te \geq G(y) - G(x)\right\} \\ &= \frac{G(y)-G(x)}{e}. \end{aligned}$$

Then,

$$\left\langle \frac{G(y)-G(x)}{e}, ky - kx \right\rangle \geq 0, \forall x, y \in \Omega.$$

Noting that $k > 0$ and $e > 0$, we have that

$$\left\langle G(y)-G(x), y-x \right\rangle \geq 0, \forall x, y \in \Omega.$$

We can see that G is reduced to the well-known monotonically increasing function.

Definition 3.3. Let $\Omega \subseteq X, e \in \operatorname{int} C$. A set-valued mapping $G : \Omega \rightrightarrows Y$ is said to be

(i) m_1-monotonically decreasing on Ω iff $\forall k \in K^* \setminus \{0_{X^*}\}$, one has that

$$\left\langle I_e^{m_1}\left(G(y), G(x)\right), k(y)-k(x)\right\rangle \leq 0, \forall x, y \in \Omega.$$

(ii) m_2-monotonically decreasing on Ω iff $\forall k \in K^* \setminus \{0_{X^*}\}$, one has that

$$\left\langle I_e^{m_2}\left(G(y), G(x)\right), k(y)-k(x)\right\rangle \leq 0, \forall x, y \in \Omega.$$

Similarly, if all the conditions in Remark 2 hold, we also can get that

$$\left\langle G(y)-G(x), y-x \right\rangle \leq 0, \forall x, y \in \Omega.$$

This shows that G is reduced to the well-known monotonically decreasing function. Next, we give an example to illustrate Definitions 3.2 and 3.3.

Example 3.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Omega = \{1, -1\}$, $K = \mathbb{R}_+$ and $C = \mathbb{R}_+^2$. Let $e = (1,1)$. A set-valued mapping $G : \Omega \rightrightarrows Y$ is defined by

$$G(x) = \begin{cases} \left\{y = (y_1, y_2)^{\mathrm{T}} : (y_1 - 1)^2 + (y_2 - 1)^2 \leq 1\right\}, & x = 1, \\ \left\{y = (y_1, y_2)^{\mathrm{T}} : (y_1 - a)^2 + (y_2 - a)^2 \leq 1, a \geq 1\right\}, & x = -1. \end{cases}$$

It follows from a direct computation that

$$I_e^{m_1}\left(G(1),G(-1)\right)=\inf\left\{t\in\mathbb{R}:(t+a-1,t+a-1)\cap C\neq\varnothing\right\}=1-a\leq 0$$

and $I_e^{m_1}\left(G(-1),G(1)\right)=a-1\geq 0$. For every $k\in K^*=\mathbb{R}_+$, we have that $k(x)=kx$. Then

$$\left\langle I_e^{m_1}\left(G(1),G(-1)\right),k(1)-k(-1)\right\rangle\leq 0$$

and

$$\left\langle I_e^{m_1}\left(G(-1),G(1)\right),k(-1)-k(1)\right\rangle\leq 0.$$

Hence, G is m_1-monotonic decreasing on Ω. When $a\leq 1$, we can find that G is m_1-monotonic increasing on Ω.

4. Semicontinuity of Solution Set Mapping

In this section, we shall discuss the upper and lower semicontinuities of the m_1-minimal solution set mappings $S_{m_i}(\cdot)$ ($i=1,2$) of (PSOP) at $\lambda_0\in\Lambda$.

Lemma 4.1. *Assume that* $F(\cdot,\lambda):X\rightrightarrows Y$ *is* m_1*-monotonically decreasing and bounded with compact values on* $M(\lambda)$ *for each* $\lambda\in\Lambda$. *Then,*

$$y-x\in K\Leftrightarrow F(y,\lambda)\preceq_C^{m_1}F(x,\lambda),\ \forall x,y\in M(\lambda).$$

Proof. "Necessity". It follows from the linearity of k, Proposition 2.1 (i) and $y-x\in K$ that $k(y)\geq k(x)$ for all $k\in K^*$. Since $F(\cdot,\lambda)$ is m_1-monotonically decreasing on $M(\lambda)$ for each $\lambda\in\Lambda$, we have that $I_e^{m_1}\left(F(y,\lambda),F(x,\lambda)\right)\leq 0$. This together with the compactness of $F(x,\lambda)$ and Proposition 3.1(i) leads to $F(y,\lambda)\preceq_C^{m_1}F(x,\lambda)$.

"Sufficiency". By $F(y,\lambda)\preceq_C^{m_1}F(x,\lambda)$ and Proposition 3.1 (i), we get that $I_e^{m_1}\left(F(y,\lambda),F(x,\lambda)\right)\leq 0$. Since $F(\cdot,\lambda)$ is m_1-monotonically decreasing on $M(\lambda)$ for each $\lambda\in\Lambda$, we have that $k(y)\geq k(x)$ for all $k\in K^*$. It follows from the linearity of $k\in K^*$ and Proposition 2.1 (i) that $y-x\in K$. □

Lemma 4.2. *Assume that* $F(\cdot,\lambda):X\rightrightarrows Y$ *is* m_1*-monotonically increasing with compact values on* $M(\lambda)$ *for each* $\lambda\in\Lambda$. *Then, for any* $x,y\in M(\lambda)$, *one has*

$$F(y,\lambda)\preceq_C^{m_1}F(x,\lambda)\Rightarrow y-x\notin K\setminus\{0\}.$$

Proof. Since $F(y,\lambda)\preceq_C^{m_1}F(x,\lambda)$ and $F(\cdot,\lambda):X\rightrightarrows Y$ is compact valued on $M(\lambda)$ for each $\lambda\in\Lambda$, we get that $I_e^{m_1}\left(F(y,\lambda),F(x,\lambda)\right)\leq 0$ by Proposition 3.1 (i). Besides $F(\cdot,\lambda)$ is m_1-monotonicaly increasing on $M(\lambda)$, so we have that $k(y)\leq k(x)$ for all $k\in K^{\#}$. It follows from Proposition 2.1(ii) that $y-x\notin K\setminus\{0\}$. □

Lemma 4.3. *Assume that* $F(\cdot,\lambda):X\rightrightarrows Y$ *is* m_1*-monotonically increasing with compact values on* $M(\lambda)$ *for each* $\lambda\in\Lambda$. *Then, for any* $x,y\in M(\lambda)$, *one has*

$$y-x\notin K\Rightarrow F(y,\lambda)\preceq_C^{m_1}F(x,\lambda).$$

Proof. It follows from the linearity of k, Proposition 2.1 (i) and $y-x\notin K$ that $k(y)<k(x)$ for some $k\in K^*$. Since $F(\cdot,\lambda)$ is m_1-monotonically in-

creasing on $M(\lambda)$ for each $\lambda \in \Lambda$, we have that $I_e^{m_1}(F(y,\lambda),F(x,\lambda)) \le 0$, which implies that $F(y,\lambda) \preceq_C^{m_1} F(x,\lambda)$ by the compact values of $F(x,\lambda)$ and Proposition 3.1 (i). □

Theorem 4.1. *Let* $\lambda_0 \in \Lambda$. *Suppose that the following conditions are satisfied:*

(i) M is continuous at λ_0 and $M(\lambda_0)$ is a compact convex set;

(ii) $F(\cdot,\lambda): X \rightrightarrows Y$ is m_1-monotonically increasing with compact values on $M(\lambda)$ for each $\lambda \in \Lambda$.

Then, $S_{m_1}(\cdot)$ is u.s.c and closed at λ_0.

Proof. Suppose that $S_{m_1}(\cdot)$ is not u.s.c at λ_0. Then there exists a neighborhood W_0 of $S_{m_1}(\lambda_0)$, and for any neighborhood V of λ_0, there exists $\lambda' \in V$ such that $S_{m_1}(\lambda') \not\subseteq W_0$. Thus, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that $S_{m_1}(\lambda_n) \not\subseteq W_0$, $\forall n \in \mathbb{N}$. This implies that there exists $x_n \in S_{m_1}(\lambda_n)$ such that

$$x_n \notin W_0,\ \forall n \in \mathbb{N}. \tag{1}$$

It follows from $x_n \in S_{m_1}(\lambda_n)$ that $x_n \in M(\lambda_n)$. Since M is u.s.c with compact values at λ_0 and by Proposition 2.2 (ii), there exist $x_0 \in M(\lambda_0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. Without any loss of generality, we assume that $x_n \to x_0$.

Now, we prove that $x_0 \in S_{m_1}(\lambda_0)$. Suppose that $x_0 \notin S_{m_1}(\lambda_0)$, then there exists $y_0 \in M(\lambda_0)$ with $F(x_0,\lambda_0) \ne F(y_0,\lambda_0)$ such that

$$F(y_0,\lambda_0) \preceq_C^{m_1} F(x_0,\lambda_0). \tag{2}$$

As M is l.s.c at λ_0 and by Proposition 2.2 (i), there exists $y_n \in M(\lambda_n)$ such that $y_n \to y_0$. It follows from (2) and Lemma 4.2 that

$$y_0 - x_0 \notin K \setminus \{0\},$$

that is, $y_0 - x_0 \in (X \setminus K) \cup \{0\}$. Noting that $x_n \to x_0$ and $y_n \to y_0$, we get that $y_n - x_n \to y_0 - x_0$. Since $y_0 \ne x_0$, we have that $y_0 - x_0 \in X \setminus K$. It follows from the closedness of K that $y_n - x_n \notin K$ for n large enough. By Lemma 4.3, we can get that

$$F(y_n,\lambda_n) \preceq_C^{m_1} F(x_n,\lambda_n). \tag{3}$$

On the other hand, from $y_n - x_n \notin K$, we have that there exists $k_0 \in K^* \setminus \{0\}$ such that $k_0(x_n) > k_0(y_n)$ by Proposition 2.1 (i). Since $F(\cdot,\lambda)$ is strictly m_1-monotonicaly increasing on $M(\lambda)$ for each $\lambda \in \Lambda$, we have $I_e^{m_1}(F(x_n,\lambda_n),F(y_n,\lambda_n)) \le 0$ and so $F(x_n,\lambda_n) \not\preceq_C^{m_1} F(y_n,\lambda_n)$ by Proposition 3.1 (i). This implies that $F(x_n) \ne F(y_n)$. Combining with (3), we have $x_n \notin S_{m_1}(\lambda_n)$, which contradicts $x_n \in S_{m_1}(\lambda_n)$. Therefore, $x_0 \in S_{m_1}(\lambda_0)$. It follows form $S_{m_1}(\lambda_0) \subseteq W_0$ that $x_0 \in W_0$. Noting that $x_n \to x_0$, we have $x_n \in W_0$ for n large enough, which contradicts with (1). Therefore, $S_{m_1}(\cdot)$ is u.s.c at λ_0.

Next, we show that S_{m_1} is closed at λ_0. Take $x_n \in S_{m_1}(\lambda_n)$ with $x_n \to x_0$ and $\lambda_n \to \lambda_0$. Since $x_n \in M(\lambda_n)$ and M is u.s.c with compact values at λ_0, we obtain that $x_0 \in M(\lambda_0)$. Then by virtue of the same proof as above, we get that $x_0 \in S_{m_1}(\lambda_0)$. Furthermore, $S_{m_1}(\lambda_0)$ is compact by assumption (i) and

$S_{m_1}(\lambda_0) \subseteq M(\lambda_0)$. □

Now, we give an example to illustrate Theorem 4.1.

Example 4.1 Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = [0,1]$, $M(\Lambda) = [0,1]$ for each $\lambda \in \Lambda$, $K = \mathbb{R}_+$ and $C = \mathbb{R}^2_+$. Let $\lambda_0 = 0$, $e = (1,1)$ and set-valued mapping $F : M(\Lambda) \times \Lambda \rightrightarrows Y$ as follows:

$$F(x,\lambda) = \begin{cases} \left\{ y = (y_1, y_2)^{\mathrm{T}} : (y_1 - 1)^2 + (y_2 - 1)^2 \le 1 \right\}, & \text{if } \lambda \ne 0, x \in (0,1], \\ \left\{ y = (y_1, y_2)^{\mathrm{T}} : (y_1 - \lambda)^2 + (y_2 - \lambda)^2 \le 1 \right\}, & \text{if } \lambda \ne 0, x = 0, \\ [0,1] \times [0,1], & \text{if } \lambda = 0, x \in [0,1]. \end{cases}$$

It is easy to see that the assumptions in Theorem 4.1 are satisfied. It follows from a direct computation that $S_{m_1}(\lambda_0) = [0,1]$ and $S_{m_1}(\lambda) = \{0\}$ when $\lambda \ne 0$. Hence, $S_{m_1}(\cdot)$ is u.s.c and closed at λ_0.

Theorem 4.2. *Let* $\lambda_0 \in \Lambda$. *Suppose that the following conditions are satisfied*:

(i) M is continuous at λ_0 and $M(\lambda_0)$ is a compact convex set;

(ii) $F(\cdot,\lambda) : X \rightrightarrows Y$ is m_1-monotonic decreasing and bounded with compact values on $M(\lambda)$ for each $\lambda \in \Lambda$.

Then, $S_{m_1}(\cdot)$ is l.s.c at λ_0.

Proof. Suppose that $S_{m_1}(\cdot)$ is not l.s.c at λ_0. Then there exist $x_0 \in S_{m_1}(\lambda_0)$ and a neighborhood W_0 of x_0, for any neighborhood U of λ_0, there exists $\lambda' \in U$ such that $W_0 \cap S(\lambda') = \varnothing$. Hence, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0$ such that

$$W_0 \cap S_{m_1}(\lambda_n) = \varnothing. \tag{4}$$

Since $x_0 \in S_{m_1}(\lambda_0) \subseteq M(\lambda_0)$ and M is l.s.c at λ_0, there exists $x_n \in M(\lambda_n)$ such that $x_n \to x_0$. Therefore, $x_n \in W_0$ for n large enough.

Now, we prove that $x_n \in S_{m_1}(\lambda_n)$. Suppose that $x_n \notin S_{m_1}(\lambda_n)$, then there exists $y_n \in M(\lambda_n)$ with $F(x_n,\lambda_n) \ne F(y_n,\lambda_n)$ such that

$$F(y_n,\lambda_n) \preceq_C^{m_1} F(x_n,\lambda_n). \tag{5}$$

Since M is u.s.c with compact values at λ_0, there exists $y_0 \in M(\lambda_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$. Without any loss of generality, we assume that $y_n \to y_0$. It follows from (5) and Lemma 4.1 that

$$y_n - x_n \in K.$$

Noting that $x_n \to x_0$, we get that $y_n - x_n = y_n - y_0 + y_0 - x_0 + x_0 - x_n \to y_0 - x_0$. By the closedness of K, we have that $y_0 - x_0 \in K$. Again from Lemma 4.1, then we get that

$$F(y_0,\lambda_0) \preceq_C^{m_1} F(x_0,\lambda_0). \tag{6}$$

Noting that $x_0 \in S_{m_1}(\lambda_0)$, then

$$F(y_0,\lambda_0) \not\preceq_C^{m_1} F(x_0,\lambda_0) \text{ for all } y_0 \in M(\lambda_0) \setminus \{y_0\} \text{ and } F(x_0,\lambda_0) \ne F(y_0,\lambda_0)$$

which contradicts with (6). Hence, $x_n \in S_{m_1}(\lambda_n) \cap W_0$ for n large enough. This contradicts with (4). Therefore, $S_{m_1}(\cdot)$ is l.s.c at λ_0. □

Now, we give an example to illustrate Theorem 4.2.

Example 4.2. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = [0,1]$, $M(\Lambda) = [0,1]$ for each $\lambda \in \Lambda$, $K = \mathbb{R}_+$ and $C = \mathbb{R}_+^2$. Let $\lambda_0 = 0$, $e = (1,1)$ and set-valued mapping $F : M(\Lambda) \times \Lambda \rightrightarrows Y$ as follows:

$$F(x,\lambda) = \begin{cases} \left\{ y = (y_1, y_2)^{\mathrm{T}} : y_1^2 + y_2^2 \le 1 \right\}, & \text{if } \lambda \in [0,1], x \in (0,1], \\ \left\{ (-1,-1)^{\mathrm{T}} \right\}, & \text{if } \lambda \in [0,1], x = 0. \end{cases}$$

It is easy to see that $F(x,\lambda)$ is m_1-monotonic decreasing on $M(\lambda)$. It follows from a direct computation that $S_{m_1}(\lambda) = \{0\}$ for each $\lambda \in [0,1]$. Hence, $S_{m_1}(\cdot)$ is l.s.c at λ_0.

Based on the similar technique as perviously shown, we also can get the following Corollaries.

Corollary 4.1. *Let* $\lambda_0 \in \Lambda$. *Suppose that the following conditions are satisfied*:

(i) M is continuous at λ_0 and $M(\lambda_0)$ is a compact convex set;

(ii) $F(\cdot,\lambda) : X \rightrightarrows Y$ is m_2-monotonic increasing and bounded with compact values on $M(\lambda)$ for each $\lambda \in \Lambda$.

Then, $S_{m_1}(\cdot)$ is u.s.c and closed at λ_0.

Corollary 4.2 *Let* $\lambda_0 \in \Lambda$. *Suppose that the following conditions are satisfied*:

(i) M is continuous at λ_0 and $M(\lambda_0)$ is a compact convex set;

(ii) $F(\cdot,\lambda) : X \rightrightarrows Y$ is m_2-monotonic decreasing and bounded with compact values on $M(\lambda)$ for each $\lambda \in \Lambda$.

Then, $S_{m_2}(\cdot)$ is l.s.c at λ_0.

5. Discussion with Other References

In this section, we shall give the comparisons between the results (Theorem 4.1 and Theorem 4.2) in Section 4 and ones in [16] [21] [22] [25] [26].

Remark 5.1. The proof of Theorem 4.1 is different from one of Theorem 3.4 of Preechasilp and Wangkeeree [25]. Moreover, in Theorem 4.1, we do not use converse m_1-property for set-valued objective mapping F, which is applied in Theorem 3.4 of Preechasilp and Wangkeeree [25]. And the order relations we used are smaller than the set less order relations in [16] [21] [22] [26].

The following example is given to show that the converse m_1-property in [25] is unnecessary in Theorem 4.1 for this paper.

Example 5.1. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $\Lambda = [0,1]$, $M(\Lambda) = [0,1]$ for each $\lambda \in \Lambda$, $K = \mathbb{R}_+$ and $C = \mathbb{R}_+^2$. Let $\lambda_0 = 0$, $e = (1,1)$ and set-valued mapping $F : M(\Lambda) \times \Lambda \rightrightarrows Y$ as follows:

$$F(x,\lambda) = \begin{cases} \left(\lambda(0,1+x)\right) \times (0,2+x), & \text{if } \lambda \ne 0, x \in [0,1], \\ \left\{ y = (y_1, y_2)^{\mathrm{T}} : y_1^2 + y_2^2 \le 1 \right\}, & \text{if } \lambda \ne 0, x \in (0,1], \\ \left\{ (0,0)^{\mathrm{T}} \right\}, & \text{if } \lambda = 0, x = 0. \end{cases}$$

It is easy to see that the assumptions in Theorem 4.1 are satisfied. It follows from a direct computation that $S_{m_1}(\lambda) = \{0\}$ for each $\lambda \in \Lambda$. Hence, $S_{m_1}(\cdot)$ is u.s.c at λ_0. Besides, F does not satisfy the converse m_1-property at $(x_0,\lambda_0)=(0,1)$ with respect to $y_0=0$. Indeed, we obtain that $F(y_0,\lambda_0)\preceq_C^{m_1} F(x_0,\lambda_0)$. However, for any sequences $\{x_n\}\subseteq\left(\frac{1}{2},1\right)$ with $x_n\to x_0$, $\{y_n\}\subseteq\left(0,\frac{1}{2}\right)$ with $y_n\to y_0$ and $\{\lambda_n\}\subseteq\left(0,\frac{1}{2}\right)$ with $\lambda_n\to\lambda_0$, one has

$$F(y_n,\lambda_n)\npreceq_C^{m_1} F(x_n,\lambda_n),\ \forall n.$$

Remark 5.2. It is worth noting that the proof of Theorem 4.2 is very different from one in Theorem 3.4 of Preechasilp and Wangkeeree [25]. In Theorem 4.2, we do not use m_1-property for set-valued objective mapping F, which is applied in Theorem 3.8 of Preechasilp and Wangkeeree [25]. Moreover, we do not use the assumption (iv) in Theorem 4.2 of Xu and Li [21] and Theorem 4.2 of Liu and Wei [26] either. And the order relations we used are smaller than the set less order relations in [16] [21] [22] [26].

The following example is given to show that the m_1-property in [25] and the assumption (iv) in Theorem 4.2 of [21] and Theorem 4.2 of [26] are dispensable in some cases.

Example 5.2. Let $X=Z=\mathbb{R}$, $Y=\mathbb{R}^2$, $\Lambda=[0,1]$, $M(\Lambda)=[0,1]$ for each $\lambda\in\Lambda$, $K=\mathbb{R}_+$ and $C=\mathbb{R}_+^2$. Let $\lambda_0=0$, $e=(1,1)$ and set-valued mapping $F:M(\Lambda)\times\Lambda\rightrightarrows Y$ as follows:

$$F(x,\lambda)=\begin{cases}\lambda[0,2+x]\times[0,1], & \text{if } \lambda\neq 0, x\in[0,1],\\ \left\{y=(y_1,y_2)^{\mathrm{T}}:(y_1+1)^2+(y_2+1)^2\le 1\right\}, & \text{if } \lambda=0, x\in(0,1],\\ \left\{y=(y_1,y_2)^{\mathrm{T}}:y_1^2+y_2^2\le 1\right\}, & \text{if } \lambda=0, x=0.\end{cases}$$

It is easy to see that the assumptions in Theorem 4.2 are satisfied. It follows from Example 4.5 of [26] that the assumption (iv) in Theorem 4.2 of [21] and Theorem 4.2 of [26] does not hold. Besides, F does not satisfy the m_1-property at $(x_0,\lambda_0)=(0,1)$ with respect to $y_0=0$. Indeed, we obtain that $F(y_0,\lambda_0)\npreceq_C^{m_1} F(x_0,\lambda_0)$. However, for any sequences $\{x_n\}\subseteq\left(\frac{1}{2},1\right)$ with $x_n\to x_0$, $\{y_n\}\subseteq\left(0,\frac{1}{2}\right)$ with $y_n\to y_0$ and $\{\lambda_n\}\subseteq\left(0,\frac{1}{2}\right)$ with $\lambda_n\to\lambda_0$, one has

$$F(y_n,\lambda_n)=\lambda_n[0,2+y_n]\times[0,1]\preceq_C^{m_1}\lambda_n[0,2+x_n]\times[0,1]=F(x_n,\lambda_n),\ \forall n.$$

By a direct computation that $S_{m_1}(\lambda)=\{0\}$ for each $\lambda\in[0,1]$. Hence, $S_{m_1}(\cdot)$ is l.s.c at λ_0.

6. Conclusions

By using two scalarizing functions which are proposed by partial order relations, some new monotonicity concepts of set-valued mapping are introduced in this paper. The continuity of the solution mapping for a parametric optimization problem is obtained under these kinds of monotonicity concepts and some suitable assumptions which do not need the continuity of the set-valued objective mapping. Especially, when we give the lower semicontinuity of the solution mapping, we do not use assumptions presented in [21] [25] [26]. And the order relations we used are smaller than the set less order relations in [16] [21] [22] [26].

The findings of the paper are summarized as follows.

(i) We put forward some new monotonicity concepts of set-valued mapping.

(ii) Under new assumptions which are different from ones presented in [21] [25] [26], we give the lower semicontinuity and the upper semicontinuity of the solution mapping of the problem (PSOP) with partial order relations.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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