Advances in Pure Mathematics, 2020, 10, 611-622
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Super Congruences Involving Alternating Harmonic Sums

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How to cite this paper: Shen, Z.Y. and Cai, T.X. (2020) Super Congruences Involving Alternating Harmonic Sums. *Advances in Pure Mathematics*, **10**, 611-622.
https://doi.org/10.4236/apm.2020.1010037

Received: September 23, 2020
Accepted: October 26, 2020
Published: October 29, 2020

Abstract

Let p be an odd prime, the harmonic congruence such as

$$\sum_{i+j+k=p}\frac{1}{ijk}\equiv -2B_{p-3}\pmod p,$$

and many different variations and generalizations have been studied intensively. In this note, we consider the congruences involving the combination of alternating harmonic sums,

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2\cdots i_n},\ \sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2\cdots i_n},\cdots,\ \sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2+\cdots+i_{\left[\frac{n}{2}\right]}}}{i_1i_2\cdots i_n}.$$

where $\mathcal{P}_p$ denotes the set of positive integers which are prime to *p*. And we establish the combinational congruences involving alternating harmonic sums for positive integer $n=3,4,5$.

Keywords

Bernoulli Numbers, Alternating Harmonic Sums, Congruences, Modulo Prime Powers

1. Introduction

At the beginning of the 21th century, Zhao (Cf. [1]) first announced the following curious congruence involving multiple harmonic sums for any odd prime $p>3$,

$$\sum_{i+j+k=p}\frac{1}{ijk}\equiv -2B_{p-3}\pmod p, \tag{1}$$

which holds when $p=3$ evidently. Here, Bernoulli numbers B_k are defined by the recursive relation:

$$\sum_{i=0}^{n}\binom{n+1}{i}B_i=0, n\geq 1.$$

A simple proof of (1) was presented in [2]. This congruence has been generalized along several directions. First, Zhou and Cai [3] established the following harmonic congruence for prime $p>3$ and integer $n\leq p-2$

$$\sum_{l_1+l_2+\cdots+l_n=p}\frac{1}{l_1l_2\cdots l_n}\equiv\begin{cases}-(n-1)!B_{p-n}\pmod{p}, & \text{if } 2\nmid n,\\ -\dfrac{n(n!)}{2(n+1)}pB_{p-n-1}\pmod{p^2}, & \text{if } 2\mid n.\end{cases} \tag{2}$$

Later, Xia and Cai [4] generalized (1) to

$$\sum_{i+j+k=p}\frac{1}{ijk}\equiv\frac{12B_{p-3}}{p-3}-\frac{3B_{2p-4}}{p-4}\pmod{p^2},$$

where $p>5$ is a prime.

Recently, Wang and Cai [5] proved for every prime $p\geq 3$ and positive integer r,

$$\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}}\frac{1}{ijk}\equiv -2p^{r-1}B_{p-3}\pmod{p^r}, \tag{3}$$

where $\mathcal{P}_p$ denotes the set of positive integers which are prime to p.

Let $n=2$ or 4, for every positive integer $r\geq\frac{n}{2}$ and prime $p>n$, Zhao [6] extended (3) to

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{1}{i_1i_2\cdots i_n}\equiv -\frac{n!}{n+1}p^rB_{p-n-1}\pmod{p^{r+1}}. \tag{4}$$

For any prime $p>5$ and integer $r>1$, Wang [7] proved that

$$\sum_{\substack{i_1+i_2+\cdots+i_5=p^r\\ i_1,i_2,\cdots,i_5\in\mathcal{P}_p}}\frac{1}{i_1i_2\cdots i_5}\equiv -\frac{5!}{6}p^{r-1}B_{p-5}\pmod{p^r}.$$

We consider the following alternating harmonic sums

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{\sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_n^{i_n}}{i_1i_2\cdots i_n},$$

where $\sigma_i\in\{1,-1\}, i=1,2,\cdots,n$. Given n, we only need to consider the following alternating harmonic sums,

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2\cdots i_n},\ \sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2\cdots i_n},\cdots,\ \sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2+\cdots+i_{\left[\frac{n}{2}\right]}}}{i_1i_2\cdots i_n}$$

where $[x]$ denotes the largest integer less than or equal to x.

In this paper, we consider the congruences involving the combination of alternating harmonic sums,

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2\cdots i_n},\ \sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2\cdots i_n},\cdots,\ \sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2+\cdots+i_{\left[\frac{n}{2}\right]}}}{i_1i_2\cdots i_n}.$$

We obtain the following theorems. Among them, Theorem 1 and Theorem 2 have been proved by Wang [8] using different method.

Theorem 1. *Let p be an odd prime and r a positive integer, then*

$$\sum_{\substack{i+j+k=2p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}\equiv p^{r-1}B_{p-3}\left(\operatorname{mod} p^r\right).$$

Remark 1. There is no solution (i,j,k) for the equation $i+j+k=2p^r$ with $i,j,k\in\mathcal{P}_{2p}$.

Theorem 2. *Let p be an odd prime and r a positive integer, then*

$$\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}\equiv \frac{1}{2}p^{r-1}B_{p-3}\left(\operatorname{mod} p^r\right).$$

Theorem 3. *Let* $p\ge 5$ *be a prime and r a positive integer, then*

$$4\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2i_3i_4}+3\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4}\equiv\begin{cases}\frac{216}{5}pB_{p-5}\left(\operatorname{mod} p^2\right), \text{if } r=1,\\ \frac{36}{5}p^rB_{p-5}\left(\operatorname{mod} p^{r+1}\right), \text{if } r>1.\end{cases}$$

Theorem 4. *Let* $p>5$ *be a prime and r a positive integer, then*

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2i_3i_4i_5}+2\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4i_5}\equiv\begin{cases}12B_{p-5}\left(\operatorname{mod} p\right), \text{if } r=1,\\ 6p^{r-1}B_{p-5}\left(\operatorname{mod} p^r\right), \text{if } r>1.\end{cases}$$

2. Preliminaries

In order to prove the theorems, we need the following lemmas.

Lemma 1 ([5]) *Let p be an odd prime and r, m positive integers, then*

$$\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}}\frac{1}{ijk}\equiv -2mp^{r-1}B_{p-3}\left(\operatorname{mod} p^r\right).$$

Lemma 2. *Let p be an odd prime and r, m positive integers, then*

$$\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}}\frac{1}{ijk}=\frac{6}{mp^r}\sum_{\substack{1\le j<l\le mp^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}.$$

Proof. It is easy to see that

$$\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}}\frac{1}{ijk}=\frac{1}{mp^r}\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}}\frac{i+j+k}{ijk}=\frac{3}{mp^r}\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}}\frac{1}{ij}.$$

Let $l=j+k$, then $1\le j<l\le mp^r$ and $j,l,l-j\in\mathcal{P}_p$. By symmetry, we have

$$\frac{3}{mp^r}\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}}\frac{1}{ij}=\frac{3}{mp^r}\sum_{\substack{i+j<mp^r\\ i,j,l\in\mathcal{P}_p}}\frac{1}{l}\frac{i+j}{ij}=\frac{6}{mp^r}\sum_{\substack{1\le j<l\le mp^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}.$$

This completes the proof of Lemma 2. □

Lemma 3. *Let* $p>4$ *be a prime and r, m positive integers, then*

$$\sum_{\substack{i_1+i_2+i_3+i_4=mp^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{1}{i_1i_2i_3i_4}=\frac{24}{mp^r}\sum_{\substack{1\le u_1<u_2<u_3\le mp^r\\ u_1,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}.$$

Proof. The proof of Lemma 3 is similar to the proof of Lemma 2. □

Lemma 4 ([3]) *Let* $r,\alpha_1,\cdots,\alpha_n$ *be positive integers,* $r=\alpha_1+\cdots+\alpha_n\le p-3$, *then*

$$\sum_{\substack{1\le l_1,\cdots,l_n\le p-1\\ l_i\ne l_j,\,\forall i\ne j}}\frac{1}{l_1^{\alpha_1}l_2^{\alpha_2}\cdots l_n^{\alpha_n}}\equiv\begin{cases}(-1)^n(n-1)!\dfrac{r(r+1)}{2(r+2)}B_{p-r-2}p^2\left(\operatorname{mod} p^3\right),\text{if } 2\nmid r,\\ (-1)^{n-1}(n-1)!\dfrac{r}{r+1}B_{p-r-1}p\left(\operatorname{mod} p^2\right),\text{if } 2\mid r.\end{cases}$$

Lemma 5 ([7]). *Let p be an odd prime, and* $\alpha_1,\cdots,\alpha_n$ *positive integers, where* $r=\alpha_1+\cdots+\alpha_n\le p-3$, *then*

$$\sum_{\substack{1\le l_1,\cdots,l_n\le 2p\\ l_i\ne l_j,\,l_i\in\mathcal{P}_p}}\frac{1}{l_1^{\alpha_1}l_2^{\alpha_2}\cdots l_n^{\alpha_n}}\equiv\begin{cases}(-1)^n(n-1)!\dfrac{2r(r+1)}{r+2}B_{p-r-2}p^2\left(\operatorname{mod} p^3\right),\text{if } 2\nmid r,\\ (-1)^{n-1}(n-1)!\dfrac{2r}{r+1}B_{p-r-1}p\left(\operatorname{mod} p^2\right),\text{if } 2\mid r.\end{cases}$$

Lemma 6. *Let* $p>4$ *be a prime, then*

$$\sum_{\substack{i_1+i_2+i_3+i_4=2p\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{1}{i_1i_2i_3i_4}\equiv-\frac{240}{5}pB_{p-5}\left(\operatorname{mod} p^2\right).$$

Proof. By Lemma 3, we have

$$\sum_{\substack{i_1+i_2+i_3+i_4=2p\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{1}{i_1i_2i_3i_4}=\frac{24}{2p}\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}. \tag{5}$$

It is easy to see that

$$\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}=\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_2,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}+\sum_{\substack{1\le u_1<p<u_3\le 2p\\ u_1,u_3\in\mathcal{P}_p}}\frac{1}{u_1pu_3}.$$

By Lemma 4, we have

$$\sum_{\substack{1\le u_1<p<u_3\le 2p\\ u_1,u_3\in\mathcal{P}_p}}\frac{1}{u_1pu_3}=\frac{1}{p}\sum_{1\le u_1<p}\frac{1}{u_1}\sum_{p<u_3<2p}\frac{1}{u_3}\equiv 0\left(\operatorname{mod} p^3\right). \tag{6}$$

Hence

$$\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}\equiv\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_2,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}$$

$$\equiv\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_2,u_3\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}-\sum_{\substack{1\le u_1<u_1+p<u_3\le 2p\\ u_1,u_3\in\mathcal{P}_p}}\frac{1}{u_1(u_1+p)u_3}$$

$$-\sum_{\substack{1\le u_1<u_2<u_2+p\le 2p\\ u_1,u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2(u_2+p)}\left(\bmod p^3\right).$$

Replace $u_3=u_2+p$, then

$$\sum_{\substack{1\le u_1<u_1+p<u_3\le 2p\\ u_1,u_3\in\mathcal{P}_p}}\frac{1}{u_1(u_1+p)u_3}=\sum_{\substack{1\le u_1<u_1+p<u_2+p\le 2p\\ u_1,u_2\in\mathcal{P}_p}}\frac{1}{u_1(u_1+p)(u_2+p)}$$

$$\equiv\sum_{1\le u_1<u_2<p}\frac{1}{u_1^2u_2}\left(1-\frac{p}{u_1}+\frac{p^2}{u_1^2}\right)\left(1-\frac{p}{u_2}+\frac{p^2}{u_2^2}\right)\left(\bmod p^3\right).$$

and

$$\sum_{\substack{1\le u_1<u_2<u_2+p\le 2p\\ u_1,u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2(u_2+p)}\equiv\sum_{1\le u_1<u_2<p}\frac{1}{u_1u_2^2}\left(1-\frac{p}{u_2}+\frac{p^2}{u_2^2}\right)\left(\bmod p^3\right).$$

Thus

$$\begin{aligned}&\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}\\&\equiv\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_2,u_3\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}-\sum_{1\le u_1<u_2<p}\left(\frac{1}{u_1^2u_2}+\frac{1}{u_1u_2^2}-p\left(\frac{1}{u_1^3u_2}+\frac{1}{u_1^2u_2^2}+\frac{1}{u_1u_2^3}\right)\right.\\&\quad\left.+p^2\left(\frac{1}{u_1^4u_2}+\frac{1}{u_1^3u_2}+\frac{1}{u_1^2u_2^3}+\frac{1}{u_1^1u_2^4}\right)\right)\\&\equiv\frac{1}{3!}\sum_{\substack{1\le u_1,u_2,u_3\le 2p\\ u_i\ne u_j,u_i\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}-\sum_{1\le u_1,u_2<p}\left(\frac{1}{u_1^2u_2}-p\left(\frac{1}{u_1^3u_2}+\frac{1}{2}\frac{1}{u_1^2u_2^2}\right)\right.\\&\quad\left.+p^2\left(\frac{1}{u_1^4u_2}+\frac{1}{u_1^3u_2}\right)\right)\left(\bmod p^3\right).\end{aligned}\tag{7}$$

Using Lemma 5 in the first sum of the right hand in (7) and using Lemma 4 in the second sum, we have

$$\begin{aligned}\sum_{\substack{1\le u_1<u_2<u_3\le 2p\\ u_1,u_3,u_2-u_1,u_3-u_2\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}&\equiv\frac{1}{3!}(-1)^3(3-1)!\frac{24}{5}B_{p-5}p^2-(-1)^2\frac{12}{10}B_{p-5}p^2\\&\quad+\frac{3p}{2}\left(-\frac{4}{5}B_{p-5}p\right)-p^2\frac{30}{14}B_{p-7}p^2\\&\equiv-\frac{20}{5}p^2B_{p-5}\left(\bmod p^3\right).\end{aligned}\tag{8}$$

Combining (5) with (8), we complete the proof of Lemma 6. □

Lemma 7. *Let* $p>4$ *be a prime and* $r>1$ *a positive integer, then*

$$\sum_{\substack{i_1+i_2+i_3+i_4=2p^r \\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}} \frac{1}{i_1i_2i_3i_4} \equiv -\frac{48}{5}p^rB_{p-5}\left(\bmod p^{r+1}\right).$$

Proof. The proof of Lemma 7 is similar to the proof method of (4) in [6]. □

Lemma 8 ([7]). *Let* $p>5$ *be a prime and* r,m *positive integers,* $(m,p)=1$, *then*

$$\sum_{\substack{i_1+i_2+\cdots+i_5=mp \\ i_1,i_2,\cdots,i_5\in\mathcal{P}_p}} \frac{1}{i_1i_2i_3i_4i_5} \equiv \begin{cases} -4\left(5m+m^3\right)B_{p-5}\left(\bmod p\right), \text{if } r=1, \\ -20mp^{r-1}B_{p-5}\left(\bmod p^r\right), \text{if } r>1. \end{cases}$$

Lemma 9. *Let* $p>5$ *be a prime and* r,m *positive integers, then*

$$\sum_{\substack{i_1+i_2+\cdots+i_5=mp^r \\ i_1,i_2,\cdots,i_5\in\mathcal{P}_p}} \frac{1}{i_1i_2i_3i_4i_5} = \frac{120}{mp^r} \sum_{\substack{1\le u_1<u_2<u_3<u_4\le mp^r \\ u_1,u_4,u_2-u_1,u_3-u_2,u_4-u_3\in\mathcal{P}_p}} \frac{1}{u_1u_2u_3u_4}.$$

Proof. The proof of Lemma 9 is similar to the proof of Lemma 2. □

3. Proofs of the Theorems

Proof of Theorem 1. It is easy to see that

$$\begin{aligned} \sum_{\substack{i+j+k=2p^r \\ i,j,k\in\mathcal{P}_p}} \frac{(-1)^i}{ijk} &= \frac{1}{2p^r} \sum_{\substack{i+j+k=2p^r \\ i,j,k\in\mathcal{P}_p}} \frac{(-1)^i(i+j+k)}{ijk} \\ &= \frac{1}{2p^r} \sum_{\substack{i+j+k=2p^r \\ i,j,k\in\mathcal{P}_p}} \left(\frac{(-1)^i}{jk} + \frac{2(-1)^i}{ij}\right). \end{aligned} \tag{9}$$

Let $l=j+k$,then $1\le j<l\le 2p^r$ and $j,l,l-j\in\mathcal{P}_p$, hence

$$\sum_{\substack{i+j+k=2p^r \\ i,j,k\in\mathcal{P}_p}} \frac{(-1)^i}{jk} = \sum_{\substack{1\le j<l\le 2p^r \\ j,l,l-j\in\mathcal{P}_p}} \frac{1}{l}\frac{(-1)^l(j+k)}{jk} = \sum_{\substack{1\le j<l\le 2p^r \\ j,l,l-j\in\mathcal{P}_p}} \frac{2(-1)^l}{jl}. \tag{10}$$

Let $l'=i+j$, then $1\le i<l'\le 2p^r$ and $i,l',l'-i\in\mathcal{P}_p$, hence

$$\sum_{\substack{i+j+k=2p^r \\ i,j,k\in\mathcal{P}_p}} \frac{2(-1)^i}{ij} = \sum_{\substack{1\le i<l'\le 2p^r \\ i,l',l'-i\in\mathcal{P}_p}} \frac{1}{l'}\frac{2(-1)^i(i+j)}{ij} = \sum_{\substack{1\le i<l'\le 2p^r \\ i,l',l'-i\in\mathcal{P}_p}} \left(\frac{2(-1)^i}{jl'} + \frac{2(-1)^i}{il'}\right). \tag{11}$$

Noting that $i=l'-j$, $(-1)^{l'-j}=(-1)^{l'+j}$ and we rename l' to l , then

$$\sum_{\substack{1\le i<l'\le 2p^r \\ i,l',l'-i\in\mathcal{P}_p}} \frac{2(-1)^i}{jl'} = \sum_{\substack{1\le j<l\le 2p^r \\ j,l,l-j\in\mathcal{P}_p}} \frac{2(-1)^{j+l}}{jl}. \tag{12}$$

Rename *i* to *j* and l' to l , then

$$\sum_{\substack{1\le i<l'\le 2p^r \\ i,l',l'-i\in\mathcal{P}_p}} \frac{2(-1)^i}{il'} = \sum_{\substack{1\le j<l\le 2p^r \\ j,l,l-j\in\mathcal{P}_p}} \frac{2(-1)^j}{jl}. \tag{13}$$

Combining (9)-(13), we have

$$\sum_{\substack{i+j+k=2p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}=\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p}}\left(\frac{(-1)^l}{jl}+\frac{(-1)^{j+l}}{jl}+\frac{(-1)^j}{jl}\right)$$

$$=\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{\left(1+(-1)^l\right)\left(1+(-1)^j\right)}{jl}-\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl} \tag{14}$$

$$=\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p,\, j\text{ even},\, l\text{ even}}}\frac{4}{jl}-\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}.$$

Let $j=2j', l=2l'$ in the first sum of (14) and noting that

$$\sum_{\substack{1\le j'<l'\le p^r\\ j',l',l'-j'\in\mathcal{P}_p}}\frac{1}{j'l'}=\sum_{\substack{1\le j<l\le p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl},$$

(14) is equal to

$$\sum_{\substack{i+j+k=2p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}=\frac{1}{p^r}\sum_{\substack{1\le j<l\le p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}-\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}. \tag{15}$$

By Lemma 1, Lemma 2 and (15), we obtain

$$\sum_{\substack{i+j+k=2p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}=\frac{1}{p^r}\sum_{\substack{1\le j<l\le p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}-\frac{1}{p^r}\sum_{\substack{1\le j<l\le 2p^r\\ j,l,l-j\in\mathcal{P}_p}}\frac{1}{jl}\equiv p^{r-1}B_{p-3}\left(\bmod p^r\right).$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. For every triple (i,j,k) of positive integers which satisfies $i+j+k=2p^r, i,j,k\in\mathcal{P}_p$, we take it to 3 cases.

Cases 1. Let $A(p^r)=\{(i,j,k)\,|\,1\le i,j,k\le p^r-1 \text{ and } i,j,k\in\mathcal{P}_p\}$. $(i,j,k)\leftrightarrow(p^r-i,p^r-j,p^r-k)$ is a bijection between the solutions of $i+j+k=2p^r, (i,j,k)\in A(p^r)$ and $i+j+k=p^r, i,j,k\in\mathcal{P}_p$, we have

$$\sum_{\substack{i+j+k=2p^r\\ (i,j,k)\in A(p^r)}}\frac{(-1)^i}{ijk}\equiv\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^{p^r-i}}{(p^r-i)(p^r-j)(p^r-k)}$$

$$\equiv\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}\left(\bmod p^r\right). \tag{16}$$

Cases 2. Let

$B(p^r)=\{(i,j,k)\,|\,p^r+1\le i\le 2p^r-1, 1\le j,k\le p^r-1 \text{ and } i,j,k\in\mathcal{P}_p\}$.

$(i,j,k)\leftrightarrow(p^r+i,j,k)$ is a bijection between the solutions of $i+j+k=2p^r, (i,j,k)\in B(p^r)$ and $i+j+k=p^r, i,j,k\in\mathcal{P}_p$, we have

$$\sum_{\substack{i+j+k=2p^r\\ (i,j,k)\in B(p^r)}}\frac{(-1)^i}{ijk}\equiv\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^{p^r+i}}{(p^r+i)jk}\equiv-\sum_{\substack{i+j+k=p^r\\ i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}\left(\bmod p^r\right). \tag{17}$$

Cases 3. Let

$$C(p^r)=\{(i,j,k)\,|\,p^r+1\le j\le 2p^r-1, 1\le i,k\le p^r-1 \text{ and } i,j,k\in\mathcal{P}_p\}$$

and

$$D\left(p^r\right)=\left\{(i,j,k)\mid p^r+1\le k\le 2p^r-1, 1\le i,j\le p^r-1 \text{ and } i,j,k\in\mathcal{P}_p\right\}.$$

$(i,j,k)\leftrightarrow(i,p^r+j,k)$ in the former and $(i,j,k)\leftrightarrow(i,j,p^r+k)$ in the later are the bijections between the solutions of
$i+j+k=2p^r,(i,j,k)\in C\left(p^r\right)$ or $(i,j,k)\in D\left(p^r\right)$ and
$i+j+k=p^r, i,j,k\in\mathcal{P}_p$, we have

$$\begin{aligned}&\sum_{\substack{i+j+k=2p^r\\(i,j,k)\in C(p^r)}}\frac{(-1)^i}{ijk}+\sum_{\substack{i+j+k=2p^r\\(i,j,k)\in D(p^r)}}\frac{(-1)^i}{ijk}\\&\equiv\sum_{\substack{i+j+k=p^r\\i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{i\left(p^r+j\right)k}+\sum_{\substack{i+j+k=p^r\\i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ij\left(p^r+k\right)}\\&\equiv 2\sum_{\substack{i+j+k=p^r\\i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}\left(\operatorname{mod}p^r\right).\end{aligned}\tag{18}$$

Combining (16)-(18), we have

$$\begin{aligned}\sum_{\substack{i+j+k=2p^r\\i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}&=\sum_{\substack{i+j+k=2p^r\\(i,j,k)\in A(p^r)}}\frac{(-1)^i}{ijk}+\sum_{\substack{i+j+k=2p^r\\(i,j,k)\in B(p^r)}}\frac{(-1)^i}{ijk}\\&\quad+\sum_{\substack{i+j+k=2p^r\\(i,j,k)\in C(p^r)}}\frac{(-1)^i}{ijk}+\sum_{\substack{i+j+k=2p^r\\(i,j,k)\in D(p^r)}}\frac{(-1)^i}{ijk}\\&\equiv 2\sum_{\substack{i+j+k=p^r\\i,j,k\in\mathcal{P}_p}}\frac{(-1)^i}{ijk}\left(\operatorname{mod}p^r\right).\end{aligned}$$

By Theorem 1, we complete the proof of Theorem 2. □

Proof of Theorem 3. By symmetry, it is easy to see that

$$\begin{aligned}\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2i_3i_4}&=\frac{1}{2p^r}\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}\left(i_1+i_2+i_3+i_4\right)}{i_1i_2i_3i_4}\\&=\frac{1}{2p^r}\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\left(\frac{(-1)^{i_1}}{i_2i_3i_4}+3\frac{(-1)^{i_1}}{i_1i_3i_4}\right).\end{aligned}\tag{19}$$

Let $u_3=i_2+i_3+i_4$ in the first sum of the last equation in (19), then $i_1=2p^r-u_3$, (19) equals to

$$\begin{aligned}&=\frac{1}{2p^r}\left[\sum_{\substack{u_3=i_2+i_3+i_4<2p^r\\u_3,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{2p^r-u_3}\left(i_2+i_3+i_4\right)}{i_2i_3i_4u_3}+3\sum_{\substack{u_3=i_1+i_3+i_4<2p^r\\u_3,i_1,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}\left(i_1+i_3+i_4\right)}{i_1i_3i_4u_3}\right]\\&=\frac{1}{2p^r}\left[3\sum_{\substack{u_3=i_2+i_3+i_4<2p^r\\u_3,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{u_3}}{i_3i_4u_3}+3\sum_{\substack{u_3=i_1+i_3+i_4<2p^r\\u_3,i_1,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_3i_4u_3}+6\sum_{\substack{u_3=i_1+i_3+i_4<2p^r\\u_3,i_1,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_3u_3}\right].\end{aligned}\tag{20}$$

Let $u_2 = i_3 + i_4$ in the second sum of the last equation in (20), since $u_3 = i_1 + i_3 + i_4$, then $i_1 = u_3 - u_2$, (20) equals to

$$= \frac{1}{2p^r}\left[3\sum_{\substack{u_2=i_3+i_4<u_3<2p^r\\ u_3,u_3-u_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{u_3}(i_3+i_4)}{i_3i_4u_2u_3}+3\sum_{\substack{u_2=i_3+i_4<u_3<2p^r\\ u_3,u_3-u_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{u_3-u_2}(i_3+i_4)}{i_3i_4u_3}\right.$$

$$\left.+6\sum_{\substack{u_2=i_1+i_3<u_3<2p^r\\ u_3,u_3-u_2,i_1,i_3\in\mathcal{P}_p}}\frac{(-1)^{i_1}(i_1+i_3)}{i_1i_3u_2u_3}\right]$$

$$= \frac{1}{2p^r}\left[6\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{(-1)^{u_3}}{u_1u_2u_3}+6\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{(-1)^{u_3-u_2}}{u_1u_2u_3}\right.$$

$$\left.+6\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{(-1)^{u_2-u_1}}{u_1u_2u_3}+6\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{(-1)^{u_1}}{u_1u_2u_3}\right]$$

$$= \frac{3}{p^r}\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{(-1)^{u_3}+(-1)^{u_2+u_3}+(-1)^{u_1+u_2}+(-1)^{u_1}}{u_1u_2u_3}.$$

Similarly, we have

$$\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4}=\frac{4}{p^r}\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{(-1)^{u_2}+(-1)^{u_1+u_2+u_3}+(-1)^{u_1+u_3}}{u_1u_2u_3}.$$

Hence

$$4\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2i_3i_4}+3\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4}$$

$$=\frac{12}{p^r}\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{\left[1+(-1)^{u_1}\right]\left[1+(-1)^{u_2}\right]\left[1+(-1)^{u_3}\right]-1}{u_1u_2u_3}$$

$$=\frac{12}{p^r}\sum_{\substack{0<u_1<u_2<u_3<p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}-\frac{12}{p^r}\sum_{\substack{0<u_1<u_2<u_3<2p^r\\ u_1,u_3,u_3-u_2,u_2-u_1\in\mathcal{P}_p}}\frac{1}{u_1u_2u_3}.$$

By Lemma 3, we have

$$4\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2i_3i_4}+3\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4}$$

$$=\frac{1}{2}\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{1}{i_1i_2i_3i_4}-\sum_{\substack{i_1+i_2+i_3+i_4=2p^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{1}{i_1i_2i_3i_4}.$$

By (2) and Lemma 6, we have

$$4\sum_{\substack{i_1+i_2+i_3+i_4=2p\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1}}{i_1i_2i_3i_4}+3\sum_{\substack{i_1+i_2+i_3+i_4=2p\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}}\frac{(-1)^{i_1+i_2}}{i_1i_2i_3i_4}$$

$$\equiv -\frac{24}{5}pB_{p-5}+\frac{240}{5}pB_{p-5}\equiv\frac{216}{5}pB_{p-5}\pmod{p^2}.$$

By (4) and Lemma 7, if $r \geq 2$, then

$$4 \sum_{\substack{i_1+i_2+i_3+i_4=2p^r \\ i_1,i_2,i_3,i_4 \in \mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4} + 3 \sum_{\substack{i_1+i_2+i_3+i_4=2p^r \\ i_1,i_2,i_3,i_4 \in \mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4}$$
$$\equiv -\frac{12}{5} p^r B_{p-5} + \frac{48}{5} p^r B_{p-5} \equiv \frac{36}{5} p^r B_{p-5} \left(\bmod p^{r+1}\right).$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. Similar to the proofs of Theorem 1 and Theorem 3, we have

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} = \frac{12}{p^r} \sum_{\substack{0<u_1<u_2<u_3<u_4<2p^r \\ u_1,u_4,u_2-u_1,u_3-u_2,u_4-u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} \Big[(-1)^{u_1}$$
$$+ (-1)^{u_4} + (-1)^{u_1+u_2} + (-1))^{u_2+u_3} + (-1)^{u_3+u_4} \Big]$$

and

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}$$
$$= \frac{6}{p^r} \sum_{\substack{0<u_1<u_2<u_3<u_4<2p^r \\ u_1,u_4,u_2-u_1,u_3-u_2,u_4-u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} \Big[(-1)^{u_2} + (-1)^{u_3}$$
$$+ (-1)^{u_1+u_3} + (-1)^{u_1+u_4} + (-1)^{u_2+u_4} + (-1)^{u_1+u_2+u_3}$$
$$+ (-1)^{u_1+u_2+u_4} + (-1)^{u_1+u_3+u_4} + (-1)^{u_2+u_3+u_4} + (-1)^{u_1+u_2+u_3+u_4} \Big]$$

Hence

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} + 2 \sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}$$
$$= \frac{12}{p^r} \sum_{\substack{0<u_1<u_2<u_3<u_4<2p^r \\ u_1,u_4,u_2-u_1,u_3-u_2,u_4-u_3 \in \mathcal{P}_p}} \frac{\left[1+(-1)^{u_1}\right]\left[1+(-1)^{u_2}\right]\left[1+(-1)^{u_3}\right]\left[1+(-1)^{u_4}\right]-1}{u_1 u_2 u_3 u_4}$$
$$= \frac{12}{p^r} \sum_{\substack{0<u_1<u_2<u_3<u_4<p^r \\ u_1,u_4,u_2-u_1,u_3-u_2,u_4-u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4} - \frac{12}{p^r} \sum_{\substack{0<u_1<u_2<u_3<u_4<2p^r \\ u_1,u_4,u_2-u_1,u_3-u_2,u_4-u_3 \in \mathcal{P}_p}} \frac{1}{u_1 u_2 u_3 u_4}.$$

By Lemma 9, we have

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} + 2 \sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}$$
$$= \frac{1}{10} \sum_{\substack{i_1+i_2+i_3+i_4+i_5=p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{1}{i_1 i_2 i_3 i_4 i_5} - \frac{2}{10} \sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r \\ i_1,i_2,i_3,i_4,i_5 \in \mathcal{P}_p}} \frac{1}{i_1 i_2 i_3 i_4 i_5}.$$

By (2) and Lemma 8 (1), we have

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} + 2 \sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}$$
$$\equiv -\frac{24}{10} B_{p-5} + \frac{144}{10} B_{p-5} \equiv 12 B_{p-5} \pmod{p}.$$

By Lemma 8 (2), if $r \ge 2$, then

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} + 2 \sum_{\substack{i_1+i_2+i_3+i_4+i_5=2p^r\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}$$
$$\equiv -2p^{r-1} B_{p-5} + 8p^{r-1} B_{p-5} \equiv 6p^{r-1} B_{p-5} \pmod{p^r}.$$

This completes the proof of Theorem 4. □

4. Conclusions

Let p be an odd prime and r, m positive integers, $(m,p)=1$, using Lemma 1 and Lemma 2, similar to the proof of Theorem 1, we can prove that

$$\sum_{\substack{i+j+k=2mp^r\\ i,j,k\in\mathcal{P}_p}} \frac{(-1)^i}{ijk} \equiv mp^{r-1} B_{p-3} \pmod{p^r}.$$

In particular, if $m=1$, it becomes Theorem 1.

Let p be odd prime and r, m positive integers, $(m,p)=1$, similar to the proof of Theorem 2, we can prove that

$$\sum_{\substack{i+j+k=mp^r\\ i,j,k\in\mathcal{P}_p}} \frac{(-1)^i}{ijk} \equiv \frac{1}{2} \sum_{\substack{i+j+k=2mp^r\\ i,j,k\in\mathcal{P}_p}} \frac{(-1)^i}{ijk} \equiv \frac{m}{2} p^{r-1} B_{p-3} \pmod{p^r}.$$

In particular, if $m=1$, it becomes Theorem 2.

Let $p>4$ be a prime and r, m positive integers, $(m,p)=1$, we can deduce the congruence $\left(\bmod\ p^{r+1}\right)$ for

$$4 \sum_{\substack{i_1+i_2+i_3+i_4=2mp^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4} + 3 \sum_{\substack{i_1+i_2+i_3+i_4=2mp^r\\ i_1,i_2,i_3,i_4\in\mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4}.$$

Let $p>5$ be a prime and r, m positive integers, $(m,p)=1$, we can deduce the congruence $\left(\bmod\ p^{r}\right)$ for

$$\sum_{\substack{i_1+i_2+i_3+i_4+i_5=2mp^r\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}} \frac{(-1)^{i_1}}{i_1 i_2 i_3 i_4 i_5} + 2 \sum_{\substack{i_1+i_2+i_3+i_4+i_5=2mp^r\\ i_1,i_2,i_3,i_4,i_5\in\mathcal{P}_p}} \frac{(-1)^{i_1+i_2}}{i_1 i_2 i_3 i_4 i_5}.$$

Similarly, we can consider the congruence $(\bmod\ p^{r+1})$ for

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r\\ i_1,i_2,\cdots,i_n\in\mathcal{P}_p}} \frac{\sigma_1^{i_1}\sigma_2^{i_2}\cdots\sigma_n^{i_n}}{i_1 i_2 \cdots i_n},$$

where $\sigma_i \in \{1,-1\}, i=1,2,\cdots,n$, but it seems much more complicated.

Founding

This work is supported by the Natural Science Foundation of Zhejiang Province, Project (No. LY18A010016) and the National Natural Science Foundation of China, Project (No. 12071421).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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