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Non Degeneration of Fibonacci Series, Pascal's Elements and Hex Series

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Abstract

Generally Fibonacci series and Lucas series are the same, they converge to golden ratio. After I read Fibonacci series, I thought, is there or are there any series which converges to golden ratio. Because of that I explored the inter relations of Fibonacci series when I was intent on Fibonacci series in my difference parallelogram. In which, I found there is no degeneration on Fibonacci series. In my thought, Pascal triangle seemed like a lower triangular matrix, so I tried to find the inverse for that. In inverse form, there is no change against original form of Pascal elements matrix. One day I played with ring magnets, which forms hexagonal shapes. Number of rings which forms Hexagonal shape gives Hex series. In this paper, I give the general formula for generating various types of Fibonacci series and its non-degeneration, how Pascal elements maintain its identities and which shapes formed by hex numbers by difference and matrices.

Keywords

Fibonacci Series, Lucas Series, Golden Ratio, Various Type of Fibonacci Series Generated by Matrices, Matrix Operations on Pascal's Elements and Hex Numbers

1. Introduction

The Fibonacci sequence is named after Leonardo of Pisa (c. 1170-c. 1250), popularly known as Fibonacci. He wrote a number of books such as *Liber Abaci* (The Book of Calculating) in 1202, *Practica Geometriae* (Practical Geometry) in 1220, *Flos* in 1225, and *Liber Quadratorum* (The Book of Squares) in 1225. Fibonacci sequence is a series of numbers in which each number is the sum of the two preceding numbers. First few numbers in the series are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 … In India, Fibonacci sequence appeared in Sanskrit prosody (a system of versification). In the Sanskrit oral tradition, there was much emphasis on how long (L) syllables that are 2 units of duration mix with the short (S) syllables that are 1 unit of duration. Counting the different patterns of L and S within a given fixed length results in the Fibonacci numbers—the number of patterns that are *m* short syllables-long is the Fibonacci number *Fm* + 1. According to Susantha Goonatilake of Royal Asiatic Society Sri Lanka, the development of the Fibonacci sequence "is attributed in part to Pingala (200 BC), later being associated with Virahanka (c. AD 700), Gopala (c. AD 1135), and Hemchandra (c.AD 1150)". To find *Fn* for a general positive integer *n*, we hope that we can see a pattern in the sequence of numbers already found. A sharp eye can now detect that any number in the sequence is always the sum of the two numbers preceding it. That is,

$$F_{n+2} = F_{n+1} + F_n \text{, for } \ n = 0,1,2,3,\cdots .$$

Fibonacci series is helix like identity. It converges to golden ratio, we can show its existence in spiral shells but its elements never construct volumetric object. Fibonacci series elements construct Area only. Pascal triangle elements (Binomial series elements) construct area, volume and volumetric objects but whatever be it remains its identity which means, if we constructed a matrix with Pascal triangle elements, which would be a square matrix, its k^{th} power or its inverse might have the same identity of Pascal triangle elements, hex series having different numbers, but all numbers will be derived by triangular series numbers.

The Fibonacci Numbers are also applied in Pascal's Triangle. Entry is sum of the two numbers either side of it, but in the row above. Diagonal sums in Pascal's Triangle are the Fibonacci numbers. We are getting some ideas from ([1] Jeffrey R. Chasnov (2016-19) - Fibonacci Numbers and the Golden ratio - Lecture Notes for Course - The Hong Kong University of Science and Technology, Department of Mathematics, Clear Water Bay, Kowloon - Hong Kong). We know Fibonacci Numbers and the Golden ratio ([2] Tom Davis, *Exploring Pascal's Triangle*- tomrdavis@earthlink.net http://www.geometer.org/mathcircles, January 1, 2010; Relation between Pascal's triangle and Fibonacci's numbers; [3] Balasubramani Prema Rangasamy - Some extensions on numbers - Advances in Pure Mathematics, 2019, 9, 944-958. Difference table and [4] https://en.wikipedia.org/w/index.php?title=Golden_ratio&oldid=83746951"8) We know more about Fibonacci's elements, Pascal's elements, Hex numbers and Golden ratio. ***The Golden Section*** represented by the Greek letter Phi (φ) = 1.6180339887.

In this paper, I give the general formula for generating various types of Fibonacci series and its non-degeneration, how Pascal elements maintain its identities and which shapes formed by hex numbers by difference and matrices.

2. Row Matrix Building for Fibonacci's Elements

Difference method

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 5 \\ 5 & 8 \\ 8 & 13 \\ 13 & 21 \\ \vdots & \vdots \end{bmatrix} \text{ is a } m \times 2 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are difference of two consecutive odd rows numbers.

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \\ 10 & 16 & 26 \\ 13 & 21 & 34 \\ 42 & 68 & 110 \\ 55 & 89 & 144 \\ \vdots & \vdots & \vdots \end{bmatrix} \text{ is a } m \times 3 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are difference of two consecutive odd rows numbers.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 4 & 7 & 11 & 18 \\ 5 & 8 & 13 & 21 \\ 29 & 47 & 76 & 123 \\ 34 & 55 & 89 & 144 \\ 199 & 322 & 521 & 843 \\ 233 & 377 & 610 & 987 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \text{ is a } m \times 4 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are difference of two consecutive odd rows numbers.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 7 & 12 & 19 & 31 & 50 \\ 8 & 13 & 21 & 34 & 55 \\ 81 & 131 & 212 & 343 & 555 \\ 89 & 144 & 233 & 377 & 610 \\ 898 & 1453 & 2351 & 3804 & 6155 \\ 987 & 1597 & 2584 & 4181 & 6765 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \text{ is a } m \times 5 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are difference of two consecutive odd rows numbers.

We do the same again and again we get

1st series:	1	1	2	3	5	8	13	21	34	55	89
2nd series:	1	2	3	5	8	13	21	34	55	89	144
3rd series:	2	4	6	10	16	26	42	68	110	178	288
4th series:	4	7	11	18	29	47	76	123	199	322	521
5th series:	7	12	19	31	50	81	131	212	343	555	898
6th series:	12	20	32	52	84	136	220	356	576	932	1508

We can generate m^{th} series by

$$ {}^{m}F_{n+2} = {}^{m}F_{n+1} + {}^{m}F_{n} \tag{1}$$

where

$$ {}^{m}F_{n} = {}^{1}F_{m+n} - {}^{1}F_{n} \text{ and } m \text{ is an } m^{\text{th}} \text{ Fibonacci's series.} \tag{2}$$

where ${}^{m}F_{n}$ an n^{th} element of a m^{th} Fibonacci series, ${}^{1}F_{n}$ is n^{th} element of a 1^{st} Fibonacci series and ${}^{1}F_{m+n}$ is $m+n^{\text{th}}$ element of a 1^{st} Fibonacci series.

1^{st} series elements are known as ***Fibonacci numbers.***

4^{th} series elements are known as ***Lucas numbers.***

Axiom 1: All the above series are converges to ***Golden ratio.***

3. Addition Method

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \\ 7 & 11 \\ 5 & 8 \\ 18 & 29 \\ 13 & 21 \\ \vdots & \vdots \end{bmatrix} \text{ is a } m \times 2 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are addition of two consecutive odd rows numbers.

$$\begin{bmatrix} 1 & 1 & 2 \\ 4 & 6 & 10 \\ 3 & 5 & 8 \\ 16 & 26 & 42 \\ 13 & 21 & 34 \\ 68 & 110 & 178 \\ 55 & 89 & 144 \\ \vdots & \vdots & \vdots \end{bmatrix} \text{ is a } m \times 3 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are addition of two consecutive odd rows numbers.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 6 & 9 & 15 & 24 \\ 5 & 8 & 13 & 21 \\ 39 & 63 & 102 & 165 \\ 34 & 55 & 89 & 144 \\ 267 & 432 & 699 & 1131 \\ 233 & 377 & 610 & 987 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \text{ is a } m \times 4 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are addition of two consecutive odd rows numbers.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 9 & 14 & 23 & 37 & 60 \\ 8 & 13 & 21 & 34 & 55 \\ 97 & 157 & 254 & 411 & 665 \\ 89 & 144 & 233 & 377 & 610 \\ 1076 & 1741 & 2817 & 4558 & 7375 \\ 987 & 1597 & 2584 & 4181 & 6765 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \text{ is a } m \times 5 \text{ matrix.}$$

In which odd rows numbers are Fibonacci series numbers and even rows numbers are addition of two consecutive odd rows numbers.

We do the same again and again we get

1^{st} series:	1	1	2	3	5	8	13	21	34	55	89
2^{nd} series:	3	4	7	11	18	29	47	76	123	199	322
3^{rd} series:	4	6	10	16	26	42	68	110	178	288	466
4^{th} series:	6	9	15	24	39	63	102	165	267	432	699
5^{th} series:	9	14	23	37	60	97	157	254	411	665	1076
6^{th} series:	14	22	36	58	94	152	246	398	644	1042	168

We can generate k^{th} series by

$$^{k}F_{k+2} = {}^{k}F_{n+1} + {}^{k}F_{n} \tag{3}$$

and

$$^{k}F_{n} = {}^{1}F_{k+n} + {}^{1}F_{n} \tag{4}$$

where $^{k}F_{n}$ an n^{th} element of a k^{th} Fibonacci series, $^{1}F_{n}$ is n^{th} element of a 1^{st} Fibonacci series and $^{1}F_{k+n}$ is $k+1^{th}$ element of a 1^{st} Fibonacci series.

1^{st} series elements are known as ***Fibonacci numbers.***

2^{nd} series elements are known as ***Lucas numbers.***

3^{rd} series elements are known as ***doubled Fibonacci numbers.***

4^{th} series elements are known as ***tripled Fibonacci numbers.***

4. Difference between All Series Diagonal Elements

1st series:	1		1		2		3		5		8		13		21		34		55		89
		2		2		4		6		10		16		26		42		68		110	
2nd series:	3		4		7		11		18		29		47		76		123		199		322
		0		1		1		2		3		5		8		13		21		34	
3rd series:	4		6		10		16		26		42		68		110		178		288		466
		0		1		1		2		3		5		8		13		21		34	
4th series:	6		9		15		24		39		63		102		165		267		432		699
		0		1		1		2		3		5		8		13		21		34	
5th series:	9		14		23		37		60		97		157		254		411		665		1076
		0		1		1		2		3		5		8		13		21		34	
6th series:	14		22		36		58		94		152		246		398		644		1042		1686

$$^{1}d_n = {}^{s+1}F_n - {}^{s}F_{n+1} \tag{6}$$

$$\text{and } {}^{s}d_n = {}^{s}F_{n+1} - {}^{s+1}F_n ; s \geq 2 \tag{7}$$

where ${}^{s}d_n$ is n^{th} element of a s^{th} different series, ${}^{s+1}F_n$ is n^{th} element of a $s+1^{th}$ Fibonacci series and ${}^{s}F_{n+1}$ is $n+1^{th}$ element of a s^{th} Fibonacci series.

From above those diagonal differences remains the extinct of Fibonacci's elements.

5. Difference Chart of Above Series

		Diff. 1		*Diff.* 2		*Diff.* 3		*Diff.* 4		…	
1st series:	1		1		2		3		5		8
		0		1		1		2		3	
2nd series:	1		2		3		5		8		13
		1		1		2		3		5	
3rd series:	2		4		6		10		16		26
		2		2		4		6		10	
4th series:	4		7		11		18		29		47
		3		4		7		11		18	
5th series:	7		12		19		31		50		81
		5		7		12		19		31	
6th series:	12		20		32		52		84		136
		8		12		20		32		52	
7th series:	20		33		53		86		139		225
		13		20		33		53		86	

From the above we chart,

	Series 1	series 2	series 3	series 4	series 5	…
Diff 1 series:	0	1	2	3	5	…
Diff 2 series:	1	1	2	4	7	…
Diff 3 series:	1	2	4	7	12	…
Diff 4 series:	2	3	6	11	19	…
Diff 5 series:	3	5	10	18	31	…

Diff t^{th} series:

$$^{t+2}D_n = {}^{t+1}D_n + {}^{t}D_n \tag{9}$$

where

$$^{t}D_n = {}^{t}F_{n+1} - {}^{t}F_n \tag{10}$$

$$\text{and } {}^{t+1}D_n = {}^{t+1}F_{n+1} - {}^{t+1}F_n \tag{11}$$

where ${}^{t}D_n$ an n^{th} element of a t^{th} different Fibonacci series, ${}^{1}F_n$ is n^{th} element of a 1st Fibonacci series and ${}^{1}F_{k+n}$ is $k+1^{th}$ element of a 1st Fibonacci series.

Axiom 3: All the above different series are converges to ***Golden ratio.***

6. Difference Parallelogram of Fibonacci Numbers

$$\begin{array}{ccccccccccccccccccccccccc}
0 & & 1 & & 1 & & 2 & & 3 & & 5 & & 8 & & 13 & & 21 & & 34 & & \cdots & & & & \\
 & 1 & & 0 & & 1 & & 1 & & 2 & & 3 & & 5 & & 8 & & 13 & & 21 & & \cdots & & & \\
 & & -1 & & 1 & & 0 & & 1 & & 1 & & 2 & & 3 & & 5 & & 8 & & 13 & & \cdots & & \\
 & & & 2 & & -1 & & 1 & & 0 & & 1 & & 1 & & 2 & & 3 & & 5 & & 8 & & \cdots & \\
 & & & & -3 & & 2 & & -1 & & 1 & & 0 & & 1 & & 1 & & 2 & & 3 & & 5 & & \cdots \\
 & & & & & 5 & & -3 & & 2 & & -1 & & 1 & & 0 & & 1 & & 1 & & 2 & & 3 & \cdots \\
 & & & & & & -8 & & 5 & & -3 & & 2 & & -1 & & 1 & & 0 & & 1 & & 1 & & 2 \;\; \cdots \\
 & & & & & & & 13 & & -8 & & 5 & & -3 & & 2 & & -1 & & 1 & & 0 & & 1 & 1 \;\; \cdots \\
 & & & & & & & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \;\; \cdots
\end{array}$$

Above difference parallelogram shows Fibonacci series never vanished, which means it exist everlastingly.

7. Matrices in Pascal's Elements

Let

$$A = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ {}^{m}C_0 & {}^{m}C_1 & {}^{m}C_2 & {}^{m}C_3 & \cdots & {}^{m}C_m \end{bmatrix}$$

be an $n \times n$ matrix having Pascal's elements. Where $m = n - 1$. We called it as Pascal's matrix.

Now we define Pascal matrix by any variable.

1) NW (North-west Pascal's matrix)

Let

$$A = \begin{bmatrix} a & & & & & \\ a & a & & & & \\ a & 2a & a & & & \\ a & 3a & 3a & a & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ {}^{m}C_0 a & {}^{m}C_1 a & {}^{m}C_2 a & {}^{m}C_3 a & \cdots & {}^{m}C_m a \end{bmatrix}$$

be an $n \times n$ matrix having Pascal's elements. Where $m = n - 1$. k is an exponent and "a" is variant.

Now,

$$A^k = \begin{bmatrix} k^0 a^k & & & & & \\ k^1 a^k & k^0 a^k & & & & \\ k^2 a^k & 2k^1 a^k & k^0 a^k & & & \\ k^3 a^k & 3k^2 a^k & 3k^1 a^k & k^0 a^k & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ {}^m C_0 k^{n-1} a^k & {}^m C_1 k^{n-2} a^k & {}^m C_2 k^{n-3} a^k & {}^m C_3 k^{n-4} a^k & \cdots & {}^m C_m k^0 a^k \end{bmatrix}$$

$$A^{-1} = \frac{1}{a} \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ -1 & 3 & -3 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ (-1)^{n-1}\left[{}^m C_0\right] & (-1)^{n-1}\left[{}^m C_1\right] & (-1)^{n-1}\left[{}^m C_2\right] & (-1)^{n-1}\left[{}^m C_3\right] & \cdots & (-1)^{n-1}\left[{}^m C_m\right] \end{bmatrix}.$$

Jordan normal matrix of A

$$J_A = \begin{bmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & a \end{bmatrix}$$

2) NE (North-East Pascal's matrix)

Let

$$A = \begin{bmatrix} & & & & & a \\ & & & & a & a \\ & & & a & 2a & a \\ & & a & 3a & 3a & a \\ & \iddots & \vdots & \vdots & \vdots & \vdots \\ {}^x C_x a & \cdots & {}^x C_3 a & {}^x C_2 a & {}^x C_1 a & {}^x C_0 a \end{bmatrix}$$

be an $y \times y$ matrix having Pascal's elements. Where $x = y - 1$. k is an exponent and "a" is variant.

Now, inverse for North-East matrix

$$A^{-1} = \frac{1}{a} \begin{bmatrix} (-1)^{y-1}\left[{}^x C_0\right] & (-1)^{y-1}\left[{}^x C_1\right] & (-1)^{y-1}\left[{}^x C_2\right] & (-1)^{y-1}\left[{}^x C_3\right] & \cdots & (-1)^{y-1}\left[{}^x C_x\right] \\ \vdots & \vdots & \vdots & \vdots & \iddots & \\ -1 & 3 & -3 & 1 & & \\ 1 & -2 & 1 & & & \\ -1 & 1 & & & & \\ 1 & & & & & \end{bmatrix}$$

3) SE (South-East Pascal's matrix)

Let

$$A=\begin{bmatrix} {}^mC_m a & \cdots & {}^mC_3 a & {}^mC_2 a & {}^mC_1 a & {}^mC_0 a \\ & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & a & 3a & 3a & a \\ & & & a & 2a & a \\ & & & & a & a \\ & & & & & a \end{bmatrix}$$

be an $n\times n$ matrix having Pascal's elements. Where $m = n - 1$. k is an exponent and "a" is variant.

Now,

$$A^{-1}=\frac{1}{a}\begin{bmatrix} (-1)^{n-1}\left[{}^mC_m\right] & & (-1)^{n-1}\left[{}^mC_3\right] & (-1)^{n-1}\left[{}^mC_2\right] & (-1)^{n-1}\left[{}^mC_1\right] & (-1)^{n-1}\left[{}^mC_0\right] \\ & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & 1 & -3 & 3 & -1 \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \\ & & & & & 1 \end{bmatrix}.$$

4) SW (South-West Pascal's matrix)

Let

$$A=\begin{bmatrix} {}^xC_0 a & {}^xC_1 a & {}^xC_2 a & {}^xC_3 a & \cdots & {}^xC_x a \\ \vdots & \vdots & \vdots & \vdots & ⋰ & \\ a & 3a & 3a & a & & \\ a & 2a & a & & & \\ a & a & & & & \\ a & & & & & \end{bmatrix}$$

be an $y\times y$ matrix having Pascal's elements. Where $x = y - 1$. k is an exponent and 'a' is variant.

Now, inverse for south-west matrix

$$A^{-1}=\frac{1}{a}\begin{bmatrix} & & & & & 1 \\ & & & & 1 & -1 \\ & & & 1 & -2 & 1 \\ & & 1 & -3 & 3 & -1 \\ & ⋰ & \vdots & \vdots & \vdots & \vdots \\ (-1)^{y-1}\left[{}^xC_x\right] & \cdots & (-1)^{y-1}\left[{}^xC_3\right] & (-1)^{y-1}\left[{}^xC_2\right] & (-1)^{y-1}\left[{}^xC_1\right] & (-1)^{y-1}\left[{}^xC_0\right] \end{bmatrix}$$

Hex numbers:

1	7	19	37	61	91	127	169	217	271	331
397	469	547	631	721	817	919	1027	1141	1261	…

Let h be any hex number. We know mod 6 of any h is equal to 1.

Mod 6 of $h_1 \equiv 1$; Mod 6 of $h_2 \equiv 1$; $\cdots$; Mod 6 of $h_k \equiv 1$;

Theorem 1: Difference between any two elements of Hex numbers is fully divided by 6.

Theorem 2: $\sum_{k=0}^{\infty} H_{6n\pm k} \equiv a \bmod 6$, where n is integer and $0 \le a < 6$.

Theorem 3: Remainder of arbitrary product of any number of Hex series is always 1 when the product is divided by 6.

Proof:

$$R = \frac{\prod_i H_i}{6} = 1.$$

We can say above as

$$R(h_1 \times h_2 \times \cdots \times h_k) \div 6 \equiv (1 \times 1 \times \cdots \times 1) \bmod 6 = 1.$$

Theorem 4: $\sum_{o} H_k \bmod(6) \equiv \sum_{E} H_k (\bmod 6) \mid k \in Z$

Theorem 5: $\sum_{k=0}^{\infty} H_k = k^3$, where H_k is Hex series elements.

Matrices of **Hex numbers**

Let we see the relation between hex numbers in matrix

1) Let $A = \begin{bmatrix} 1 & h+1 \\ 3h+1 & 6h+1 \end{bmatrix}$ be a 2 × 2 matrix which elements are hex numbers (where $h = 6$) then $|A| = \begin{vmatrix} 1 & h+1 \\ 3h+1 & 6h+1 \end{vmatrix} = 6h+1-3h^2-4h+1 = -3h^2+2h$

2) Let $A = \begin{bmatrix} h+1 & 3h+1 \\ 6h+1 & 10h+1 \end{bmatrix}$ be a 2 × 2 matrix which elements are hex numbers (where $h = 6$) then $|A| = \begin{vmatrix} h+1 & 3h+1 \\ 6h+1 & 10h+1 \end{vmatrix} = -8h^2+2h$

By above way we get, $-15h^2+2h$; $-24h^2+2h$; $\cdots$; $-n(n+2)h^2+2h$

1) Let we construct a difference triangle about above determinants

Initial:	$-3h^2+2h$		$-8h^2+2h$		$-15h^2+2h$		$-24h^2+2h$		$-35h^2+2h$		…
1st diff:		$5h^2$		$7h^2$		$9h^2$		$11h^2$		…	
2nd diff:			$2h^2$		$2h^2$		$2h^2$		…		
3rd diff:				0		0		…			

a) Let $A = \begin{bmatrix} 1 & h+1 & 3h+1 \\ 6h+1 & 10h+1 & 15h+1 \\ 21h+1 & 28h+1 & 36h+1 \end{bmatrix}$ be a 3 × 3 matrix which elements are hex numbers then $|A| = \begin{vmatrix} 1 & h+1 & 3h+1 \\ 6h+1 & 10h+1 & 15h+1 \\ 21h+1 & 28h+1 & 36h+1 \end{vmatrix} = -27h^3$

b) $|A| = \begin{vmatrix} h+1 & 3h+1 & 6h+1 \\ 10h+1 & 15h+1 & 21h+1 \\ 28h+1 & 36h+1 & 45h+1 \end{vmatrix} = -27h^3$

2) Let we construct a difference triangle about above determinants

Initial:	$27h^3$		$27h^3$		$27h^3$		$27h^3$		$27h^3$		…
1^{st} diff:		0		0		0		0		…	

a) Let $A=\begin{bmatrix} 1 & h+1 & 3h+1 & 6h+1 \\ 10h+1 & 15h+1 & 21h+1 & 28h+1 \\ 36h+1 & 45h+1 & 55h+1 & 66h+1 \\ 78h+1 & 91h+1 & 105h+1 & 120h+1 \end{bmatrix}$ be a 4 × 4 matrix which elements are hex numbers then $|A|=\begin{vmatrix} 1 & h+1 & 3h+1 & 6h+1 \\ 10h+1 & 15h+1 & 21h+1 & 28h+1 \\ 36h+1 & 45h+1 & 55h+1 & 66h+1 \\ 78h+1 & 91h+1 & 105h+1 & 120h+1 \end{vmatrix}=0$

b) $|A|=\begin{vmatrix} h+1 & 3h+1 & 6h+1 & 10h+1 \\ 15h+1 & 21h+1 & 28h+1 & 36h+1 \\ 45h+1 & 55h+1 & 66h+1 & 78h+1 \\ 91h+1 & 105h+1 & 120h+1 & 136h+1 \end{vmatrix}=0$

3) Let we construct a difference triangle about above determinants

Initial:	0	0	0	0	0	…

From the above we can state:

a) Determinants of 2 × 2 matrix with hex series elements vanished at 2nd difference.

b) Determinant of 3 × 3 matrix with hex series elements vanished at 0th difference.

c) Determinant of 4 × 4 matrix with hex series elements and above are 0.

Which means hex series elements are forming hexagonal only.

8. Conclusions

1) *Fibonacci series never dies. We can generate so many series like Fibonacci series, they also converge to golden ratio. By this way we find so many golden ratio pairs.*

2) *Matrix with Pascal elements never vanished at any "n" dimensional matrix calculation. For all arithmetic and matrix operation of matrix with Pascal elements never give up its frame. Here frame means the structure of matrix.*

3) Sum of k^{th} elements of hex series gives k^3 and hex series elements form hexagonal only.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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