Scientific Research Publishing

Advances in Pure Mathematics, 2020, 10, 155-173
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

# Perturbation Analysis of Continuous-Time Linear Time-Invariant Systems

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**How to cite this paper:** Weng, P.C.-Y. and Phoa, F.K.H. (2020) Perturbation Analysis of Continuous-Time Linear Time-Invariant Systems. *Advances in Pure Mathematics*, **10**, 155-173.
https://doi.org/10.4236/apm.2020.104010

**Received:** March 2, 2020
**Accepted:** April 6, 2020
**Published:** April 9, 2020



## Abstract

In this paper, we consider the perturbation analysis of linear time-invariant systems, which arise from the linear optimal control in continuous-time. We provide a method to compute condition numbers of continuous-time linear time-invariant systems. It solves the perturbed linear time-invariant systems via Riccati differential equations and continuous-time algebraic Riccati equations in finite and infinite time horizons. We derive the explicit expressions of measuring the perturbation bounds of condition numbers with respect to the solution of the linear time-invariant systems. Furthermore, condition numbers and their upper bounds of Riccati differential equations and continuous-time algebraic Riccati equations are also discussed. Numerical simulations show the sharpness of the perturbation bounds computed via the proposed methods.

## Keywords

Continuous-Time Linear Time-Invariant System, Condition Number, Perturbation Bound, Riccati Differential Equation, Continuous-Time Algebraic Riccati Equation

## 1. Introduction

Many mathematical models of physical, biological and social systems involve partial differential equations (PDEs). In order to understand these systems, we consider problems of control and optimization, leading to PDE boundary control, optimization constrained by stochastic PDEs, model order reduction and some related applications.

Consider the continuous-time linear time-invariant system (CLTI) by discretizing the PDE,

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0, t \in [0, t_1], \\ &y(t) = Cx(t), \end{aligned} \tag{1}$$

with coefficient matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, state vector $x(t) \in \mathbb{R}^{n}$, control vector $u(t) \in \mathbb{R}^{m}$ and output vector $y(t) \in \mathbb{R}^{r}$. We can apply the optimal control $u$ to influence the state vector $x$ for output vector $y$. From control theory, we seek to find the optimal control via solving the Riccati differential equation (RDE) in the finite time. For infinite time, we solve the continuous-time algebraic Riccati equation (CARE).

We solve the (perturbed) CLTI to get the relative errors in the exact solutions via RDEs and CAREs in the finite and infinite time horizons respectively. For solving RDEs, Leipnik [1] used the canonical form of the self-adjoint RDEs to obtain a convenient explicit solution. Rusnak [2] proposed an almost analytic representation for the solution of the nonhomogeneous and homogeneous, time invariant, and time variant RDEs to discuss the behavior of the optimal estimator on a finite time interval. For solving CAREs, it has been an extremely active area of research in various years. Laub [3] proposed the Schur method. Byers [4] suggested a stable symplectic orthogonal method as well as the matrix sign function method [5]. Guo and Lancaster [6] applied the Newton's method. Benner and Byers [7] adopted a modified Newton's method for solving CAREs that used exact line search to improve the convergence behavior of Newton's method. Furthermore, Chu *et al.* [8] used the SDA.

Perturbation analysis considers the sensitivity of the solution to the small perturbations in the input data of a problem. A condition number, which is a measurement of the sensitivity, is important in the numerical computation. Furthermore, perturbation bounds are usually discussed. Kenney and Hewer studied the sensitivity of the RDEs developed by Byers [9] in [10]. Konstantinov and Pelova presented linear and nonlinear methods for estimating the sensitivity of the solution to RDEs in [11]. Konstantinov *et al.* [12] [13] proposed new methods to improve the sensitivity estimate of RDEs in 2-norm. For the sensitivity analysis of the linear differential system, we refer papers [14] [15] [16] and their references therein. For the perturbation analysis and perturbation bounds of CAREs, please see [9] [10] [17] [18] [19] [20] [21]. In this paper, it is the first to consider the perturbation analysis of CLTI via RDEs and CAREs.

The paper is organized as follows. We introduce the CLTI, solve the perturbed CLTI with only one perturbed coefficient matrix via RDEs, discuss the sensitivity of the RDEs, compute condition numbers and perturbation bounds of the CLTI via RDEs and apply backward differentiation formula (BDF) to solve differential Lyapunov matrix equation (DLE) in Section 2. Section 3 discusses the CLTI via CAREs, the sensitivity of the CAREs, condition numbers and perturbation bounds of the CLTI via CAREs. The illustrative numerical examples are presented in Section 4. Section 5 concludes the paper.

## 2. Solving Continuous-Time Linear Time-Invariant System Via Riccati Differential Equation

In order to guarantee the existence and uniqueness of the state and output vectors, respectively in the CLTI (1), we assume that the condition $\det(rI_n - A) \neq 0$ holds, for some *r*. The linear-quadratic regulator (LQR) problem for finite time horizon seeks the optimal control $u(t)$ to minimize the cost function:

$$J(u, Q_1, t_1) \equiv \int_0^{t_1} \left[ y^{\mathrm{T}}(t) y(t) + u^{\mathrm{T}}(t) R u(t) \right] \mathrm{d}t + x^{\mathrm{T}}(t_1) Q_1 x(t_1),$$

for some $R > 0$ and $Q_1 \geq 0$. The optimal control is given by

$$u(t) = -R^{-1} B^{\mathrm{T}} X(t) x(t), t \in [0, t_1], \tag{2}$$

with $X(t)$ being the solution to the RDE:

$$\dot{X}(t) = -A^{\mathrm{T}} X(t) - X(t) A - H + X(t) G X(t), X(t_1) = Q_1, \tag{3}$$

where $G \equiv BR^{-1}B^{\mathrm{T}}$ and $H \equiv C^{\mathrm{T}}C$.

In this paper, the Bernoulli substitution technique is applied to solve RDEs (3), then we can take the optimal control $u(t)$ (2) into the CLTI (1) and solve the ordinary differential equation (ODE) to get the state vector $x(t)$. Furthermore, the output vector $y(t)$ can be also obtained. Please refer to Weng and Phoa [22] about the details of solving the CLTI (1) via RDEs (3).

### 2.1. Sensitivity of the Riccati Differential Equation

As we solve the CLTI (1) by applying RDEs (3), then the sensitivity of RDEs (3) is studied. We first derive two kinds of condition numbers and perturbation bounds before we present the sensitivity of CLTI (1).

The RDEs (3) study nonlinear matrix differential equations arising in optimal control, optimal filtering, $H_\infty$-control of linear-time varying systems, differential games, etc.; see, e.g. [23] [24] [25] [26]. Moreover, there is a variety of methods in the literature to compute the solution of RDEs (3); see, e.g. [27] [28] [29].

First, we transform from the RDEs (3) with terminal condition into initial value condition. Let $P(t) = X(t_1 - t)$, then for $0 \leq t \leq t_1$

$$\dot{P}(t) = H + A^{\mathrm{T}} P(t) + P(t) A - P(t) G P(t), P(0) = Q_1. \tag{4}$$

Suppose we add some small perturbations only to coefficient matrix *A* in the RDEs (3) due to some applications like electric circuit simulation and multibody dynamics [30]. Other two coefficient matrices *B* and *C* in the CLTI (1) are treated similarly step by step. The solution to perturbed Riccati differential equation (pRDE) is $\tilde{P}(t) = P(t) + \Delta P(t)$, then we get

$$\dot{\tilde{P}}(t) = H + \tilde{A}^{\mathrm{T}} \tilde{P}(t) + \tilde{P}(t) \tilde{A} - \tilde{P}(t) G \tilde{P}(t), \tilde{P}(0) = Q_1 + \Delta Q_1, \tag{5}$$

where $\tilde{A} \equiv A + \Delta A$ is the perturbed coefficient matrix.

Dropping the second and high-order terms in (5) yields

$$\Delta \dot{P}(t) = A_g(t) \Delta P(t) + \Delta P(t) A_g^{\mathrm{T}}(t) + \Delta A^{\mathrm{T}} P(t) + P(t) \Delta A, \tag{6}$$

$$\Delta P(0) = \Delta Q_1, 0 \leq t \leq t_1, Ag(t) \equiv A^{\mathrm{T}} - P(t) G. \tag{7}$$

Let $\Phi_g$ satisfy

$$\dot{\Phi}_g(t) = A_g(t)\Phi_g(t), \Phi_g(0) = I, 0 \le t \le t_1. \tag{8}$$

Define

$$\Omega_g^{-1}(Z) = \int_0^t \Phi_g(t)\Phi_g^{-1}(s)Z(s)\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)\mathrm{d}s \tag{9}$$

for any continuous matrix function $Z = Z(s)$, $s \in [0,t]$. By variation method, $\Delta P(t)$ in (6) can be solved

$$\Delta P(t) = \Phi_g(t)\Delta Q_1\Phi_g^{\mathrm{T}}(t) + \Theta_g(\Delta A), \tag{10}$$

where

$$\Theta_g(Z) \equiv \Omega_g^{-1}\left(Z^{\mathrm{T}}P(t) + P(t)Z\right). \tag{11}$$

Since we only perturb the coefficient matrix *A*, we modify the condition theory of Rice [31] into

$$C_\epsilon^A(t) = \sup\left\{\frac{\|\Delta X(t)\|}{\epsilon\|X(t)\|} \mid \|\Delta A\| \le \epsilon\|A\|, \|\Delta Q_1\| \le \epsilon\|Q_1\|\right\}.$$

Taking the limit as $\epsilon$ goes to zero, the condition number is defined:

$$K_{RDE}^A = \lim_{\epsilon\to 0} C_\epsilon^A(t).$$

That is,

$$K_{RDE}^A = \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta P(t_1-t)\|}{\epsilon\|P(t_1-t)\|} \mid \|\Delta A\| \le \epsilon\|A\|, \|\Delta Q_1\| \le \epsilon\|Q_1\|\right\}. \tag{12}$$

The following theorem describes the condition numbers of RDEs using 2- and $\infty$-norm.

**Theorem 2.1.** *Using the notations given above, we can derive the explicit expressions and perturbation bounds for two kinds of condition numbers of the RDEs according to only perturbed matrix A*

$$K_{2\_RDE}^A = \frac{m_1}{\|P(t_1-t)\|}, \tag{13}$$

$$K_{\infty\_RDE}^A = \frac{m_2}{\|P(t_1-t)\|_\infty}, 0 \le t \le t_1, \tag{14}$$

where

$$m_1 = \|\Phi_g(t_1-t)\|^2\|Q_1\| + \|\Theta_g\|\|A\|,$$

$$m_2 = \|\Phi_g(t_1-t)\|_\infty^2\|Q_1\|_\infty + \|\Theta_g\|_\infty\|A\|_\infty.$$

**Proof.** According to the above definition about the condition number of RDEs (12), we take 2-norm in (10) and substitute $t$ into $t_1 - t$, then obtain

$$\|\Delta P(t_1-t)\| \le \|\Phi_g(t_1-t)\|^2\|\Delta Q_1\| + \|\Theta_g\|\|\Delta A\|.$$

For $\epsilon$ sufficiently small, with $\|\Delta A\|/\|A\|$, $\|\Delta Q_1\|/\|Q_1\| \le \epsilon$, we can get

$$\left\|\Delta P(t_1-t)\right\| \le \left\|\Phi_g(t_1-t)\right\|^2 \epsilon\left\|Q_1\right\| + \left\|\Theta_g\right\|\epsilon\left\|A\right\|.$$

Divide by $\epsilon\left\|P(t_1-t)\right\|$ to get

$$\frac{\left\|\Delta P(t_1-t)\right\|}{\epsilon\left\|P(t_1-t)\right\|} \le \frac{\left\|\Phi_g(t_1-t)\right\|^2\left\|Q_1\right\| + \left\|\Theta_g\right\|\left\|A\right\|}{\left\|P(t_1-t)\right\|}.$$

From (12), let $\epsilon \to 0$ give

$$K_{2\_RDE}^A \le \frac{m_1}{\left\|P(t_1-t)\right\|}, 0 \le t \le t_1.$$

Analogously, we take $\infty$-norm in (10) and change $t$ into $t_1-t$, then obtain

$$\frac{\left\|\Delta P(t_1-t)\right\|_\infty}{\epsilon\left\|P(t_1-t)\right\|_\infty} \le \frac{\left\|\Phi_g(t_1-t)\right\|_\infty^2\left\|Q_1\right\|_\infty + \left\|\Theta_g\right\|_\infty\left\|A\right\|_\infty}{\left\|P(t_1-t)\right\|_\infty}.$$

Let $\epsilon \to 0$, we can get

$$K_{\infty\_RDE}^A \le \frac{m_2}{\left\|P(t_1-t)\right\|_\infty}, 0 \le t \le t_1.$$

In order to compute two kinds of condition numbers and perturbation bounds of the RDEs efficiently, we let $F_l = F_l(t)$ be the solution to the differential Lyapunov matrix equation (DLE)

$$\dot{F}_l = A_g(t)F_l + F_lA_g(t)^{\mathrm{T}} + P^l, F_l(0) = 0, l = 0,2, \tag{15}$$

where $A_g(t)$ is defined in (7) and $P = P(t)$ is the solution of RDEs (4). We assume that $A_g^{\mathrm{T}}(t)$ is a $c$-stable matrix and therefore (15) has a unique symmetric solution $F_l(t) = F_l^{\mathrm{T}}(t)$ [32]. The following theorem is the connection between DLE (15) and partial condition numbers (13) and (14).

**Theorem 2.2.** *For* $\Omega_g^{-1}$, $\Theta_g$, *and* $F_l$ *as in* (9), (11), *and* (15), *respectively,*

$$\left\|\Theta_g\right\| \le 2\left\|F_0\right\|^{\frac{1}{2}}\left\|F_2\right\|^{\frac{1}{2}}.$$

**Proof.** For $Z \in \mathbb{R}^{m\times m}$, we let $u$ and $v$ be unit left and right singular vectors of $\Theta_g(Z)$ such that

$$\left\|\Theta_g(Z)\right\| = u^{\mathrm{T}}\Theta_g(Z)v.$$

Using (9) and (11), we can obtain

$$\begin{aligned}\left\|\Theta_g(Z)\right\| &= \int_0^t u^{\mathrm{T}}\Phi_g(t)\Phi_g^{-1}(s)\left(Z^{\mathrm{T}}P(s)+P(s)Z\right)\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)v\mathrm{d}s,\\ &\le \int_0^t\left\|u^{\mathrm{T}}\Phi_g(t)\Phi_g^{-1}(s)\right\|\left\|Z^{\mathrm{T}}P(s)+P(s)Z\right\|\left\|\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)v\right\|\mathrm{d}s,\\ &\le 2\left\|Z\right\|\int_0^t\left\|u^{\mathrm{T}}\Phi_g(t)\Phi_g^{-1}(s)\right\|\left\|P(s)\right\|\left\|\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)v\right\|\mathrm{d}s.\end{aligned}$$

Applying the Cauchy-Schwarz inequality [21], we get

$$\left\|\Theta_g(Z)\right\| \le 2\left\|Z\right\|\left[\int_0^t\left\|u^{\mathrm{T}}\Phi_g(t)\Phi_g^{-1}(s)\right\|^2\mathrm{d}s\right]^{\frac{1}{2}}\left[\int_0^t\left\|P(s)\right\|^2\left\|\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)v\right\|^2\mathrm{d}s\right]^{\frac{1}{2}}.$$

We can express the solution $F_l$ to (15) explicitly using (8)

$$F_l(t)=\int_0^t \Phi_g(t)\Phi_g^{-1}(s)P^l(s)\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)\mathrm{d}s.$$

However,

$$\int_0^t \left\|u^{\mathrm{T}}\Phi_g(t)\Phi_g^{-1}(s)\right\|^2 \mathrm{d}s = u^{\mathrm{T}}\int_0^t \Phi_g(t)\Phi_g^{-1}(s)\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)\mathrm{d}s u \tag{16}$$

$$\Rightarrow u^{\mathrm{T}}F_0 u \le \|F_0\|, \tag{17}$$

where $u$ is a unit vector. Moreover,

$$\begin{aligned}&\int_0^t \|P(s)\|^2 \left\|\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)v\right\|^2 \mathrm{d}s\\ &= v^{\mathrm{T}}\int_0^t \Phi_g(t)\Phi_g^{-1}(s)P^2(s)\Phi_g^{-\mathrm{T}}(s)\Phi_g^{\mathrm{T}}(t)\mathrm{d}s v\\ &= v^{\mathrm{T}}F_2 v \le \|F_2\|,\end{aligned} \tag{18}$$

where $v$ is a unit vector. Combining (17) and (18), we have

$$\left\|\Theta_g(Z)\right\| \le 2\|Z\|\|F_0\|^{\frac{1}{2}}\|F_2\|^{\frac{1}{2}}.$$

Thus,

$$\left\|\Theta_g\right\| \le 2\|F_0\|^{\frac{1}{2}}\|F_2\|^{\frac{1}{2}}.$$

### 2.2. Sensitivity of the CLTI via RDEs

In this subsection, we discuss the perturbation analysis of the CLTI (1) using RDEs (3) and derive two kinds of condition numbers. Furthermore, we also present their perturbation bounds.

Suppose we introduce some small perturbation $\Delta A$ only to coefficient matrix $A$ and the state vector to the perturbed system is $\tilde{x}(t)=x(t)+\Delta x(t)$, then the perturbed CLTI is

$$\dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t)+B\tilde{u}(t), \tilde{x}(0)=x_0+\Delta x_0. \tag{19}$$

We can replace the perturbed optimal control

$$\tilde{u}(t)=-R^{-1}B^{\mathrm{T}}\tilde{X}(t)\tilde{x}(t),$$

and obtain

$$\dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t)-G\tilde{X}(t)\tilde{x}(t), \tag{20}$$

where $\tilde{A}=A+\Delta A$ is the perturbed coefficient matrix and $\tilde{X}(t)=X(t)+\Delta X(t)$ is the solution of perturbed Riccati differential equation (pRDE):

$$\dot{\tilde{X}}(t)=-H-\tilde{A}^{\mathrm{T}}\tilde{X}(t)-\tilde{X}(t)\tilde{A}+\tilde{X}(t)G\tilde{X}(t), \tilde{X}(t_1)\equiv\tilde{Q}_1, 0\le t\le t_1. \tag{21}$$

Dropping the second and higher-order terms in (20) yields

$$\Delta\dot{x}(t)=\left(A-GX(t)\right)\Delta x(t)+\left(\Delta A x(t)-G\Delta X(t)x(t)\right), \tag{22}$$

$$\Delta x(0)=\Delta x_0, 0\le t\le t_1,$$

where the pRDEs (21) are solved. Let $\Phi_d$ satisfy

$$\dot{\Phi}_d(t)=A_d(t)\Phi_d(t), \Phi_d(0)=I, 0\le t\le t_1, \tag{23}$$

where

$$A_d(t) \equiv A - GX(t). \tag{24}$$

Define

$$\Omega_d^{-1}(Z) = \int_0^t \Phi_d(t)\Phi_d^{-1}(s)Z(s)\Phi_d^{-\mathrm{T}}(s)\Phi_d^{\mathrm{T}}(t)\mathrm{d}s \tag{25}$$

for any continuous matrix function $Z = Z(s)$, $s \in [0,t]$. By variation method, we can solve (22) and get

$$\Delta x(t) = \Phi_d(t)\Delta x_0 + \Omega_d^{-1}(\Delta A x(t)) - \Omega_d^{-1}(G\Delta X(t)x(t)). \tag{26}$$

The above relation discusses a first-order perturbation $\Delta x(t)$ in the state vector corresponding to the perturbation $\Delta A$. Based on the perturbation analysis for $0 \le t \le t_1$, we modify the condition theory of Rice [31] into

$$K_\epsilon^A(t) = \sup\left\{\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|} \mid \|\Delta A\| \le \epsilon\|A\|, \|\Delta X(t)\| \le \epsilon\|X(t)\|, \|\Delta x_0\| \le \epsilon\|x_0\|\right\}.$$

Taking the limit as $\epsilon$ goes to zero, we can get the condition number

$$K^A(t) = \lim_{\epsilon\to 0} K_\epsilon^A(t).$$

The following theorem describes the condition numbers of the CLTI (1) via RDEs and perturbation bounds in 2- and $\infty$-norm according to only perturbed matrix $A$.

**Theorem 2.3.** *Using the notations given above, we can derive the explicit expressions and perturbation bounds for two kinds of condition numbers of the CLTI* (1) *via RDEs*

$$K_{2\_CLTI-RDE}^A = \frac{u_1}{\|x(t)\|},$$

$$K_{\infty\_CLTI-RDE}^A = \frac{u_2}{\|x(t)\|_\infty}, 0 \le t \le t_1,$$

where

$$u_1 = \|\Phi_d(t)\|\|x_0\| + \|\Omega_d^{-1}\|\|x(t)\|(\|A\| + \|G\|\|X(t)\|),$$

$$u_2 = \|\Phi_d(t)\|_\infty\|x_0\|_\infty + \|\Omega_d^{-1}\|_\infty\|x(t)\|_\infty(\|A\|_\infty + \|G\|_\infty\|X(t)\|_\infty).$$

**Proof.** We can investigate condition numbers in 2- and $\infty$-norm according to only perturbed matrix $\tilde{A}$ defined by

$$K_{2\_CLTI-RDE}^A = \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|} \mid \|\Delta A\| \le \epsilon\|A\|, \|\Delta X(t)\| \le \epsilon\|X(t)\|, \|\Delta x_0\| \le \epsilon\|x_0\|\right\},$$

$$K_{\infty\_CLTI-RDE}^A = \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta x(t)\|_\infty}{\epsilon\|x(t)\|_\infty} \mid |\Delta A| \le \epsilon|A|, |\Delta X(t)| \le \epsilon|X(t)|, |\Delta x_0| \le \epsilon|x_0|\right\}. \tag{27}$$

For $\epsilon$ sufficiently small, with $\|\Delta A\|/\|A\|$, $\|\Delta X(t)\|/\|X(t)\|$, $\|\Delta x_0\|/\|x_0\| \le \epsilon$, we take 2-norm in (26) and get

$$\begin{aligned}\|\Delta x(t)\| &\le \|\Phi_d(t)\|\|\Delta x_0\| + \|\Omega_d^{-1}\|\|\Delta A\|\|x(t)\| + \|\Omega_d^{-1}\|\|G\|\|\Delta X(t)\|\|x(t)\|,\\ &\le \|\Phi_d(t)\|\epsilon\|x_0\| + \|\Omega_d^{-1}\|\epsilon\|A\|\|x(t)\| + \|\Omega_d^{-1}\|\|G\|\epsilon\|X(t)\|\|x(t)\|.\end{aligned}$$

Therefore, we can obtain

$$\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|} \le \frac{\|\Phi_d(t)\|\|x_0\|}{\|x(t)\|} + \|\Omega_d^{-1}\|\|A\| + \|\Omega_d^{-1}\|\|G\|\|X(t)\|.$$

Thus $\epsilon \to 0$ gives

$$K_{2\_CLTI-RDE}^{A} \le \frac{u_1}{\|x(t)\|}, 0 \le t \le t_1.$$

Analogously, we take $\infty$-norm in (26) and apply (27), then obtain

$$\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|_\infty} \le \frac{\|\Phi_d(t)\|_\infty\|x_0\|_\infty}{\|x(t)\|_\infty} + \|\Omega_d^{-1}\|_\infty\|A\|_\infty + \|\Omega_d^{-1}\|_\infty\|G\|_\infty\|X(t)\|_\infty.$$

Let $\epsilon \to 0$, we can get

$$K_{\infty\_CLTI-RDE}^{A} \le \frac{u_2}{\|x(t)\|_\infty}, 0 \le t \le t_1.$$

When we compute condition numbers and perturbation bounds of CLTI efficiently via solving RDEs, we let $X_{\Omega_d} = X_{\Omega_d}(t)$ be the solution to the following DLEs

$$\dot{X}_{\Omega_d}(t) = A_d(t)X_{\Omega_d} + X_{\Omega_d}A_d^{\mathrm{T}}(t) + I, X_{\Omega_d}(0) = 0, \tag{28}$$

where $A_d(t)$ is defined in (24). We assume that $A_d(t)$ is a c-stable matrix and therefore (28) has a unique symmetric solution $X_{\Omega_d}(t) = X_{\Omega_d}^{\mathrm{T}}(t)$ [32]. The following theorem states the solution $X_{\Omega_d}$ of DLEs (28) that is equivalent to $\Omega_d^{-1}(I)$ defined in (25).

**Theorem 2.4.** [23] *For* $A_d(t)$ *and* $\Omega_d^{-1}$ *as in* (24) *and* (25), *respectively, the unique solution of the DLEs* (28) *is defined by*

$$X_{\Omega_d}(t) = \int_0^t \Phi_d(t)\Phi_d^{-1}(s)\Phi_d^{-\mathrm{T}}(s)\Phi_d^{\mathrm{T}}(t)\mathrm{d}s,$$

where $\Phi_d(t)$ is defined in (23). Furthermore, we can obtain

$$\Omega_d^{-1}(I) = \int_0^t \Phi_d(t)\Phi_d^{-1}(s)\Phi_d^{-\mathrm{T}}(s)\Phi_d^{\mathrm{T}}(t)\mathrm{d}s = X_{\Omega_d}(t).$$

Therefore,

$$\|\Omega_d^{-1}\| = \|X_{\Omega_d}(t)\| \text{ and } \|\Omega_d^{-1}\|_\infty = \|X_{\Omega_d}(t)\|_\infty.$$

## 2.3. Backward Differentiation Formula Method for Solving DLEs

There is a large variety of methods to compute the solution of DLEs, see, e.g. [27] [28] [29]. In this paper, we apply the efficient method called Backward

Differentiation Formula (BDF) to (15), which can be treated (28) similarly.

Consider

$$\dot{F}_l = \mathcal{R}(t, F_l),$$
$$\mathcal{R}(t, F_l) = A_g F_l + F_l A_g^{\mathrm{T}} + P^l, l = 0, 2,$$
$$F_l(0) = 0. \tag{29}$$

Applying the fixed-coefficients BDF method to the DLEs (29), we obtain the matrix valued BDF scheme

$$(F_l)_{k+1} = \sum_{j=1}^{p} -\alpha_j (F_l)_{k+1-j} + h_k \beta \mathcal{R}\left(t_{k+1}, (F_l)_{k+1}\right),$$

where $h_k = t_{k+1} - t_k$ is the time step size, $(F_l)_{k+1} \equiv F_l(t_{k+1})$, $\alpha_j$, $\beta$ are the determining coefficients of the p-step BDF method as listed in **Table 1** (see, e.g. [33]).

It leads to solving the following Lyapunov-BDF difference equation

$$-(F_l)_{k+1} + h_k \beta \left( (A_g)_{k+1} (F_l)_{k+1} + (F_l)_{k+1} (A_g)_{k+1}^{\mathrm{T}} + P_{k+1}^l \right) - \sum_{j=1}^{p} \alpha_j (F_l)_{k+1-j} = 0,$$

with $(A_g)_{k+1} \equiv A_g(t_{k+1})$, $P_{k+1}^l \equiv P^l(t_{k+1})$, which can be written as the following Lyapunov equation

$$A_b (F_l)_{k+1} + (F_l)_{k+1} A_b^{\mathrm{T}} + \left( h_k \beta P_{k+1}^l - \sum_{j=1}^{p} \alpha_j (F_l)_{k+1-j} \right) = 0, \tag{30}$$

for $(F_l)_{k+1}$ and $A_b \equiv h_k \beta (A_g)_{k+1} - \frac{1}{2} I$.

The Lyapunov Equation (30) can be solved by applying various methods such as the Schur vector method, symplectic SR methods, the matrix sign function, the matrix disk function or the doubling method; see, e.g. [34] [35] [36]. In this paper, we used the MATLAB function "lyap" to compute the unique symmetric positive semidefinite solution to the Lyapunov Equation (30).

**Table 1.** Coefficients of the p-step BDF method with $p \le 5$.

| $p$ | $\beta$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ |
|---|---|---|---|---|---|---|
| 1 | 1 | $-1$ | | | | |
| 2 | $\frac{2}{3}$ | $-\frac{4}{3}$ | $\frac{1}{3}$ | | | |
| 3 | $\frac{6}{11}$ | $-\frac{18}{11}$ | $\frac{9}{11}$ | $-\frac{2}{11}$ | | |
| 4 | $\frac{12}{25}$ | $-\frac{48}{25}$ | $\frac{36}{25}$ | $-\frac{16}{25}$ | $\frac{3}{25}$ | |
| 5 | $\frac{60}{137}$ | $-\frac{300}{137}$ | $\frac{300}{137}$ | $-\frac{200}{137}$ | $\frac{75}{137}$ | $-\frac{12}{137}$ |

## 3. Solving Continuous-Time Linear Time-Invariant System via Continuous-Time Algebraic Riccati Equation

For infinite time horizon, $t_1 \to \infty$ and we search for the steady state solution of the RDEs (3), which leads to the continuous-time algebraic Riccati equation (CARE):

$$\mathcal{C}(X) \equiv A^{\mathrm{T}}X + XA + H - XGX = 0. \tag{31}$$

In this case, the time-invariant solution $X$ leads to the optimal control

$$u(t) = -R^{-1}B^{\mathrm{T}}Xx(t), t \in [0, \infty). \tag{32}$$

In this paper, we used the MATLAB function "care" to compute the unique symmetric positive semidefinite solution $X$ to the CAREs (31), then replace the optimal control $u(t)$ (32) in the CLTI (1) and solve the ODE to get the state vector $x(t)$. Moreover, we can also obtain the output vector $y(t)$. For the details about the solvable conditions of CAREs (31) and solving the CLTI (1) via CAREs (31), please see Weng and Phoa [22].

### 3.1. Sensitivity of the Continuous-Time Algebraic Riccati Equation

Before we discuss the sensitivity of the CLTI (1) via solving CAREs (31), we first consider the sensitivity of the CAREs. Suppose we add some small perturbations only to the coefficient matrix $A$ in the CAREs (31) similar to that in the RDEs (4), then we get the perturbed continuous-time algebraic Riccati equation (pCARE):

$$\tilde{A}^{\mathrm{T}}\tilde{X} + \tilde{X}\tilde{A} + H - \tilde{X}G\tilde{X} = 0, \tag{33}$$

where $\tilde{X} \equiv X + \Delta X$. Dropping the second and high-order terms in (33) yields

$$\left(A^{\mathrm{T}} - XG\right)\Delta X + \Delta X\left(A - GX\right) + \Delta A^{\mathrm{T}}X + X\Delta A = 0. \tag{34}$$

Set

$$A_c = A - GX, \tag{35}$$

and let $\Omega_c$ satisfy

$$\Omega_c(Z) = A_c^{\mathrm{T}}Z + ZA_c.$$

Due to solvable conditions of CAREs (31), it is known that the matrix $A_c = A - GX$ is c-stable [3] [9]. Furthermore, we can get that $\Omega_c$ is invertible [37] and

$$\Omega_c^{-1}(Z) = \int_0^{\infty} e^{A_c^{\mathrm{T}}t} Z e^{A_c t}\,\mathrm{d}t. \tag{36}$$

Therefore, we can solve the Lyapunov Equation (34)

$$\Delta X = -\Theta_c(\Delta A), \tag{37}$$

where

$$\Theta_c(Z) = \Omega_c^{-1}\left(Z^{\mathrm{T}}X + XZ\right). \tag{38}$$

To connect $\|\Delta X\|$ to only $\|\Delta A\|$, we modify the condition theory of Rice [31]

DOI: 10.4236/apm.2020.104010

into

$$C_\epsilon^A = \sup\left\{\frac{\|\Delta X\|}{\epsilon\|X\|} \mid \|\Delta A\| \le \epsilon\|A\|\right\}.$$

Taking the limit as $\epsilon$ goes to zero, we obtain the condition number

$$K_{CARE}^A = \lim_{\epsilon\to 0} C_\epsilon^A. \tag{39}$$

The following theorem derives two kinds of condition numbers of CAREs (31) in 2- and $\infty$-norm.

**Theorem 3.1.** *Using the notations given above, we can derive the explicit expressions and perturbation bounds for two kinds of condition numbers of CAREs* (31) *according to only perturbed matrix A*

$$K_{2\_CARE}^A - \frac{n_1}{\|X\|}, \tag{40}$$

$$K_{\infty\_CARE}^A = \frac{n_2}{\|X\|_\infty}, \tag{41}$$

where

$$n_1 = \|\Theta_c\|\|A\|,$$

$$n_2 = \|\Theta_c\|_\infty \|A\|_\infty.$$

**Proof.** For $\epsilon$ sufficiently small, with $\|\Delta A\|/\|A\| \le \epsilon$, we take 2-norm in (37) according to the definition of the condition number (39) and get

$$\|\Delta X\| = \|\Theta_c(\Delta A)\| \le \|\Theta_c\|\|\Delta A\| \le \|\Theta_c\|\epsilon\|A\|.$$

Divide by $\epsilon\|X\|$ to get

$$\frac{\|\Delta X\|}{\epsilon\|X\|} \le \frac{\|\Theta_c\|\|A\|}{\|X\|}.$$

Take $\epsilon \to 0$ and obtain

$$K_{2\_CARE}^A \le \frac{n_1}{\|X\|}.$$

Analogously, we take $\infty$-norm in (37) and divide $\epsilon\|X\|_\infty$, then we obtain

$$\frac{\|\Delta X\|_\infty}{\epsilon\|X\|_\infty} \le \frac{\|\Theta_c\|_\infty\|\Delta A\|_\infty}{\epsilon\|X\|_\infty} \le \frac{\|\Theta_c\|_\infty\|A\|_\infty}{\|X\|_\infty}.$$

Let $\epsilon \to 0$ give

$$K_{\infty\_CARE}^A \le \frac{n_2}{\|X\|_\infty}.$$

To solve two kinds of condition numbers and perturbation bounds of CAREs (31) efficiently, we let $E_k$ be the solution to the Lyapunov equation

$$A_c^{\mathrm{T}} E_k + E_k A_c = -X^k, k = 0, 2, \tag{42}$$

where $A_c$ is defined in (35) and $X$ is the solution of CAREs (31). The following theorem applies the Lyapunov equation (42) to compute the condition numbers (40) and (41) efficiently.

**Theorem 3.2.** *For* $\Omega_c^{-1}$, $\Theta_c$ *and* $E_k$ *as in* (36), (38) *and* (42), *respectively*

$$\|\Theta_c\| \le 2\|E_0\|^{\frac{1}{2}}\|E_2\|^{\frac{1}{2}}.$$

**Proof.** For $Z \in \mathbb{R}^{m\times m}$, we let $u$ and $v$ be unit left and right singular vectors of $\Theta_c(Z)$ such that

$$\|\Theta_c(Z)\| = u^{\mathrm{T}}\Theta_c(Z)v.$$

By (36) and (38), we get

$$\begin{aligned}\|\Theta_c(Z)\| &= \int_0^\infty u^{\mathrm{T}} e^{A_c^{\mathrm{T}}t}\left(Z^{\mathrm{T}}X + XZ\right)e^{A_c t}v\mathrm{d}t,\\ &\le \int_0^\infty \left\|u^{\mathrm{T}}e^{A_c^{\mathrm{T}}t}\right\|\left\|Z^{\mathrm{T}}X + XZ\right\|\left\|e^{A_c t}v\right\|\mathrm{d}t,\\ &\le 2\|Z\|\int_0^\infty \left\|u^{\mathrm{T}}e^{A_c^{\mathrm{T}}t}\right\|\|X\|\left\|e^{A_c t}v\right\|\mathrm{d}t.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\|\Theta_c(Z)\| \le 2\|Z\|\left[\int_0^\infty \left\|u^{\mathrm{T}}e^{A_c^{\mathrm{T}}t}\right\|^2\mathrm{d}t\right]^{\frac{1}{2}}\left[\int_0^\infty \|X\|^2\left\|e^{A_c t}v\right\|^2\mathrm{d}t\right]^{\frac{1}{2}}.$$

We can express the solution $E_k$ of (42) explicitly [37]

$$E_k = \int_0^\infty e^{A_c^{\mathrm{T}}t}X^k e^{A_c t}\mathrm{d}t. \tag{43}$$

But

$$\int_0^\infty \left\|u^{\mathrm{T}}e^{A_c^{\mathrm{T}}t}\right\|^2\mathrm{d}t = u^{\mathrm{T}}\int_0^\infty e^{A_c^{\mathrm{T}}t}e^{A_c t}\mathrm{d}tu = u^{\mathrm{T}}E_0u \le \|E_0\|, \tag{44}$$

where $u$ is a unit vector. Moreover,

$$\int_0^\infty \|X\|^2\left\|e^{A_c t}v\right\|^2\mathrm{d}t = v^{\mathrm{T}}\int_0^\infty e^{A_c^{\mathrm{T}}t}X^2e^{A_c t}\mathrm{d}tv = v^{\mathrm{T}}E_2v \le \|E_2\|, \tag{45}$$

where $v$ is a unit vector. Therefore, we combine (44) and (45), so

$$\|\Theta_c(Z)\| \le 2\|Z\|\|E_0\|^{\frac{1}{2}}\|E_2\|^{\frac{1}{2}}.$$

Thus,

$$\|\Theta_c\| \le 2\|E_0\|^{\frac{1}{2}}\|E_2\|^{\frac{1}{2}}.$$

### 3.2. Sensitivity of the CLTI via CAREs

We consider the perturbed CLTI (19) and take the perturbed optimal control

$$\tilde{u}(t) = -R^{-1}B^{\mathrm{T}}\tilde{X}\tilde{x}(t)$$

in (19), then obtain

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) - G\tilde{X}\tilde{x}(t),\ \tilde{x}(0) = x_0 + \Delta x_0. \tag{46}$$

By dropping the second and higher-order terms in (46), we apply the similar technique such as variation method to solve

$$\Delta\dot{x}(t) = A_c\Delta x(t) + \Delta A x(t) - G\Delta X x(t),$$

and we obtain

$$\Delta x(t) = \Psi_c(t)\Delta x_0 + \Pi_c^{-1}(\Delta A x(t)) - \Pi_c^{-1}(G\Delta X x(t)), \tag{47}$$

with $\Psi_c(t)$ and $\Pi_c^{-1}$ being the following differentiation and integral functions, respectively

$$\dot{\Psi}_c(t) = A_c\Psi_c(t), \Psi_c(0) = I, 0 \le t < \infty,$$

$$\Pi_c^{-1}(Y) = \int_0^{\infty} e^{A_c^{\mathrm{T}}t} Y e^{A_c t}\mathrm{d}t, \tag{48}$$

where $A_c$ is defined in (35).

The above relation (47) states a first-order perturbation $\Delta x(t)$ in the state vector corresponding to only one perturbation matrix $\Delta A$. From the perturbation analysis, we investigate two kinds of condition numbers according to only perturbed matrix $A$ in the CLTI (1) via CAREs (31) in the following theorem.

**Theorem 3.3.** *Using the above notations, the explicit expressions and perturbation bounds for two kinds of condition numbers in the CLTI* (1) *via CAREs* (31) *according to only perturbed matrix A are*

$$K_{2\_CLTI-CARE}^{A} = \frac{v_1}{\|x(t)\|}, \tag{49}$$

$$K_{\infty\_CLTI-CARE}^{A} = \frac{v_2}{\|x(t)\|_\infty}, 0 \le t < \infty, \tag{50}$$

where

$$v_1 = \|\Psi_c(t)\|\|x_0\| + \|\Pi_c^{-1}\|\|x(t)\|(\|A\| + \|G\|\|X\|),$$

$$v_2 = \|\Psi_c(t)\|_\infty\|x_0\|_\infty + \|\Pi_c^{-1}\|_\infty\|x(t)\|_\infty(\|A\|_\infty + \|G\|_\infty\|X\|_\infty).$$

**Proof.** We consider condition numbers of the CLTI (1) via CAREs (31) according to only perturbed matrix $A$ in 2- and $\infty$-norm defined by

$$K_{2\_CLTI-CARE}^{A} = \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|} \mid \|\Delta A\| \le \epsilon\|A\|, \|\Delta X\| \le \epsilon\|X\|, \|\Delta x_0\| \le \epsilon\|x_0\|\right\},$$

$$K_{\infty\_CLTI-CARE}^{A} = \lim_{\epsilon\to 0}\sup\left\{\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|_\infty} \mid |\Delta A| \le \epsilon|A|, |\Delta X| \le \epsilon|X|, |\Delta x_0| \le \epsilon|x_0|\right\}. \tag{51}$$

For $\epsilon$ sufficiently small, with $\|\Delta A\|/\|A\|$, $\|\Delta X\|/\|X\|$, $\|\Delta x_0\|/\|x_0\| \le \epsilon$, we take 2-norm in (47) and get

$$\|\Delta x(t)\| \le \|\Psi_c(t)\|\|\Delta x_0\| + \|\Pi_c^{-1}\|\|\Delta A\|\|x(t)\| + \|\Pi_c^{-1}\|\|G\|\|\Delta X\|\|x(t)\|,$$

$$\le \|\Psi_c(t)\|\epsilon\|x_0\| + \|\Pi_c^{-1}\|\epsilon\|A\|\|x(t)\| + \|\Pi_c^{-1}\|\|G\|\epsilon\|X\|\|x(t)\|.$$

According to the above definition of the condition number, we obtain

$$\frac{\|\Delta x(t)\|}{\epsilon\|x(t)\|} \le \frac{\|\Psi_c(t)\|\|x_0\|}{\|x(t)\|} + \|\Pi_c^{-1}\|\|A\| + \|\Pi_c^{-1}\|\|G\|\|X\|.$$

Let $\epsilon \to 0$ give

$$K_{2\_CLTI-CARE}^{A} \le \frac{v_1}{\|x(t)\|}, 0 \le t < \infty.$$

Analogously, we take $\infty$-norm in (47) by applying (51) and obtain

$$\frac{\|\Delta x(t)\|_\infty}{\|x(t)\|_\infty} \le \frac{\|\Psi_c(t)\|_\infty\|x_0\|_\infty}{\|x(t)\|_\infty} + \|\Pi_c^{-1}\|_\infty\|A\|_\infty + \|\Pi_c^{-1}\|_\infty\|G\|_\infty\|X\|_\infty.$$

Take $\epsilon \to 0$, we can get

$$K_{\infty\_CLTI-CARE}^{A} \le \frac{v_2}{\|x(t)\|_\infty}, 0 \le t < \infty.$$

To solve two kinds of condition numbers of CLTI (1) via CAREs (31) efficiently, we apply Theorem 3.2 to compute condition numbers (49) and (50) efficiently.

**Theorem 3.4.** *For* $A_c$ *and* $\Pi_c^{-1}$ *as in* (35) *and* (48), *respectively, the unique solution of the Lyapunov* Equation (42) *is represented in* (43), *then we can get*

$$\Pi_c^{-1}(I) = \int_0^\infty e^{A_c^{\mathrm{T}}t}e^{A_c t}\mathrm{d}t = E_0.$$

Thus,

$$\|\Pi_c^{-1}\| = \|E_0\| \text{ and } \|\Pi_c^{-1}\|_\infty = \|E_0\|_\infty.$$

## 4. Numerical Examples

The numerical simulations are conducted on a desktop with a 3.40 GHz Intel Core 2 Duo processor and 32 GB RAM, with machine accuracy $eps = 2.22 \times 10^{-16}$. We compute with MATLAB [38] Version R2017b.

We have chosen one example for demonstration:

1) The example 1 illustrates condition numbers and perturbation bounds of CLTI via solving RDEs and CAREs with finite and infinite time horizons, respectively to present the effectiveness of the theoretical results.

**Example 1 (CLTI)**

Consider the CLTI (1) with $n = m = 2$ and $r = 1$:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, R = I_2, C = [1, 0],$$

satisfying

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0, 0 \le t \le t_f,$$
$$y(t) = Cx(t),$$

with the optimal controls $u(t)$ chosen through minimizing the cost functional

$$J=\frac{1}{2}x^{\mathrm{T}}\left(t_f\right)Q_1x\left(t_f\right)+\frac{1}{2}\int_0^{t_f}u^{\mathrm{T}}\left(t\right)Ru\left(t\right)\mathrm{d}t.$$

In the example, the perturbed coefficient matrix is constructed such as $\tilde{E}=E+\Delta E$ and $\Delta E=10^{-j}\times randn(n,m)$, for $E\in\mathbb{K}^{n\times m}$, $10^{-j}$ being the weighted coefficient. From the tables, $\tilde{X}(t)=X(t)+\Delta X(t)$, $\tilde{x}(t)=x(t)+\Delta x(t)$ and $\tilde{X}=X+\Delta X$ are solutions of pRDEs, pCLTI and pCAREs, respectively, then we obtain relative differences of solutions between original and perturbed equations in 2- and $\infty$-norm such as $\frac{\|\Delta X(t)\|_2}{\|X(t)\|_2}$, $\frac{\|\Delta X(t)\|_\infty}{\|X(t)\|_\infty}$, $\frac{\|\Delta x(t)\|_2}{\|x(t)\|_2}$, $\frac{\|\Delta x(t)\|_\infty}{\|x(t)\|_\infty}$, $\frac{\|\Delta X\|_2}{\|X\|_2}$ and $\frac{\|\Delta X\|_\infty}{\|X\|_\infty}$ and the corresponding perturbation bounds $K_{2\_RDE}^A$, $K_{\infty\_RDE}^A$, $K_{2\_CLTI-RDE}^A$, $K_{\infty\_CLTI-RDE}^A$, $K_{2\_CLTI-CARE}^A$, $K_{\infty\_CLTI-CARE}^A$, $K_{2\_CARE}^A$ and $K_{\infty\_CARE}^A$ according to only perturbed coefficient matrix $\tilde{A}$.

Moreover, some parameters are set below:

$$\epsilon_1^{RDE}=\max\left\{\frac{\|\Delta A\|_F}{\|A\|_F},\frac{\|\Delta Q_1\|_F}{\|Q_1\|_F}\right\},$$

$$\epsilon_2^{RDE}=\min\left\{\epsilon:|\Delta A|\le\epsilon|A|,|\Delta Q_1|\le\epsilon|Q_1|,\epsilon>0\right\},$$

$$\epsilon_1^{CLTI-RDE}=\max\left\{\frac{\|\Delta A\|_F}{\|A\|_F},\frac{\|\Delta X(t)\|_F}{\|X(t)\|_F},\frac{\|\Delta x_0\|_F}{\|x_0\|_F}\right\},$$

$$\epsilon_2^{CLTI-RDE}=\min\left\{\epsilon:|\Delta A|\le\epsilon|A|,|\Delta X(t)|\le\epsilon|X(t)|,|\Delta x_0|\le\epsilon|x_0|,\epsilon>0\right\},$$

$$\epsilon_1^{CLTI-CARE}=\max\left\{\frac{\|\Delta A\|_F}{\|A\|_F},\frac{\|\Delta X\|_F}{\|X\|_F},\frac{\|\Delta x_0\|_F}{\|x_0\|_F}\right\},$$

$$\epsilon_2^{CLTI-CARE}=\min\left\{\epsilon:|\Delta A|\le\epsilon|A|,|\Delta X|\le\epsilon|X|,|\Delta x_0|\le\epsilon|x_0|,\epsilon>0\right\},$$

$$\epsilon_1^{CARE}=\left\{\frac{\|\Delta A\|_F}{\|A\|_F}\right\},$$

$$\epsilon_2^{CARE}=\left\{\epsilon:|\Delta A|\le\epsilon|A||,\epsilon>0\right\},$$

for perturbation bounds of RDEs, CLTI and CAREs, respectively; the time range is $[0,1]$ for CLTI via solving RDEs and CAREs with a division 15 parts; the parameter of the weighted coefficient is fixed into $j=3$; the terminal time is $t_f=1$.

From Table 2, we skip the relative differences of RDEs in $\infty$-norm as $\frac{\|\Delta X(t)\|_2}{\|X(t)\|_2}=\frac{\|\Delta X(t)\|_\infty}{\|X(t)\|_\infty}$. We can observe sharper perturbation bounds of the relative differences in RDEs and CLTI such as $\frac{\|\Delta X(t)\|_2}{\|X(t)\|_2}\lesssim\epsilon_1^{RDE}K_{2\_RDE}^A$,

**Table 2.** Example 1 (Condition numbers and perturbation bounds of CLTI solved by RDEs; $n=2$, $m=2$, $r=1$).

| $t$ | $\frac{\|\Delta X(t)\|_2}{\|X(t)\|_2}$ | $\epsilon_1^{RDE} K_{2\_RDE}^A$ | $\epsilon_2^{RDE} K_{\infty\_RDE}^A$ | $\frac{\|\Delta x(t)\|_2}{\|x(t)\|_2}$ | $\epsilon_1^{CLTI} K_{2\_CLTI}^A$ | $\frac{\|\Delta x(t)\|_\infty}{\|x(t)\|_\infty}$ | $\epsilon_2^{CLTI} K_{\infty\_CLTI}^A$ |
|---|---|---|---|---|---|---|---|
| 0 | 2.5870e−04 | 7.6082e−04 | 8.0831e−04 | 9.2242e−04 | 9.2242e−04 | 9.2978e−04 | 9.2978e−04 |
| 0.0667 | 2.5856e−04 | 7.6823e−04 | 8.1539e−04 | 8.6703e−04 | 1.0657e−03 | 8.7716e−04 | 1.1148e−03 |
| 0.1333 | 2.5803e−04 | 7.7703e−04 | 8.2377e−04 | 8.1119e−04 | 1.2269e−03 | 8.4001e−04 | 1.3251e−03 |
| 0.2000 | 2.5677e−04 | 7.8723e−04 | 8.3342e−04 | 7.5528e−04 | 1.3740e−03 | 8.0345e−04 | 1.5178e−03 |
| 0.2667 | 2.5430e−04 | 7.9861e−04 | 8.4408e−04 | 6.9985e−04 | 1.5005e−03 | 7.6736e−04 | 1.6842e−03 |
| 0.3333 | 2.4983e−04 | 8.1055e−04 | 8.5505e−04 | 6.4563e−04 | 1.6089e−03 | 7.3152e−04 | 1.8274e−03 |
| 0.4000 | 2.4218e−04 | 8.2175e−04 | 8.6497e−04 | 5.9361e−04 | 1.7044e−03 | 6.9567e−04 | 1.9545e−03 |
| 0.4667 | 2.2956e−04 | 8.2989e−04 | 8.7140e−04 | 5.4514e−04 | 1.7931e−03 | 6.5941e−04 | 2.0740e−03 |
| 0.5333 | 2.0947e−04 | 8.3130e−04 | 8.7058e−04 | 5.0210e−04 | 1.8820e−03 | 6.2220e−04 | 2.1966e−03 |
| 0.6000 | 1.7867e−04 | 8.2098e−04 | 8.5741e−04 | 4.6699e−04 | 1.9797e−03 | 5.8328e−04 | 2.3359e−03 |
| 0.6667 | 1.3375e−04 | 7.9373e−04 | 8.2674e−04 | 4.4295e−04 | 2.0980e−03 | 5.4158e−04 | 2.5112e−03 |
| 0.7333 | 7.2418e−05 | 7.4724e−04 | 7.7649e−04 | 4.3366e−04 | 2.2528e−03 | 4.9563e−04 | 2.7494e−03 |
| 0.8000 | 4.5173e−06 | 6.8682e−04 | 7.1256e−04 | 4.4279e−04 | 2.4637e−03 | 4.4333e−04 | 3.0829e−03 |
| 0.8667 | 9.2024e−05 | 6.3045e−04 | 6.5392e−04 | 4.7364e−04 | 2.7408e−03 | 3.9502e−04 | 3.5256e−03 |
| 0.9333 | 1.8188e−04 | 6.1786e−04 | 6.4218e−04 | 5.2947e−04 | 3.0334e−03 | 5.1072e−04 | 3.9783e−03 |
| 1.0000 | 2.6472e−04 | 7.9124e−04 | 8.2675e−04 | 6.1487e−04 | 3.0669e−03 | 6.3318e−04 | 3.9355e−03 |

$\frac{\|\Delta X(t)\|_\infty}{\|X(t)\|_\infty} \lesssim \epsilon_2^{RDE} K_{\infty\_RDE}^A$, $\frac{\|\Delta x(t)\|_2}{\|x(t)\|_2} \lesssim \epsilon_1^{CLTI} K_{2\_CLTI}^A$ and $\frac{\|\Delta x(t)\|_\infty}{\|x(t)\|_\infty} \lesssim \epsilon_2^{CLTI} K_{\infty\_CLTI}^A$.

**Table 3** shows that condition numbers of CLTI via solving CAREs are closely bounded by perturbation bounds such as

$$\frac{\|\Delta X\|_2}{\|X\|_2} = 2.5692\text{e}{-04} \lesssim 7.2261\text{e}{-04} = \epsilon_1^{CARE} K_{2\_CARE}^A,$$

$$\frac{\|\Delta X\|_\infty}{\|X\|_\infty} = 2.5692\text{e}{-04} \lesssim 7.7101\text{e}{-04} = \epsilon_2^{CARE} K_{\infty\_CARE}^A, \quad \frac{\|\Delta x(t)\|_2}{\|x(t)\|_2} \lesssim \epsilon_1^{CLTI} K_{2\_CLTI}^A$$

and $\frac{\|\Delta x(t)\|_\infty}{\|x(t)\|_\infty} \lesssim \epsilon_2^{CLTI} K_{\infty\_CLTI}^A$.

To sum up, perturbation bounds of CLTI are tight around $O(10^{-3})$ according to the weighted coefficient $10^{-3}$ whatever we solve via RDEs or CAREs.

## 5. Conclusion

We have proposed, tested and analyzed CLTI for the condition numbers and perturbation bounds according to only one perturbed coefficient matrix via solving RDEs and CAREs. Numerical simulations show that condition numbers provide tight perturbation bounds of the solutions to CLTI under some small

**Table 3.** Example 1 (Condition numbers and perturbation bounds of CLTI solved by CAREs; $n=2$, $m=2$, $r=1$).

| $t$ | $\frac{\|\Delta x(t)\|_2}{\|x(t)\|_2}$ | $\epsilon_1^{CLTI} K_{2\_CLTI}^{A}$ | $\frac{\|\Delta x(t)\|_\infty}{\|x(t)\|_\infty}$ | $\epsilon_2^{CLTI} K_{\infty\_CLTI}^{A}$ |
|---|---|---|---|---|
| 0 | 9.2242e−04 | 1.7511e−03 | 9.2978e−04 | 2.0709e−03 |
| 0.0667 | 8.6124e−04 | 1.7768e−03 | 8.8827e−04 | 2.1070e−03 |
| 0.1333 | 8.0087e−04 | 1.8030e−03 | 8.6181e−04 | 2.1434e−03 |
| 0.2000 | 7.4210e−04 | 1.8297e−03 | 8.3565e−04 | 2.1801e−03 |
| 0.2667 | 6.8599e−04 | 1.8570e−03 | 8.0980e−04 | 2.2171e−03 |
| 0.3333 | 6.3401e−04 | 1.8848e−03 | 7.8425e−04 | 2.2543e−03 |
| 0.4000 | 5.8810e−04 | 1.9132e−03 | 7.5903e−04 | 2.2917e−03 |
| 0.4667 | 5.5074e−04 | 1.9421e−03 | 7.3413e−04 | 2.3294e−03 |
| 0.5333 | 5.2480e−04 | 1.9716e−03 | 7.0956e−04 | 2.3673e−03 |
| 0.6000 | 5.1314e−04 | 2.0016e−03 | 6.8534e−04 | 2.4054e−03 |
| 0.6667 | 5.1787e−04 | 2.0322e−03 | 6.6146e−04 | 2.4437e−03 |
| 0.7333 | 5.3970e−04 | 2.0633e−03 | 6.3793e−04 | 2.4822e−03 |
| 0.8000 | 5.7779e−04 | 2.0948e−03 | 6.1476e−04 | 2.5208e−03 |
| 0.8667 | 6.3025e−04 | 2.1269e−03 | 5.9196e−04 | 2.5596e−03 |
| 0.9333 | 6.9480e−04 | 2.1595e−03 | 7.0861e−04 | 2.5984e−03 |
| 1.0000 | 7.6931e−04 | 2.1926e−03 | 8.3791e−04 | 2.6374e−03 |

change in the only one coefficient matrix. In summary, we introduce some efficient measurement tools for the sensitivity analysis of CLTI via solving RDEs and CAREs respectively.

## Acknowledgements

This work was supported by Academia Sinica (Taiwan) grant number 103-CDA-M04, and Ministry of Science and Technology (Taiwan) grant numbers 104-2118-M-001-016-MY2 and 105-2118-M-001-007-MY2.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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