Advances in Pure Mathematics, 2020, 10, 1-11
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Approximation by Complex Meyer-König and Zeller Operators

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How to cite this paper: Qi, Q.L. and Ma, J.S. (2020) Approximation by Complex Meyer-König and Zeller Operators. *Advances in Pure Mathematics*, **10**, 1-11.
https://doi.org/10.4236/apm.2020.101001

Received: November 25, 2019
Accepted: January 4, 2020
Published: January 7, 2020

Abstract

The Meyer-König and Zeller operator is one of the most challenging operators. Sometimes the study of its properties will rely on the weighted approximation by Baskakov operator. In this paper, this relation is extended to complex space; the quantitative estimates and the Voronovskaja type results for analytic functions by complex Meyer-König and Zeller operators were obtained.

Keywords

Complex Meyer-König and Zeller Operators, Complex Baskakov Operators, Voronovskaja Type Result, Analytic Function

1. Introduction

The well known Meyer-König and Zeller operators are defined for functions $f(x) \in C[0,1)$ by [1]-[7]

$$M_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x),$$

where $m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}$.

The Meyer-König and Zeller operators [1]-[7], the Durrmeyer-type [8]-[15] have been the object of several investigations in approximation theory. The estimation of moments, the direct and inverse approximation properties were studied. Recently, many new modified types [12]-[19] have been constructed for different function spaces. Gal, Mahmudov, Opris etc. [16] [17] [18] [19] obtained the quantitative approximation estimates by complex Bernstein-type, Szász-type operators in compact disks.

The goal of this paper is to extend the results to complex Meyer-König and Zeller operators defined as follows: For analytic functions $f:\bar{D}_R\cup[R,1)\to C, 0\le R<1$,

$$M_n(f,z)=\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)m_{n,k}(z),$$

where $m_{n,k}(z)=\binom{n+k}{k}z^k(1-z)^{n+1}, D_R=\{z\in C:|z|<R\}$.

We will obtain the following estimates for the complex Meyer-König and Zeller operators.

Theorem 1. *Suppose that* $f:\bar{D}_R\cup[R,1)\to C$ *is analytic in* $\bar{D}_R$ *and continuous in* $[R,1)$, *that is,* $f(z)=\sum_{p=0}^{\infty}c_pz^p$, *for all* $z\in\bar{D}_R$. *Let* $\sqrt{2}-1\le r<R<1$, *for all* $|z|\le r$ *and* $n\ge 2$, *we have*

$$|M_n(f,z)-f(z)|\le\frac{M_r(f)}{n},$$

where $M_r(f)=\sum_{p=1}^{\infty}|c_p|(2p)!r^{p-1}<+\infty$.

Theorem 2. *Under the conditions of Theorem* 1, *for all* $|z|\le r$ *and* $n\ge 2$, *we have the following Voronovskaja type results*

$$\left|M_n(f,z)-f(z)-\frac{z}{2n}f''(z)\right|\le\frac{N_r(f)}{n^2},$$

where

1) $N_r(f)=\sum_{p=1}^{\infty}|c_{p+1}|5p^2(2p)!r^{p-1}<+\infty$, for $\sqrt{2}-1\le r\le\frac{\sqrt{5}-1}{2}$;

2) $N_r(f)=\sum_{p=1}^{\infty}|c_{p+1}|5p^2(2p)!r^{p+1}<+\infty$, for $\frac{\sqrt{5}-1}{2}\le r<1$.

Theorem 3. *Under the hypothesis of Theorem* 2, *if f is not a polynomial of degree* ≤ 1 *and the series* $N_r(f)<+\infty$, *then for* $\sqrt{2}-1\le r<R<1$, *we have*

$$\|M_n(f,z)-f(z)\|_r\sim\frac{1}{n}, n\in N,$$

here $\|f\|_r=\sup\{|f(z)|:z\in D_r\}$.

The paper is organized as the following: In Section 2, we are going to promote the relationship between the Meyer-König and Zeller and Baskakov operators to complex space. In Section 3, we will study the approximation by the complex Baskakov operators. In Section 4, we will give the proof of Theorems 1 - 3. In Section 5, we will give the conclusion of this paper.

2. The Connection between the Complex Meyer-König and Zeller and Baskakov Operators

The proof is based on the connection between Meyer-König and Zeller and Baskakov operators. V. Totik was the first to use it [2] in the study of Meyer-König and Zeller operators, and many other afterward, see e.g. [6] [7] [8] [9] [11]. In this section, the connection will be extended to complex space. We will study a transformation τ mapping functions defined on $D_l = \left\{ t \in C : |t - t_0| < l, 0 \le l < +\infty, Ret > -\frac{1}{2} \right\}$ into functions defined on $D_r = \{ z \in C : |z| < r, 0 \le r < 1 \}$. The operator τ will allow us to relate the results for the complex Baskakov operators to their counterparts for the complex Meyer-König and Zeller operators. We will consider variables and functions defined on D_r as $z, f(z)$ respectively, and their analogs defined on D_l, as the later will be denoted with $t, g(t)$. We consider the weight functions

$$w_1(z) = w_1(\alpha_0, \alpha_1, z) = z^{\alpha_0}(1-z)^{\alpha_1}, z \ne 0, 1, z \in D_r$$

defined for real values of the parameters $\alpha_0, \alpha_1 \in [-1, 0]$. We will utilize the change $\sigma : D_r \to D_l$ given by

$$t = \sigma(z) = \frac{z}{1-z}, z \in D_r.$$

Remark 1.

$$\sigma : D_r = \{ z \in C : |z| < r, 0 \le r < 1 \} \to D_l = \left\{ t \in C : |t - t_0| < l, 0 \le l < +\infty, Ret > -\frac{1}{2} \right\},$$

where $l = \frac{r}{1-r^2}$, $t_0 = \frac{r^2}{1-r^2}$. For example:

$$\sigma : D_{\frac{1}{2}} = \left\{ z \in C : |z| < \frac{1}{2} \right\} \to D_{\frac{2}{3}} = \left\{ t \in C : \left| t - \frac{1}{3} \right| < \frac{2}{3} \right\}.$$

Then, its inverse change $\sigma^{-1} : D_l \to D_r$ is

$$z = \sigma^{-1}(t) = \frac{t}{1+t}. \quad (1)$$

Remark 2. From the definition of σ and σ^{-1}, we have that the change σ and σ^{-1} are linear fractional transformations and conformal mappings.

A function g defined on D_l is transformed to a function f defined on D_r by $\tau : g \to f$

$$f(z) = \tau(g)(z) = \lambda(z)(g \circ \sigma)(z), \lambda(z) = 1 - z. \quad (2)$$

The inverse operator τ^{-1} transforming a function f defined on D_r to a function g defined on D_l is $\tau^{-1} : f \to g$

$$g(t) = \tau^{-1}(f)(t) = \frac{1}{(\lambda \circ \sigma^{-1})(t)}(f \circ \sigma^{-1})(t), t \ne -1. \quad (3)$$

When a product of two functions is treated, that means, the associated operator Υ is defined by

$$\Upsilon : w_1(z) = \Upsilon(w)(z) = \frac{1}{\lambda(z)}(w \circ \sigma)(z), \quad (4)$$

and its inverse Υ^{-1} is defined by

$$\Upsilon^{-1} : w(t) = \Upsilon^{-1}(w_1)(t) = (\lambda \circ \sigma^{-1})(t)(w_1 \circ \sigma^{-1})(t).$$

For $f = \tau(g)$, $w_1 = \Upsilon(w)$, we have

$$w_1 f = \Upsilon(w)\tau(g) = (w\circ\sigma)(g\circ\sigma),$$
$$wg = \Upsilon^{-1}(w_1)\tau^{-1}(f) = (w_1\circ\sigma^{-1})(f\circ\sigma^{-1}). \tag{5}$$

The operators τ and Υ have the following properties. From the definition (1)-(3), we yield immediately.

Proposition 1. *Let* F_r, F_l *denote the spaces of all functions defined on* D_r *and* D_l *respectively. Then* $\tau: F_l \to F_r$ *and* τ^{-1} *are linear operators.*

Proposition 2. *Let* w_1 *be a weight in* D_r, $w = \Upsilon^{-1}(w_1)$,

$$F_{w_1} = \{f \in F_r : w_1 f \in L_\infty(D_r)\};$$
$$F_w = \{g \in F_l : wg \in L_\infty(D_l)\}.$$

Then the mapping $\tau: F_w \to F_{w_1}$ is a linear correspondence with $\|w_1\tau(g)\|_r = \|wg\|_l$, $\|w\tau^{-1}(f)\|_l = \|w_1 f\|_r$.

Proof. From the definition of the mapping τ (2) and the operator Υ (4), combining the Proposition 1, we get the mapping $\tau: F_w \to F_{w_1}$ is a linear correspondence.

Noting that the relation

$$w_1\tau(g)(z) = \frac{1}{\lambda(z)}(w\circ\sigma)(z)\cdot\lambda(z)(g\circ\sigma)(z) = w(t)g(t),$$

one can get the desired result.

The following proposition is very important, it gives the connection between the complex Meyer-König and Zeller operators and the complex Baskakov operators

$$V_n(g,t) = \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)v_{n,k}(t),$$

where $v_{n,k}(t) = \binom{n+k-1}{k}t^k(1+t)^{-n-k}$.

Proposition 3. *For every f such that one of the series in* (6) *is convergent, for every* $n\in N$, *we have*

$$M_n(f,z) = \tau\left(V_n\left(\tau^{-1}(f)\right)\right)(z), z\in D_r. \tag{6}$$

Proof. From the definition of the operator $V_n(g,t), M_n(f,z)$, Proposition 1 and the identities

$$\frac{n+k}{n}\tau(v_{n,k})(z) = m_{n,k}(z),$$
$$\tau^{-1}(f)\left(\frac{k}{n}\right) = \frac{n+k}{n}f\left(\frac{k}{n+k}\right)$$

valid for $k \in N\cup\{0\}$, we have (6).

Proposition 4. *Under the conditions of Proposition* 3, *we have*

$$\|w_1(M_n f - f)\|_r = \|w(V_n g - g)\|_l.$$

Proof. From Proposition 3, relations ((4), (1), (3)), we obtain for $g = \tau^{-1} f$

and $w = \Upsilon^{-1} w_1$,

$$w_1(M_n f - f) = (w(V_n g - g)) \circ \sigma$$

and hence

$$\|w_1(M_n f - f)\|_r = \|w(V_n g - g)\|_l .$$

Remark 3. If the weight $w_1(z) = 1$ (*i.e.* $\alpha_0 = \alpha_1 = 0$), the corresponding weight to $w_1(z) = 1$ is $w(t) = \frac{1}{1+t}$.

Then, we have the following auxiliary results.

Lemma 2.1. *Under the conditions of Proposition* 3, $w_1(z) = 1$, $w(t) = \frac{1}{1+t}$, *we have*

$$\|M_n f - f\|_r = \|w(t)(V_n g - g)\|_l .$$

Lemma 2.2. [16] *Denoting* $e_p(t) = t^p$ *and* $T_{n,p}(t) = V_n(e_p, t)$, $T_{n,p}(t)$ *is a polynomial of degree p*, $p = 0,1,2,\cdots$, *we have the recurrence formula*

$$T_{n,p+1}(t) = \frac{t(t+1)}{n} T'_{n,p}(t) + t T_{n,p}(t).$$

3. Weighted Approximation by the Complex Baskakov Operators

Theorems 1 - 3 will be proved in Section 4 by transferring the corresponding results for the complex Baskakov operators. In this section, we will prove some properties of the complex Baskakov operators. The first main result of this section is the following theorem for upper bound.

Theorem 3.1. *Suppose that* $g : \bar{D}_L \cup [L, +\infty) \to C$ *is continuous in* $\bar{D}_L \cup [L, +\infty)$ *and analytic in* $\bar{D}_L$, *i.e.* $g(t) = \sum_{p=0}^{\infty} c_p t^p$. *Let* $\frac{1}{2} \le l < L < +\infty$, *for all* $|t| \le l, n \ge 2$, *we have*

$$\|w(t)(V_n(g,t) - g(t))\|_l \le \frac{M_l(g)}{n},$$

where $M_l(g) = \sum_{p=1}^{\infty} |c_p| (2p)! l^{p-1} < +\infty, w(t) = \frac{1}{1+t}$.

Proof. By using the recurrence relation of Lemma 2.2, for all $t \in C, p = 0,1,2,\cdots, n \ge 2$, we have

$$T_{n,p+1}(t) = \frac{t(t+1)}{n} T'_{n,p}(t) + t T_{n,p}(t).$$

From this we immediately get the recurrence formula

$$w(t)(T_{n,p}(t) - t^p) = \frac{t}{n}(T_{n,p-1}(t) - t^{p-1})' + t\left[w(t)(T_{n,p-1}(t) - t^{p-1})\right] + \frac{p-1}{n} t^{p-1}.$$

To estimate $\|w(t)(T_{n,p}(t) - e_p(t))\|_l$, we wil use the relation [16] p. 7:

$\left|B_k'(t)\right| \le \frac{k}{l}\left\|B_k\right\|_l$ for all $|t| \le l$, where $B_k(t)$ is a polynomial of degree $\le k$. Then, we get

$$\begin{aligned}&\left\|w(t)\left(T_{n,p}(t)-e_p(t)\right)\right\|_l \\ &\le \frac{l}{n}\left\|T_{n,p-1}(t)-e_{p-1}(t)\right\|_l \frac{p-1}{l} + l\left\|w(t)\left(T_{n,p-1}(t)-e_{p-1}(t)\right)\right\|_l + \frac{p-1}{n}l^{p-1},\end{aligned}$$

which implies

$$\begin{aligned}&\left\|w(t)\left(T_{n,p}(t)-e_p(t)\right)\right\|_l \\ &\le \left(\frac{3l(p-1)}{n}+l\right)\left\|w(t)\left(T_{n,p-1}(t)-e_{p-1}(t)\right)\right\|_l + \frac{p-1}{n}l^{p-1}.\end{aligned} \tag{7}$$

We will prove the following relation by mathematical induction with respect to *p*:

$$\left\|w(t)\left(T_{n,p}(t)-e_p(t)\right)\right\|_l \le \frac{(2p)!}{n}l^{p-1}.$$

Indeed for $p=1$, $\left\|w(t)\left(T_{n,1}(t)-e_1(t)\right)\right\|_l = 0 \le \frac{2}{n}$. Suppose that it is true for $p>1$, that is,

$$\left\|w(t)\left(T_{n,p}(t)-e_p(t)\right)\right\|_l \le \frac{(2p)!}{n}l^{p-1}. \tag{8}$$

Now for $p+1$, by the relations ((7), (8)), we have

$$\left\|w(t)\left(T_{n,p+1}(t)-e_{p+1}(t)\right)\right\|_l \le \left(\frac{3lp}{n}+l\right)\frac{(2p)!}{n}l^{p-1} + \frac{p}{n}l^p.$$

It remains to prove that for $n \ge 2$

$$\left(\frac{3lp}{n}+l\right)\frac{(2p)!}{n}l^{p-1} + \frac{p}{n}l^p \le \frac{(2(p+1))!}{n}l^p.$$

By mathematical induction that the last inequality holds true for all $p \ge 1$ and $n \ge 2$. From the hypothesis on *g*, it follows that $V_n(g,t)$ is analytic in D_l, we write

$$\left\|w(t)\left(V_n(g,t)-g(t)\right)\right\|_l \le \sum_{p=1}^{\infty}\left|c_p\right| \cdot \left\|w(t)\left(T_{n,p}(t)-e_p(t)\right)\right\|_l \le \sum_{p=1}^{\infty}\left|c_p\right|\frac{(2p)!}{n}l^{p-1}.$$

Theorem 3.2. *Under the conditions of Theorem* 3.1, *let* $\frac{1}{2} \le l < L < +\infty$, *for all* $|t| \le l, n \ge 2$, *we have the following Voronovskaja type formula*

$$\left|w(t)\left(V_n(g,t)-g(t)-\frac{t(1+t)}{2n}g''(t)\right)\right| \le \frac{N_l(g)}{n^2},$$

where

1) for $\frac{1}{2} \le l < L < 1$, $N_l(g) = \sum_{p=1}^{\infty}\left|c_{p+1}\right|5p^2(2p)!l^{p-1} < +\infty$;

2) for $1 \le l < L < +\infty$, $N_l(g) = \sum_{p=1}^{\infty} |c_{p+1}| 5p^2 (2p)! l^{p+1} < +\infty$.

Proof. Case I. For $\frac{1}{2} \le l < L < 1$, noting that $e_p(t) = t^p, p = 0,1,2,\cdots$ and $T_{n,p}(t) = V_n(e_p,t)$ and $V_n(g,t) = \sum_{p=0}^{\infty} c_p V_n(e_p,t)$, we have

$$\left| w(t)\left(V_n(g,t) - g(t) - \frac{t(1+t)}{2n} g''(t) \right) \right|$$
$$\le \sum_{p=1}^{\infty} |c_p| \left| w(t)\left(T_{n,p}(t) - e_p(t) - \frac{p(p-1)(1+t)}{2n} t^{p-1} \right) \right|.$$

Using the recurrence relation of Lemma 2.2, we write

$$T_{n,p+1}(t) = \frac{t(t+1)}{n} T'_{n,p}(t) + tT_{n,p}(t).$$

Denote that

$$E_{n,p}(t) = T_{n,p}(t) - e_p(t) - \frac{p(p-1)(1+t)}{2n} t^{p-1}.$$

Noting that $T_{n,1}(t) - e_1(t) = 0$, for $p \ge 2$, we have

$$E'_{n,p}(t) = \frac{n}{t(1+t)} T_{n,p+1}(t) - \frac{n}{1+t} T_{n,p}(t) - pt^{p-1} - \frac{p^2(p-1)}{2n} t^{p-1} - \frac{p(p-1)^2}{2n} t^{p-2}.$$

By simple computation, we get

$$E_{n,p+1}(t) = \frac{t(1+t)}{n} E'_{n,p}(t) + tE_{n,p}(t) + \frac{p^2(p-1)(1+t)}{2n^2} t^p + \frac{p(p-1)^2(1+t)}{2n^2} t^{p-1}.$$

Thus, for all $p, n \in N$, $|t| < l$, $\frac{1}{2} \le l < L < 1$, we have

$$|w(t)E_{n,p+1}(t)| \le \frac{l(1+l)}{n} |w(t)E'_{n,p}(t)| + l|w(t)E_{n,p}(t)| + \frac{2p^3}{n^2} l^{p-1}. \tag{9}$$

Using the estimate in the proof of Theorem 3.1, for all $p \in N, n \ge 2$ and $\frac{1}{2} \le l < L < 1$, we have

$$\|w(t)(T_{n,p}(t) - e_p(t))\|_l \le \frac{(2p)!}{n} l^{p-1}.$$

Now we shall estimate $|w(t)E'_{n,p}(t)|$ for $p \ge 2$. Noting that $E_{n,p}(t)$ is a polynomial of degree $\le p$, combining the Bernstein's inequality, we have

$$|w(t)E'_{n,p}(t)| \le \frac{p}{l} \|w(t)E_{n,p}(t)\|_l$$
$$\le \frac{p}{l} \left[\|w(t)(T_{n,p}(t) - e_p(t))\|_l + \left\| w(t) \frac{p(p-1)(1+t)}{n} t^{p-1} \right\|_l \right]$$
$$\le \frac{p}{l} \left(\frac{(2p)!}{n} l^{p-1} + \frac{p^2}{n} l^{p-1} \right) \le \frac{2p(2p)!}{n} l^{p-2},$$

thus,

$$\frac{l(1+l)}{n}\left|w(t)E'_{n,p}(t)\right| \le \frac{4p(2p)!}{n^2} l^{p-1},$$

$$\left|w(t)E_{n,p+1}(t)\right| \le \frac{4p(2p)!}{n^2} l^{p-1} + l\left|w(t)E_{n,p}(t)\right| + \frac{2p^3}{n^2} l^{p-1}, \tag{10}$$

$$\left|w(t)E_{n,p+1}(t)\right| \le \frac{5p(2p)!}{n^2} l^{p-1} + l\left|w(t)E_{n,p}(t)\right|,$$

we obtain step by step following

$$\left|w(t)E_{n,p+1}(t)\right| \le \frac{5pl^{p-1}}{n^2}\sum_{j=1}^{p}(2j)! \le \frac{5p^2(2p)!}{n^2} l^{p-1},$$

which follows that

$$\left|w(t)\left(V_n(g,t)-g(t)-\frac{t(1+t)}{2n}g''(t)\right)\right| \le \frac{N_l(g)}{n^2},$$

where $N_l(g)=\sum_{p=2}^{\infty}\left|c_p\right|5(p-1)^2\left[2(p-1)\right]!l^{p-2} < +\infty$.

Case 2. For $1 \le l < L$, in the proof of Case 1, the relation (9) should be changed to

$$\left|w(t)E_{n,p+1}(t)\right| \le \frac{l(1+l)}{n}\left|w(t)E'_{n,p}(t)\right| + l\left|w(t)E_{n,p}(t)\right| + \frac{2p^3}{n^2} l^{p+1},$$

and the relation (10) should be changed to

$$\begin{aligned}\left|w(t)E_{n,p+1}(t)\right| &\le \frac{4p(2p)!}{n^2} l^{p-1} + l\left|w(t)E_{n,p}(t)\right| + \frac{2p^3}{n^2} l^{p+1} \\ &\le \frac{5p(2p)!}{n^2} l^{p+1} + l\left|w(t)E_{n,p}(t)\right|,\end{aligned}$$

then,

$$N_l(g)=\sum_{p=1}^{\infty}\left|c_{p+1}\right|5p^2(2p)!l^{p+1} < +\infty.$$

4. The Proof of Theorems 1 - 3

The Proof of Theorem 1. Combining Lemma 2.1 and Theorem 3.1, we can obtain Theorem 1.

The Proof of Theorem 2. From Lemma 2.1 and Theorem 3.2, we have Theorem 2.

In what follows we obtain the exact degree in the approximation by $M_n(f,z)$.

Theorem 4.1. *Suppose that the hypothesis on the function f and Theorem 2. If f is not a polynomial of degree* ≤ 1 *and the series* $N_r(f) < +\infty$, *then* $\left\|M_n f - f\right\|_r \ge \frac{C_r(f)}{n}$ *holds, where* $C_r(f)$ *depends only on f and r.*

Proof. For all $|z| \le r$, we can write

$$M_n(f,z)-f(z)=\frac{1}{n}\left[zf''(z)+\frac{1}{n}n^2\left(M_n(f,z)-f(z)-\frac{z}{n}f''(z)\right)\right].$$

Applying the inequality $\|F+G\|_r \ge \left|\|F\|_r-\|G\|_r\right| \ge \|F\|_r-\|G\|_r$, we obtain

$$\|M_n(f,z)-f(z)\|_r \ge \frac{1}{n}\left[\|zf''(z)\|_r-\frac{1}{n}n^2\left\|\left(M_n(f,z)-f(z)-\frac{z}{n}f''(z)\right)\right\|_r\right].$$

Since f is not a polynomial of degree ≤ 1 in D_R, we get $\|zf''(z)\|_r>0$. Indeed, supposing the contrary, it follows that $zf''(z)=0$ for all $|z|\le r$, which implies $f''(z)=0$ for all $z\in \bar{D}_R\setminus\{0\}$. Since f is analytic in $\bar{D}_R$, this means that $f''(z)=0$ for all $z\in\bar{D}_R$, that is f is a polynomial of degree ≤ 1, a contradiction with the hypothesis.

Now by Theorem 2, for $N_r(f)<+\infty$, we have

$$n\left|M_n(f,z)-f(z)-\frac{z}{2n}f''(z)\right|\le\frac{N_r(f)}{n}\to 0\ (n\to\infty).$$

Choose n_1, such that for all $n\ge n_1$, we have

$$\|zf''(z)\|_r-\left\|n\left(M_n(f,z)-f(z)-\frac{z}{n}f''\right)(z)\right\|_r\ge\frac{1}{2}\|zf''(z)\|_r,$$

which implies for all $n\ge n_1$,

$$\|M_n(f,z)-f(z)\|_r\ge\frac{1}{2n}\|zf''(z)\|_r.$$

For $1\le n\le n_1-1$, we have

$$\|M_n(f,z)-f(z)\|_r\ge\frac{1}{n}n\|M_n(f,z)-f(z)\|_r=\frac{C_{n,r}(f)}{n}>0,$$

i.e. $\|M_n(f,z)-f(z)\|_r\ge\frac{C_r(f)}{n}$, here

$C_r(f)=\min\left\{C_{1,r}(f),C_{2,r}(f),\cdots,C_{n_1-1,r}(f),\frac{1}{2}\|zf''(z)\|_r\right\}$.

The Proof of Theorem 3. From Lemma 2.1, Theorem 4.1 and Theorem 1, we can obtain Theorem 3.

5. Conclusion

In this paper, the properties of approximation are studied by using the general relation between the Meyer-König and Zeller and Baskakov operators. The geometric properties (the shap-preserving) of such complex operators still remain to be studied.

Acknowledgements

We thank the Editor and the referee for their comments. The work is partially supported by NSF of China (11571089, 11871191) and NSF of Hebei Province (2012205028; ZD2019053). The project supported by science foundation of Hebei Normal University.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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