

Some Extensions on Numbers

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Abstract

My previous work dealt finding numbers which relatively prime to factorial value of certain number, high exponents and also find the way for finding mod values on certain number's exponents. Firstly, I retreat my previous works about Euler's phi function and some works on Fermat's little theorem. Next, I construct exponent parallelogram to find coherence numbers of Euler's phi functioned numbers and apply to Fermat's little theorem. Then, I test the primality of prime numbers on Pascal's triangle and explore new ways to construct Pascal's triangle. Finally, I find the factorial value for certain number by using exponent triangle.

Keywords

Factorial, Fermat's Little Theorem, Fermat's Last Theorem, Euler's Totient Function, Totient Function of n th Factorial, Totient Function of n th Exponent, Division on Exponents, Prime Bases on Fermat's Last Theorem, Exponent Parallelogram, Addition Triangle, Difference Triangle, Multiplication Triangle, Division Triangle, Exponent Triangle

1. Introduction

We know **Fermat's** little theorem and **Euler's** φ (phi) function. Such are well defined operations on number theory and algebra. **Euler's** φ (phi) function is considered as general proof of Fermat's little theorem. We seek other ways to find mod values on Fermat's little theorem, and generalize φ (phi) function for a certain integer's exponentiation and factorial value. We construct the exponent parallelogram to find the coherence values of Euler's φ (phi) function. We find higher valued exponents on Fermat's little theorem according to this. We also specify Fermat's last theorem by using prime numbers. Also we know binomial coefficients are constructing Pascal's triangle, in which we see

the divisibility of prime numbers (primality test) in prime number exponentiation on Pascal's triangle. In addition, we construct Pascal's triangle and seek other ways except for binomial coefficients, *i.e.* and construct Pascal's triangle by arithmetic operations triangle. Finally instead of binomial coefficients in Pascal' triangle, we use exponents value of certain integer to construct Pascal's triangle, and then use " n "th expansion to find factorial of such certain number.

First Blaise Pascal (1623-1662) introduced Pascal's triangle, after that, **Isaac Newton (1643-1727)** used the facts of Pascal's triangle he developed binomial expansion. He and his followers used binomial theorem for Probability and Statistical problems. Factorial were used to count permutations at as early as the 12th century, by Indian scholars. In 1677, Fabian Stedman described factorial as applied to change ringing, a musical art involving the ringing of many tuned bells. In his words "Now the nature of these methods is such that the change of one number comprehends (includes) changes on lesser numbers". In that mean period, **James Stirling (1692-1770)** first introduced one approximation for finding n th factorial of a certain number. Then **Adrien-Marrie Legendre** used **Leonhard Euler's (1707-1783)** second integral formula and notated a symbol for it and then named it as Gamma function. It was a good approximation finding factorial of Real numbers. **Jacques Philippe Marie Binet (1786-1856)**, modified **James Stirling's** approximation. Finally, the notation $n!$ was introduced by the French mathematician **Christian Kramp** in 1808. **Pierre de Fermat (1601-1665)** stated Little theorem, for any two relatively prime numbers, in which exponent should be prime number; after that **Leonhard Euler (1707-1783)** found Totient function and then generalized Fermat's little theorem for any two relatively prime numbers.

From this book "Prime numbers a computational Perspective" [1], we know prime numbers and primality test. From this paper "Fermat's little theorem" [2], we know various types of explanations about Fermat's little theorem.

Propositions 2 to 6 are worked by me. They are noted as PRB which means **Prema. R. Balasubramani [3]**. They are published in **Fermat's theorem one extension: Mathematical Sciences International Journal ISSN 2278-8697 VOLUME 8 ISSUE 1 (JUNE 2019), P. 6-10.**

In this paper,

- 1) I retreat my previous work Fermat's theorem one extension. Here I extend my works to finding the coherence numbers (constructing exponent parallelogram) for Euler's phi function and then generalize it for Fermat's little theorem.
- 2) I test the primality of prime numbers on Pascal's triangle.
- 3) I specify Fermat's last theorem by prime numbers.
- 4) I generate Pascal's triangle by arithmetic operations.
- 5) I find factorial value for certain number by using exponent triangle.

2. Discussion and Results

Hint 1: Define $G_p = \{r \in N \mid 0 < r < pn \text{ and } \gcd(n, r) = 1\}$. Then $|G_p| = p\Phi(n)$.

Hint 2: Define $G_p = \{r \in N \mid (p-1)n < r < pn \text{ and } \gcd(n, r) = 1\}$. Then $|G| = (p-1)\Phi(n)$.

2.1. Let's Now Examine $\varphi(pn)$ When p Is a Factor of n

Lemma 1: Let p be a prime and p divides n , then $\Phi(pn) = p\Phi(n)$.

Proof: Notice that all the numbers that are relatively prime to pn are also relatively prime to n . since $\gcd(pn, n) = n$ and p divides n the following result follows: $\gcd(n, r) = 1$ if and only if $\gcd(pn, r) = 1$ for any natural number r .

There are p intervals, each with $\Phi(n)$ numbers relatively prime to pn , hence by the hint 1: the set $G_p = \{r \in N \mid 0 < r < pn \text{ and } \gcd(n, r) = 1\}$ has $|G_p| = p\Phi(n)$ elements. \square

For our example we choose 20, so let's consider 2×20 ; $\Phi(2 \times 20) = 2\Phi(20) = 2 \times 8 = 16$. Putting together the two sets mentioned in our previous examples we have $\{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39\}$, exactly all 16 numbers are relatively prime to 40.

2.2. Let's Now Examine $\varphi(pn)$ When p Is Not a Factor of n

Lemma 2: Let p be a prime and p does not divide n , then $\Phi(pn) = (p-1)\Phi(n)$.

Proof: We know that $p\Phi(n)$ is the number of numbers relatively prime to n and less than pn . Notice that all the multiples of p whose factors are relatively prime to n are counted, since $\gcd(p, n) = 1$. Notice the conditions imply $\gcd(pn, r) = 1$ iff $\gcd(n, r) = 1$ and $\gcd(p, r) = 1$.

Suppose the list of multiples is $\{r_1p, r_2p, r_3p, \dots, r_{\Phi(n)}p\}$, where all the r 's are relatively prime to n . the set has $\Phi(n)$ numbers relatively prime to n and 0 relatively prime to p , because they are all multiples of p . we subtract this many from our original count and we have $\Phi(pn) = p\Phi(n) - \Phi(n) = (p-1)\Phi(n)$. \square

For our examples we choose 20, so let's consider 3×20 ; $\Phi(3 \times 20) = (3-1)\Phi(20) = 2 \times 8 = 16$. Putting together the two sets mentioned in our previous examples we have $\{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\}$ exactly all 16 numbers relatively prime to 60.

Proposition 1: Let n be a positive integer. Then $\varphi(n!) = n\varphi((n-1)!)$ when n is composite number and $\varphi(n!) = (n-1)\varphi((n-1)!)$ when n is prime number.

Proof: Let n be a positive integer.

When n be a composite number and n divides $(n-1)!$, then

$$\varphi(n!) = \varphi[n(n-1)!] = n\varphi[(n-1)!].$$

Notice that all the numbers that are relatively prime to $n(n-1)!$ are also relatively prime to $(n-1)!$. Since $\gcd[n(n-1)!, (n-1)!] = (n-1)!$ And n divides

$(n-1)!$ The following result follows: $\gcd[(n-1)!, r] = 1$ if and only if $\gcd[n(n-1)!, r] = 1$ for any natural number r .

There are n intervals, each with $\varphi((n-1)!)$ numbers relatively prime to $n(n-1)!$, hence by the hint 1: the set

$G_n = \{r \in N \mid 0 < r < n(n-1)! \text{ and } \gcd((n-1)!, r) = 1\}$ has $|G_n| = n\varphi((n-1)!)$ elements.

When n be a prime and n does not divide $(n-1)!$, then $\varphi(n!) = (n-1)\varphi[(n-1)!]$.

We know that $n\varphi((n-1)!)$ is the number of numbers relatively prime to $(n-1)!$ and less than $n(n-1)!$. Notice that all the multiples of n whose factors are relatively prime to $(n-1)!$ are counted, since $\gcd[n, (n-1)!] = 1$. Notice the conditions imply $\gcd[n(n-1)!, r] = 1$ if and only if $\gcd[(n-1)!, r] = 1$ and $\gcd(n, r) = 1$.

Suppose the list of multiples is $\{r_1n, r_2n, r_3n, \dots, r_{\varphi((n-1)!)}n\}$, where all the r 's are relatively prime to $(n-1)!$. The set has $\varphi((n-1)!)$ numbers relatively prime to $(n-1)!$ and 0 relatively prime to n , because they are all multiples of n . by this way we get $\varphi(n!) = \varphi[n(n-1)!] = \varphi(n)\varphi[(n-1)!] = (n-1)\varphi[(n-1)!]$.

Preposition 2 (PRB): Let n be a positive integer. Then $\varphi(n!) = \prod n_i(n-1)_i \varphi(2)$ where n_i 's are composite numbers and $(n-1)_i$'s are prime numbers not exceeding n .

Proof:

Using preposition 1, we obtained $\varphi(n!) = n\varphi((n-1)!)$ when n is composite number and

$\varphi(n!) = (n-1)\varphi[(n-1)!]$ when n is prime number. Since all even numbers are composites except 2 because 2 is prime. So we cannot find an even composite number less than four. And two is the only prime number less than three. Also 1 is the only number relatively prime to two and below it. So we obtained from these two equations we get

$$\varphi(n!) = \prod n_i(n-1)_i \varphi(2) \tag{1}$$

Example 1: Find the value of $\varphi(8!)$

Solution:

Let $\varphi(8!)$ then we can write $\varphi(8!) = \varphi(8 \times 7!)$. So

$$\varphi(8 \times 7!) = 8\varphi(7 \times 6!) = 8\varphi(7)\varphi(6!) = 8 \times 6 \times \varphi(6!) = 8 \times 6 \times 192 = 9216.$$

Example 2: Find the value of $\varphi(13!)$

Solution:

Let $\varphi(13!)$ then $\varphi(13!) = \varphi(13 \times 12!)$. So

$$\begin{aligned} \varphi(13 \times 12!) &= 12\varphi(12!) \\ &= 12 \times 12 \times 10 \times 10 \times 9 \times 8 \times 6 \times 6 \times 4 \times 4 \times 2 \times \varphi(2) \\ &= 1194393600 \end{aligned}$$

Proposition 3 (PRB): Let n be a positive integer and “ a ” be an exponent to n . Then $\varphi(n^a) = n^{a-1}\varphi(n)$.

Proof: The positive integers less than n^a that are not relatively prime to n are those integers not exceeding n^a that are divisible by n . There are exactly n^{a-1} such integers, so there are $n^{a-1}\varphi(n)$ integers less than n^a that are relatively prime to n^a .

Hence,

$$\varphi(n^a) = n^{a-1}\varphi(n) \tag{2}$$

Example 4: Find the value of $\varphi(10^4)$.

Solution:

Let $\varphi(10^4)$ then $\varphi(10^4) = \varphi(10^3 \times 10) = 10^3\varphi(10) = 1000 \times 4 = 4000$.

Since 10 is a composite, $10^4 = 10,000$ so $\varphi(10,000) = 4000$.

Example 5: Find the value of $\varphi(331^5)$.

Solution:

Let $\varphi(331^5)$ then

$$\begin{aligned} \varphi(331^5) &= \varphi(331^4 \times 331) = 331^4\varphi(331) \\ &= 12003612721 \times 330 = 3961192197930 \end{aligned}$$

Since 331 is a prime, $331^5 = 3,973,195,810,651$ so $\varphi(3,973,195,810,651) = 3,961,192,197,930$.

2.3. Exponent Division on Fermat’s Little Theorem

Proposition 4 (PRB): If p is prime and “ a ” is a positive integer with p does not divides “ a ”, $\gcd(a, p) = 1$ and n be an exponent to “ a ” then $a^n \equiv r^{s+t} \pmod{p}$. r is a congruent of “ a ” for mod p , where “ s ” is a quotient and “ t ” is a residue when “ n ” divided by p and $n \in \mathbb{N}$ is any exponent.

Proof: Let p be a prime, and a is a positive integer with p does not divides a , $\gcd(a, p) = 1$ and n be an exponent to a then

$$a^n \equiv (qp + r)^n \equiv r^n \pmod{p}; \quad q \in \mathbb{Z}$$

$$r^n \pmod{p} = r^{s_1 p + t_1} \equiv r^{s_1 + t_1} \pmod{p} \quad \text{if } s_1 + t_1 \geq p \text{ then}$$

$$r^{s_1 + t_1} \equiv r^{s_2 p + t_2} \equiv r^{s_2 + t_2} \pmod{p} \quad \text{if } s_2 + t_2 \geq p \text{ then}$$

Do this again and again until we get $s_k + t_k < p, k \in \mathbb{N}$

$$r^{s_{k-1} + t_{k-1}} \equiv r^{s_k p + t_k} \equiv r^{s_k + t_k} \pmod{p} \quad \text{if } s_k + t_k < p.$$

Hence we get,

$$a^n \equiv r^{s+t} \pmod{p} \tag{3}$$

2.4. Proving Fermat’s Little Theorem, Using Proposition 4

If p is prime and a is a positive integer with p does not divides “ a ” and $\gcd(a, p) = 1$ then $a^{p-1} \equiv 1 \pmod{p}$.

$$\Rightarrow a^{p-1} \equiv a^{(1+0)-1} \equiv a^{1-1} \equiv a^0 \equiv 1 \pmod{p}.$$

Example 6: Find the value of $3^{1900} \pmod{13}$.

Solution: We can write

$$1900 = 146.13 + 2$$

$$\equiv 146 + 2 = 148 \text{ here } 148 \geq 13 \text{ so,}$$

$$148 = 11.13 + 5$$

$$\equiv 16 \text{ here } 16 \geq 13 \text{ so,}$$

$$19 = 1.13 + 3$$

$$\equiv 4 \text{ here } 4 < 13 \text{ so,}$$

Apply this algorithm, then we get

$$3^{1900} \equiv 3^4 = 81 \equiv 3 \pmod{13}.$$

Proposition 5 (PRB): If m is a positive integer and a is an integer with $(a, m) = 1$,

$$\text{Then } a^{[\varphi(m)+w]} \equiv 1 \pmod{m}.$$

where

$$w \in W \tag{4}$$

Proof:

Let $a^{[\varphi(m)+w]} = a^k$. So we can write $k = n\varphi(m) + k'$ for some integer m . now we can write $a^k = a^{n\varphi(m)}$. Here $k' = 0$, since k value has $\varphi(m)$ as a one factor and n is a positive integer.

It gives

$$\left(a^{\varphi(m)}\right)^n \equiv 1^n \equiv 1 \pmod{m}. \square$$

Example 8: Find the value of $5^{[\varphi(9)+5]} \pmod{9}$

Solution:

$$5^{[\varphi(9)+5]} = 5^{[6+5]} = 5^{11} \equiv [5^6]^{6652800} \equiv [1]^{6652800} \equiv 1 \pmod{9}.$$

Proposition 6 (PRB): If m is a positive integer and a is an integer with $(a, m) = 1$,

$$\text{Then } a^{\varphi(m)^n} \equiv 1 \pmod{m}.$$

where

$$n \in N \tag{5}$$

Proof: Let $[\varphi(m)]^n = k$ then $a^{\varphi(m)^n} = a^k \pmod{m}$. So we can write $k = h\varphi(m) + k'$ for some integer m . Now $a^k = a^{h\varphi(m)}$. Here $k' = 0$. Since k has $\varphi(m)$ as a one factor and h is a positive integer. It gives $\left(a^{\varphi(m)}\right)^h \equiv 1^h \equiv 1 \pmod{m}. \square$

2.5. Exponent Parallelogram

Definition 1: Let $m \in Z - \{0\}$ and $n \in Z$ be the exponent to m then do 1st operation is subtracting each element with its successive element of 1st line elements. Result will be $(m-1)m^n$, we shall name $(m-1)$ as “ a ”. 2nd operation is subtracting each element with its successive element of 1st operation, result will be $(m-1)^2 m^n$ then we shall name $(m-1)^2$ as “ a^2 ”. 3rd operation is subtracting

3) Let 5 is a first line prime exponents, $(4, 5 \times 4, 25 \times 4, 125 \times 4, \dots, 16, 5 \times 16, 25 \times 16, \dots, 64, 5 \times 64, 25 \times 64, \dots)$ are 1st operation to k th operation and $4 \in I$ with $(4, 5) = 1$, then $4^{5^{\lceil \varphi(5) \rceil k}} \equiv 1$, $4^{25^{\lceil \varphi(5) \rceil k}} \equiv 1$, $4^{125^{\lceil \varphi(5) \rceil k}} \equiv 1$, \dots , $4^{5^n \lceil \varphi(5) \rceil k} \equiv 1 \pmod{5}$. Where $(4, 5 \times 4, 25 \times 4, 125 \times 4, \dots, 16, 5 \times 16, 25 \times 16, \dots, 64, 5 \times 64, 25 \times 64, \dots)$ are coherence numbers of $\varphi(5)$ and $k, n \in I$.

2.6. Prime Bases on Fermat's Last Theorem

Let we see following summations.

Let p_i are prime numbers then

$$2 + 3 + 5 + 7 + 11 + 13 = 41;$$

For squared primes:

$$4 + 9 = 13 = (3.6055\dots)^2 = (3 + 0.6055\dots)^2;$$

$$4 + 9 + 25 = 38 = (6.1644\dots)^2 = (5 + 1.1644\dots)^2;$$

$$4 + 9 + 25 + 49 = 87 = (9.3273\dots)^2 = (7 + 2.3273\dots)^2;$$

$$4 + 9 + 25 + 49 + 121 = 208 = (14.4222\dots)^2 = (11 + 3.4222\dots)^2;$$

For cubed primes:

$$8 + 27 = 35 = (3.2710\dots)^3 = (3 + 0.2710\dots)^3;$$

$$8 + 27 + 125 = 160 = (5.4288\dots)^3 = (5 + 0.4288\dots)^3;$$

$$8 + 27 + 125 + 343 = 503 = (7.9528\dots)^3 = (7 + 0.9528\dots)^3;$$

$$8 + 27 + 125 + 343 + 1331 = 1724 = (12.2405\dots)^3 = (11 + 1.2405\dots)^3;$$

For fourth exponent primes:

$$16 + 81 = 97 = (3.1382\dots)^4 = (3 + 0.1382\dots)^4;$$

$$16 + 81 + 625 = 722 = (5.1836\dots)^4 = (5 + 0.1836\dots)^4;$$

$$16 + 81 + 625 + 2401 = 3123 = (7.4755\dots)^4 = (7 + 0.4755\dots)^4;$$

$$16 + 81 + 625 + 2401 + 14641 = 17764 = (11.5447\dots)^4 = (11 + 0.5447\dots)^4;$$

By this way we concluded,

$$2^n + 3^n + 5^n + 7^n + 11^n + 13^n + \dots + p_i^n = (p_i + B.b_1b_2b_3b_4\dots)^n.$$

where $B.b_1b_2b_3b_4\dots \in R$.

From the above recursion, we formulate the result then we get,

$$\sum_{n=2}^k p_i^n = (p_i + B.b_1b_2b_3b_4\dots)^n.$$

where

$$B.b_1b_2b_3b_4\dots \in R \tag{6}.$$

Theorem 1: Let p_i are prime numbers then $\sum_{n=2}^k p_i^n \neq q^n$. Where q is any prime.

Proof:

Let $\sum_{n=2}^k p_i^n = P$ then

Case 1: If P is prime, result is obvious.

Case 2: If P is composite, we can write $P = st + k$. if $k \neq 0$ then result is obvious.

Case 3: If P is composite and $k = 0$, then we can write $P = st$. If s, t are

distinct primes then result is obvious. But if $s = t$ we get $\sum_{n=2}^k p_i^n = q^n$. This result contradict with (1). So $\sum_{n=2}^k p_i^n \neq q^n$. Where q is any prime.

2.7. Primality of Pascal's Triangle

Definition 2: For all $n \in \mathbb{N}$ and for all reals $a, b \in \mathbb{R}$ we have the formula $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$. For every $p = n$ we have $(a+b)^p = \sum_{k=0}^p \binom{p}{k} a^{p-k} b^k$ then p divides $\sum_{k=1}^p \binom{p}{k}$ and every $\binom{p}{k}$; where $k = 1, 2, \dots, p-1$ is called Primality of binomial expansion.

Prime number Pascal's triangle coefficients

0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
...									
11	1	11	55	165	330	462	462	330	...
...									
p	$\binom{p}{0}$	$\binom{p}{1}$	$\binom{p}{2}$...	$\binom{p}{k}$	where $0 \leq k \leq p$			

Examples:

- 1) 7 divides $7 + 21 + 35 + 35 + 21 + 7$ i.e. $126/7 = 18$
- 2) 11 divides $2(11 + 55 + 165 + 330 + 462)$ i.e. $2046/11 = 186$.

2.8. Constructing Pascal's Triangle by Arithmetic Triangles

Addition triangle

Definition 3: Let $A, B, C, \dots \in \mathbb{Z}$ then do 1st operation is adding each element with its successive element of 1st line elements, 2nd operation is adding each element with its successive element of 1st operation, and 3rd operation is adding each element with its successive element of 2nd operation. By this way we do the same up to n th operation. These 1st line to n th operation diagonal elements coefficients construct Pascal's triangle.

Now we construct addition triangle.

1 st line:	A	B	C	D	E	F
1 st operation:	$B + A$	$C + B$	$D + C$	$E + D$	$F + D$	
2 nd operation:	$C + 2B + A$	$D + 2C + B$	$E + 2D + C$	$F + 2E + D$		
3 rd operation:	$D + 3C + 3B + A$	$E + 3D + 3C + B$	$F + 3E + 3D + C$			
4 th operation:	$E + 4D + 6C + 4B + A$	$F + 4E + 6D + 4C + B$				
5 th operation:	$F + 5E + 10D + 10C + 5B + A$					

From the above, using the colored diagonal we can construct a Pascal's triangle:

$$\begin{array}{cccccccc}
 1 & & & & & & & \\
 1 & 1 & & & & & & \\
 1 & 2 & 1 & & & & & \\
 1 & 3 & 3 & 1 & & & & \\
 1 & 4 & 6 & 4 & 1 & & & \\
 1 & 5 & 10 & 10 & 5 & 1 & & \\
 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
 - & - & - & - & - & - & - & - \\
 \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & - & - & - & \binom{n}{k-1} & \binom{n}{k}
 \end{array}$$

2.9. Backward Difference Triangle

Definition 4: Let $A, B, C, \dots \in Z$ then do 1st operation is subtracting each element with its predecessor element of 1st line elements, 2nd operation is subtracting each element with its predecessor element of 1st operation, and 3rd operation is subtracting each element with its predecessor element of 2nd operation. By this way we do the same up to nth operation. These 1st line to nth operation diagonal elements coefficients construct Pascal's triangle with negative coefficients.

Now we construct backward difference triangle:

1 st line:	A	B	C	D	E	F
1 st operation:	$B - A$	$C - B$	$D - C$	$E - D$	$F - E$	
2 nd operation:	$C - 2B + A$	$D - 2C + B$	$E - 2D + C$	$F - 2E + D$		
3 rd operation:	$D - 3C + 3B - A$	$E - 3D + 3C - B$	$F - 3E + 3D - C$			
4 th operation:	$E - 4D + 6C - 4B + A$	$F - 4E + 6D - 4C + B$				
5 th operation:	$F - 5E + 10D - 10C + 5B - A$					

From the above, using the colored diagonal we can construct a negative Pascal's triangle:

$$\begin{array}{cccccccc}
 1 & & & & & & & \\
 -1 & 1 & & & & & & \\
 1 & -2 & 1 & & & & & \\
 -1 & 3 & -3 & 1 & & & & \\
 1 & -4 & 6 & -4 & 1 & & & \\
 -1 & 5 & -10 & 10 & -5 & 1 & & \\
 1 & -6 & 15 & -20 & 15 & -6 & 1 & \\
 - & - & - & - & - & - & - & - \\
 \binom{n}{0} & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & - & - & - & \binom{n}{k-1} & \binom{n}{k}
 \end{array}$$

where $\binom{n}{0}$ sign depends upon whether n is odd or even. If n is odd we get

$$-\binom{n}{0}, \text{ else we get } \binom{n}{0}.$$

2.10. Forward Difference Triangle

Definition 5: Let $A, B, C, \dots \in Z$ then do 1st operation is subtracting each element with its successive element of 1st line elements, 2nd operation is subtracting each element with its successive element of 1st operation, and 3rd operation is subtracting each element with its successive element of 2nd operation. By this way we do the same up to n th operation. These 1st line to n th operation diagonal elements coefficients construct Pascal's triangle with negative coefficients.

Now we construct forward difference triangle:

A	B	C	D	E	F
$A - B$	$B - C$	$C - D$	$D - E$	$E - F$	
$A - 2B + C$	$B - 2C + D$	$C - 2D + E$	$D - 2E + F$		
$A - 3B + 3C - D$	$B - 3C + 3D - E$	$C - 3D + 3E - F$			
$A - 4B + 6C - 4D + E$	$B - 4C + 6D - 4E + F$				
$A - 5B + 10C - 10D + 5E - F$					

From the above, using the colored diagonal we can construct a negative Pascal's triangle:

1								
1	-1							
1	-2	1						
1	-3	3	-1					
1	-4	6	-4	1				
1	-5	10	-10	5	-1			
1	-6	15	-20	15	-6	1		
-	-	-	-	-	-	-	-	-
$\binom{n}{0}$	$-\binom{n}{1}$	$\binom{n}{2}$	$-\binom{n}{3}$	-	-	-	$\binom{n}{k-1}$	$\binom{n}{k}$

where $\binom{n}{k}$ sign depends upon whether n is odd or even. If n is odd we get $-\binom{n}{k}$, else we get $\binom{n}{k}$.

2.11. Multiplication Triangle

Definition 6: Let $A, B, C, \dots \in Z$ then do 1st operation is multiplying each element with its successive element of 1st line elements, 2nd operation is multiplying each element with its successive element of 1st operation, and 3rd operation is multiplying each element with its successive element of 2nd operation. By this way we do the same up to n th operation. These 1st line to n th operation diagonal elements degrees construct Pascal's triangle.

Now we construct multiplication triangle:

A	B	C	D	E	F
AB	BC	CD	DE	EF	
AB^2C	BC^2D	CD^2E	DE^2F		
AB^3C^3D	BC^3D^3E	CD^3E^3F			
$AB^4C^6D^4E$	$BC^4D^6E^4F$				
$AB^5C^{10}D^{10}E^5F$					

From the above, using the colored diagonal exponents, we can construct a Pascal's triangle:

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
-	-	-	-	-	-	-	-	-
$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	-	-	-	$\binom{n}{k-1}$	$\binom{n}{k}$

2.12. Forward Division Triangle

Definition 7: Let $A, B, C, \dots \in Z$ then do 1st operation is dividing each element with its successive element of 1st line elements, 2nd operation is dividing each element with its successive element of 1st operation, and 3rd operation is dividing each element with its successive element of 2nd operation. By this way we do the same up to nth operation. These 1st line to nth operation diagonal elements degrees construct Pascal's triangle.

Now we construct forward division triangle:

A	B	C	D	E	F
$\frac{A}{B}$	$\frac{B}{C}$	$\frac{C}{D}$	$\frac{D}{E}$	$\frac{E}{F}$	
	$\frac{AC}{B^2}$	$\frac{BD}{C^2}$	$\frac{CE}{D^2}$	$\frac{DF}{E^2}$	
		$\frac{AC^3}{B^3D}$	$\frac{BD^3}{C^3E}$	$\frac{CE^3}{D^3F}$	
		$\frac{AC^6E}{B^4D^4}$	$\frac{BD^6F}{C^4E^4}$		
		$\frac{AC^{10}E^5}{B^5D^{10}F}$			

From the above, using the colored diagonal exponents, we can construct a Pascal's triangle:

1			
1	-1		
1	-2	1	
1	-3	3	-1

where $\binom{n}{k}$ sign depends

2.14. Backward Exponent Difference Triangle

Definition 9: Let $A, B, C, \dots \in Z$ then do 1st operation is multiplying each element with its successive element of 1st line elements, 2nd operation is multiplying each element with its successive element of 1st operation, and 3rd operation is multiplying each element with its successive element of 2nd operation. By this way we do the same up to nth operation. These 1st line to nth operation diagonal elements degrees construct Pascal's triangle.

Theorem 2: Let $n \in I$ be an exponent of any $m \in Z$ then “n”th difference of m^n would be $n!$.

Let we construct backward difference triangle, in which first line numbers are “n”th exponent of whole numbers. For any $0 \leq k \leq n$,

$$\begin{array}{l}
 \text{1st line:} \quad 0^n \qquad 1^n \qquad 2^n \qquad 3^n \qquad 4^n \qquad 5^n \\
 \text{1st operation:} \quad 1^n - 0^n \qquad 2^n - 1^n \qquad 3^n - 2^n \qquad 4^n - 3^n \qquad 5^n - 4^n \\
 \text{2nd operation:} \quad 2^n - 2 \times 1^n + 0^n \qquad 3^n - 2 \times 2^n + 1^n \qquad 4^n - 2 \times 3^n + 2^n \qquad 5^n - 2 \times 4^n + 3^n \\
 \text{3rd operation:} \quad 3^n - 3 \times 2^n + 3 \times 1^n - 0^n \qquad 4^n - 3 \times 3^n + 3 \times 2^n - 1^n \qquad 5^n - 3 \times 4^n + 3 \times 3^n - 2^n \\
 \dots \\
 \text{nth operation:} \quad \binom{n}{0} m^n - \binom{n}{1} (m-1)^n + \binom{n}{2} (m-2)^n - \dots + \binom{n}{k} (m-k)^n. \quad \square
 \end{array}$$

2.15. Forward Exponent Difference Triangle

Definition 10: Let $A, B, C, \dots \in Z$ then do 1st operation is subtracting each element with its successive element of 1st line elements, 2nd operation is subtracting each element with its successive element of 1st operation, and 3rd operation is subtracting each element with its successive element of 2nd operation. By this way we do the same up to nth operation. These 1st line to nth operation diagonal elements degrees construct Pascal's triangle.

Theorem 3: Let $n \in I$ be an exponent of any $m \in Z$ then “n”th difference of m^n would be $(-1)^n (n!)$.

Proof:

Let we construct forward difference triangle, in which first line numbers are “n”th exponent of whole numbers. For any $0 \leq k \leq n$,

$$\begin{array}{l}
 \text{1st line:} \quad 0^n \qquad 1^n \qquad 2^n \qquad 3^n \qquad 4^n \qquad 5^n \\
 \text{1st operation:} \quad 0^n - 1^n \qquad 1^n - 2^n \qquad 2^n - 3^n \qquad 3^n - 4^n \qquad 4^n - 5^n \\
 \text{2nd operation:} \quad 0^n - 2 \times 1^n + 2^n \qquad 1^n - 2 \times 2^n + 3^n \qquad 2^n - 2 \times 3^n + 4^n \qquad 3^n - 2 \times 4^n + 5^n \\
 \text{3rd operation:} \quad 0^n - 3 \times 1^n + 3 \times 2^n - 3^n \qquad 1^n - 3 \times 2^n + 3 \times 3^n - 4^n \qquad 2^n - 3 \times 3^n + 3 \times 4^n - 5^n \\
 \dots \\
 \text{nth operation:} \quad \binom{n}{0} m^n - \binom{n}{1} (m+1)^n + \binom{n}{2} (m+2)^n - \dots + \binom{n}{k} (m+k)^n. \quad \blacksquare
 \end{array}$$

Examples for backward exponent difference method:

- 1) Let $m = 0$ and $n = 5$ then

$$1 \times 0^5 - 5 \times (-1)^5 + 10 \times (-2)^5 - 10 \times (-3)^5 + 5 \times (-4)^5 - (-5)^5$$

$$= 0 + 5 - 320 + 2430 - 5120 + 3125 = 120 = 5!$$

2) Let $m = -1$ and $n = 5$ then

$$1 \times (-1)^5 - 5 \times (-2)^5 + 10 \times (-3)^5 - 10 \times (-4)^5 + 5 \times (-5)^5 - (-6)^5$$

$$= -1 + 160 - 2430 + 10240 - 15625 + 7776 = 120 = 5!$$

3) Let $m = 1$ and $n = 5$ then

$$1 \times 1^5 - 5 \times 0^5 + 10 \times (-1)^5 - 10 \times (-2)^5 + 5 \times (-3)^5 - (-4)^5$$

$$= 1 - 0 - 10 + 320 - 1215 + 1024 = 120 = 5!$$

Examples for backward exponent difference method:

1) Let $m = 0$ and $n = 4$ then

$$1 \times 0^4 - 4 \times 1^4 + 6 \times 2^4 - 4 \times 3^4 + 4^4$$

$$= 0 - 4 + 96 - 324 + 256 = 24 = 4!$$

2) Let $m = -1$ and $n = 4$ then

$$1 \times (-1)^4 - 4 \times 0^4 + 6 \times 1^4 - 4 \times 2^4 + 3^4$$

$$= 1 - 0 + 6 - 64 + 81 = 24 = 4!$$

3) Let $m = 1$ and $n = 5$ then

$$1 \times 1^5 - 5 \times 2^5 + 10 \times 3^5 - 10 \times 4^5 + 5 \times 5^5 - 6^5$$

$$= 1 - 160 + 2430 - 10240 + 15625 - 7776 = -120 = -5!$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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