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Some Properties of First Order Differential Operators

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Abstract

We study some properties of first order differential operators from an algebraic viewpoint. We show this last can be decomposed in sum of an element of a module and a derivation. From a geometric viewpoint, we give some properties on the algebra of smooth functions. The Dirac mass at a point is the best example of first order differential operators at this point. This allows to construct a basis of this set and its dual basis.

Keywords

Commutative Algebra, Differential Operators, Module of Differentials

1. Introduction

When $\mathbb{L}$ is a commutative graded algebra with unit $1_{\mathbb{L}}$ over a commutative field $\mathbb{K}$ with characteristic *zero*, a $\mathbb{K}$-linear map

$$\sigma : \mathbb{L} \to \mathbb{L}$$

is a first order differential operator and of degree r ($r \in \mathbb{Z}$), if σ fulfills

$$\sigma(x \cdot y) = \sigma(x) \cdot y + (-1)^{r \cdot |x|} x \cdot \sigma(y) - (-1)^{r \cdot |x \cdot y|} x \cdot y \cdot \sigma(1_{\mathbb{L}}) \tag{1}$$

for homogeneous elements $x, y \in \mathbb{L}$, where $|x|$ denotes the degree of x. Let σ be a first order differential operator and of degree r, let σ' be another first order differential operator and of degree r', then the bracket

$$[\sigma, \sigma'] : \mathbb{L} \to \mathbb{L}$$

such that

$$[\sigma, \sigma'] = \sigma \circ \sigma' - (-1)^{r \cdot r'} \sigma' \circ \sigma \tag{2}$$

is a first order differential operator and of degree $r + r'$.

Thus the pair $(\mathbb{L}, \sigma)$ is a differential algebra if $\mathbb{L}$ is a commutative graded algebra with unit $1_{\mathbb{L}}$ and

$$\sigma : \mathbb{L} \to \mathbb{L}$$

is a first order differential operator and of degree +1 such that $\sigma \circ \sigma = 0$.

We consider a commutative algebra $\mathbb{A}$ with unit $1_{\mathbb{A}}$ over a commutative field $\mathbb{K}$ with characteristic *zero*.

We denote $\mathbb{E}$ an $\mathbb{A}$-module. For any $x \in \mathbb{E}$,

$$L_x : \mathbb{A} \to \mathbb{E}, a \mapsto a \cdot x$$

denotes the multiplication by x.

The set, $\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E})$, of all first order differential operators from $\mathbb{A}$ into $\mathbb{E}$ is an $\mathbb{A}$-module and admits a $\mathbb{K}$-Lie algebra structure.

We denote, $Der_{\mathbb{K}}(\mathbb{A},\mathbb{E})$, the set of $\mathbb{K}$-derivations from $\mathbb{A}$ into $\mathbb{E}$ which is an $\mathbb{A}$-submodule and a $\mathbb{K}$-Lie subalgebra of $\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E})$. The main goal of this paper is to study some properties of first order differential operators from an algebraic viewpoint and geometric viewpoint. We also give an example of these applications in the last case at a point and we construct the basis of the set of all these maps and its dual basis.

2. Differential Operators from $\mathbb{A}$ into $\mathbb{E}$

A differential operator of order $\leq r, r \in \mathbb{N}$, from $\mathbb{A}$ into $\mathbb{E}$ is a $\mathbb{K}$-linear map

$$D : \mathbb{A} \to \mathbb{E}$$

such that for any $a \in \mathbb{A}$, the map

$$\mathbb{A} \to \mathbb{E}, b \mapsto D(a \cdot b) - a \cdot D(b),$$

is a differential operator of order $\leq (r-1)$ from $\mathbb{A}$ into $\mathbb{E}$.

A differential operator of order *zero* is an $\mathbb{A}$-linear map from $\mathbb{A}$ into $\mathbb{E}$.

Proposition 1 *A* $\mathbb{K}$*-linear map*

$D : \mathbb{A} \to \mathbb{E}$

is a first order differential operator from $\mathbb{A}$ into $\mathbb{E}$ if and only if

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b) - a \cdot b \cdot D(1_{\mathbb{A}}) \tag{3}$$

for all a and b elements of $\mathbb{A}$.

Proof. Assume that $D : \mathbb{A} \to \mathbb{E}$ is a first order differential operator from $\mathbb{A}$ into $\mathbb{E}$, then for any $a \in \mathbb{A}$, the map

$$D_a : \mathbb{A} \to \mathbb{E}, b \mapsto D(a \cdot b) - a \cdot D(b)$$

is $\mathbb{A}$-linear. Thus for any $a, b \in \mathbb{A}$, $D_a(b) = b \cdot D_a(1_{\mathbb{A}})$. Therefore we have the following

$$D_a(b) = b \cdot D_a(1_{\mathbb{A}})$$

$$\begin{aligned} D(a \cdot b) - a \cdot D(b) &= b \cdot \left(D(a \cdot 1_{\mathbb{A}}) - a \cdot D(1_{\mathbb{A}})\right) \\ &= b \cdot \left(D(a) - a \cdot D(1_{\mathbb{A}})\right) \\ &= b \cdot D(a) - a \cdot b \cdot D(1_{\mathbb{A}}) \end{aligned}$$

We deduce that

$$D(a\cdot b)=D(a)\cdot b+a\cdot D(b)-a\cdot b\cdot D(1_{\mathbb{A}}).$$

Conversely if

$$D(a\cdot b)=D(a)\cdot b+a\cdot D(b)-a\cdot b\cdot D(1_{\mathbb{A}})$$

for any $a,b\in\mathbb{A}$, then we obtain

$$D_a(b)=b\cdot D_a(1_{\mathbb{A}})$$

That ends the proof.

For any $x\in\mathbb{E}$, we can easily see that the map

$$L_x:\mathbb{A}\to\mathbb{E},a\mapsto a\cdot x,$$

is a first order differential operator from $\mathbb{A}$ into $\mathbb{E}$.

Proposition 2 [1] *The* $\mathbb{K}$*-linear map* D *from* $\mathbb{A}$ *into* $\mathbb{E}$ *is a first order differential operator if and only if the map*

$$D-L_{D(1_{\mathbb{A}})}:\mathbb{A}\to\mathbb{E},a\mapsto D(a)-a\cdot D(1_{\mathbb{A}}),$$

is a derivation.

Proof. As D is a first order differential operator from $\mathbb{A}$ into $\mathbb{E}$, for any $a,b\in\mathbb{A}$, we verify that

$$\left(D-L_{D(1_{\mathbb{A}})}\right)(a\cdot b)-\left(D-L_{D(1_{A})}\right)(a)\cdot b-a\cdot\left(D-L_{D(1_{\mathbb{A}})}\right)(b)=0.$$

Conversely, let

$$D-L_{D(1_{\mathbb{A}})}:\mathbb{A}\to\mathbb{E},a\mapsto D(a)-a\cdot D(1_{\mathbb{A}})$$

be a derivation, then by straightforward calculation do that D is a first order differential operator.

Theorem 3 *The map*

$$\tau:\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E})\to\mathbb{E}\times Der_{\mathbb{K}}(\mathbb{A},\mathbb{E}),D\mapsto\left(D(1_{\mathbb{A}}),D-L_{D(1_{\mathbb{A}})}\right)$$

is an isomorphism of $\mathbb{A}$-modules.

Proof. The map

$$\tau:\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E})\to\mathbb{E}\times Der_{\mathbb{K}}(\mathbb{A},\mathbb{E}),D\mapsto\left(D(1_{\mathbb{A}}),D-L_{D(1_{\mathbb{A}})}\right)$$

is $\mathbb{A}$-linear.

For any $x\in\mathbb{E},d\in Der_{\mathbb{K}}(\mathbb{A},\mathbb{E})$, the map

$$\tau'_{(x,d)}:\mathbb{A}\to E,a\mapsto a\cdot x+d(a),$$

is $\mathbb{K}$-linear and is a first order differential operator. Indeed for any $a,b\in\mathbb{A}$,

$$\tau'_{(x,d)}(ab)-\tau'_{(x,d)}(a)\cdot b-a\cdot\tau'_{(x,d)}(b)+a\cdot b\cdot\tau'_{(x,d)}(1_{\mathbb{A}})=0.$$

Then

$$\tau'_{(x,d)}(ab)=\tau'_{(x,d)}(a)\cdot b+a\cdot\tau'_{(x,d)}(b)-a\cdot b\cdot\tau'_{(x,d)}(1_{\mathbb{A}}).$$

We also verify that the map

$$\tau':\mathbb{E}\times Der_{\mathbb{K}}(\mathbb{A},\mathbb{E})\to\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E}),(x,d)\mapsto\tau'_{(x,d)},$$

is $\mathbb{A}$-linear. For any pair $(x,d),(x',d') \in \mathbb{E} \times Der_{\mathbb{K}}(\mathbb{A},\mathbb{E})$ and for $a,b \in \mathbb{A}$

$$\tau'\left((x,d)+(x',d')\right)(a) = \left(\tau'_{(x,d)+(x',d')}\right)(a) = \left(\tau'(x,d)+\tau'(x',d')\right)(a).$$

We deduce that

$$\tau'\left((x,d)+(x',d')\right) = \tau'(x,d)+\tau'(x',d')$$

and

$$\tau'\left(a\cdot(x,d)\right)(b) = b\cdot(a\cdot x)+a\cdot d(b) = \left[a\cdot\tau'\left((x,d)\right)\right](b).$$

Thus

$$\tau'\left(a\cdot(x,d)\right) = a\cdot\tau'\left((x,d)\right).$$

We have

$$\begin{aligned}(\tau'\circ\tau)(D)(a) &= \tau'\left(D(1_{\mathbb{A}}), D-L_{D(1_{\mathbb{A}})}\right)(a)\\ &= a\cdot D(1_{\mathbb{A}})+\left(D-L_{D(1_{\mathbb{A}})}\right)(a)\\ &= D(a).\end{aligned}$$

So $\tau'\circ\tau = id_{\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E})}$ for any $a\in\mathbb{A}$ and $D\in\mathcal{D}_{\mathbb{K}}(\mathbb{A},\mathbb{E})$.

And

$$\begin{aligned}(\tau\circ\tau')\left((x,d)\right)(a) &= \tau\left[\tau'_{(x,d)}(a)\right]\\ &= a\cdot x+d(a)\\ &= (x,d)(a).\end{aligned}$$

So $\tau\circ\tau' = id_{\mathbb{E}\times Der_{\mathbb{K}}(\mathbb{A},\mathbb{E})}$.

Thus the map τ is an isomorphism of $\mathbb{A}$-modules.

In the following, we shall consider the $\mathbb{A}$-module $\mathbb{A}\times\Omega_{\mathbb{K}}(\mathbb{A})$ where $\Omega_{\mathbb{K}}(\mathbb{A})$ is the $\mathbb{A}$-module of Kähler $\mathbb{K}$-differential of $\mathbb{A}$ and

$$d_{\mathbb{A}/\mathbb{K}}:\mathbb{A}\to\Omega_{\mathbb{K}}(\mathbb{A}), a\mapsto d_{\mathbb{A}/\mathbb{K}}(a)$$

is the canonical derivation [1] [2] [3] and the term differential operator will mean first order differential operator.

Theorem 4 [1] [3] *The map*

$$D_{\mathbb{A}/\mathbb{K}}:\mathbb{A}\to\mathbb{A}\times\Omega_{\mathbb{K}}(\mathbb{A}), a\mapsto\left(a, d_{\mathbb{A}/\mathbb{K}}(a)\right),$$

is a differential operator and the image of $D_{\mathbb{A}/\mathbb{K}}$ generates the $\mathbb{A}$-module $\mathbb{A}\times\Omega_{\mathbb{K}}(\mathbb{A})$. Furthermore the pair $\left(\mathbb{A}\times\Omega_{\mathbb{K}}(\mathbb{A}), D_{\mathbb{A}/\mathbb{K}}\right)$ possesses the following universal property: for any $\mathbb{A}$-module $\mathbb{E}$ and for any differential operator

$$\varphi:\mathbb{A}\to\mathbb{E},$$

then there exists one and only one $\mathbb{A}$-linear map

$$\tilde{\varphi}:\mathbb{A}\times\Omega_{\mathbb{K}}(\mathbb{A})\to\mathbb{E}$$

such that

$$\tilde{\varphi}\circ D_{\mathbb{A}/\mathbb{K}} = \varphi. \tag{4}$$

The map

$$Hom_{\mathbb{A}}\left(\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right),\mathbb{E}\right)\to\mathcal{D}_{\mathbb{K}}\left(\mathbb{A},\mathbb{E}\right),\psi\mapsto\psi\circ D_{\mathbb{A}/\mathbb{K}},$$

is an isomorphism of $\mathbb{A}$-modules [1] [3].

We recall that an alternating p-differential operator from $\mathbb{A}$ into $\mathbb{E}$ is an alternating $\mathbb{K}$-multilinear map

$$\varphi:\mathbb{A}^p\to\mathbb{E}$$

such that for all p elements $a_1,a_2,\cdots,a_p\in\mathbb{A}$, the map

$$\varphi_{\left(a_1,a_2,\cdots,\widehat{a_i},\cdots,a_p\right)}:\mathbb{A}\to\mathbb{E},a_i\mapsto\varphi\left(a_1,a_2,\cdots,a_i,\cdots,a_p\right),$$

is a differential operator for any $i=1,2,\cdots,p$.

Theorem 5 [1] *Let*

$$\varphi:\mathbb{A}^p\to\mathbb{E}$$

be an alternating p-differential operator, then there exists an unique alternating $\mathbb{A}$-multilinear map

$$\tilde{\varphi}:\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right]^p\to\mathbb{E}$$

such that

$$\tilde{\varphi}\left(D_{\mathbb{A}/\mathbb{K}}\left(a_1\right),D_{\mathbb{A}/\mathbb{K}}\left(a_2\right),\cdots,D_{\mathbb{A}/\mathbb{K}}\left(a_p\right)\right)=\varphi\left(a_1,a_2,\cdots,a_p\right) \tag{5}$$

for all $a_1,a_2,\cdots,a_p\in\mathbb{A}$.

We note $\Lambda_{\mathbb{A}}^p\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right]$, the p-exterior power of the $\mathbb{A}$-module $\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)$.

Theorem 6 [1] *The map*

$$D_{\mathbb{A}/\mathbb{K}}^{(p)}:\mathbb{A}^p\to\Lambda_{\mathbb{A}}^p\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right],$$
$$\left(a_1,a_2,\cdots,a_p\right)\mapsto D_{\mathbb{A}/\mathbb{K}}\left(a_1\right)\wedge D_{\mathbb{A}/\mathbb{K}}\left(a_2\right)\wedge\cdots\wedge D_{\mathbb{A}/\mathbb{K}}\left(a_p\right),$$

is a p-alternating differential operator and the image of $D_{\mathbb{A}/\mathbb{K}}^{(p)}$ generates the $\mathbb{A}$-module $\Lambda_{\mathbb{A}}^p\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right]$. Moreover, the pair $\left(\Lambda_{\mathbb{A}}^p\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right],D_{\mathbb{A}/\mathbb{K}}^{(p)}\right)$ possesses the following universal property: for any $\mathbb{A}$-module $\mathbb{E}$ and for any alternating p-differential operator

$$\varphi:\mathbb{A}^p\to\mathbb{E},$$

there exists an unique $\mathbb{A}$-linear map

$$\tilde{\varphi}:\Lambda_{\mathbb{A}}^p\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right]\to\mathbb{E}$$

such that

$$\tilde{\varphi}\circ D_{\mathbb{A}/\mathbb{K}}^{(p)}=\varphi. \tag{6}$$

We deduce that

$$Hom_{\mathbb{A}}\left(\Lambda_{\mathbb{A}}^p\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right],\mathbb{E}\right)\to\mathcal{D}_{\mathbb{K}}^{(p)}\left(\mathbb{A},\mathbb{E}\right),\psi\mapsto\psi\circ D_{\mathbb{A}/\mathbb{K}}^{(p)},$$

is an isomorphism of $\mathbb{A}$-modules.

Let $\Lambda_{\mathbb{A}}\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right]=\bigoplus_{n\in\mathbb{N}}\Lambda_{\mathbb{A}}^n\left[\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)\right]$ be the exterior $\mathbb{A}$-algebra of $\mathbb{A}$-module $\mathbb{A}\times\Omega_{\mathbb{K}}\left(\mathbb{A}\right)$ [3].

Theorem 7 *The differential operator* $D_{\mathbb{A}/\mathbb{K}} : \mathbb{A} \to \mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})$ *extends along a differential operator, denoted once again* $D_{\mathbb{A}/\mathbb{K}}$, *from* $\Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$ *into* $\Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$ *of degree* $+1$ *and of square zero such that the pair* $(\Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})], D_{\mathbb{A}/\mathbb{K}})$ *is the differential algebra in the sense of Okassa* [1].

Proposition 8 *For any integer* p $(p \geq 1)$, *the map*

$$[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]^p \to \Lambda_{\mathbb{A}}^{p-1}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})],$$

$$(x_1, x_2, \cdots, x_p) \mapsto \sum_{i=1}^{p} (-1)^{i-1} \tilde{\varphi}(x_i) \cdot x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_p$$

is alternating $\mathbb{A}$-multilinear and induces an $\mathbb{A}$-linear map

$$i_\varphi : \Lambda_{\mathbb{A}}^p[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})] \to \Lambda_{\mathbb{A}}^{p-1}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$$

which extends along an $\mathbb{A}$-endomorphism of $\Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$, noticed once again i_φ, of degree -1.

Theorem 9 *For any* $\varphi \in \mathcal{D}_{\mathbb{K}}(\mathbb{A})$, *the inner product*

$$i_\varphi : \Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})] \to \Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$$

satisfies

$$i_\varphi(\eta_1 \wedge \eta_2) = [i_\varphi(\eta_1)] \wedge \eta_2 + (-1)^{|\eta_1|} \eta_1 \wedge [i_\varphi(\eta_2)] \tag{7}$$

for all $\eta_1, \eta_2 \in \Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$ and where $|\eta_1|$ denotes the degree of η_1. Moreover, the bracket

$$\theta_\varphi = i_\varphi \circ D_{\mathbb{A}/\mathbb{K}} + D_{\mathbb{A}/\mathbb{K}} \circ i_\varphi : \Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})] \to \Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$$

is a differential operator of $\Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$ and of degree *zero*.

Proof. It is obvious.

Proposition 10 *For any* $\varphi \in \mathcal{D}_{\mathbb{K}}(\mathbb{A})$, *we get*

$$\theta_\varphi \circ D_{\mathbb{A}/\mathbb{K}} = D_{\mathbb{A}/\mathbb{K}} \circ \theta_\varphi. \tag{8}$$

Proof. As $D_{\mathbb{A}/\mathbb{K}} \circ D_{\mathbb{A}/\mathbb{K}} = 0$ and $\theta_\varphi = i_\varphi \circ D_{\mathbb{A}/\mathbb{K}} + D_{\mathbb{A}/\mathbb{K}} \circ i_\varphi$, then we have

$$\theta_\varphi \circ D_{\mathbb{A}/\mathbb{K}} = D_{\mathbb{A}/\mathbb{K}} \circ i_\varphi \circ D_{\mathbb{A}/\mathbb{K}} = D_{\mathbb{A}/\mathbb{K}} \circ \theta_\varphi.$$

That ends the proof.

Theorem 11 *For all* $\varphi \in \mathcal{D}_{\mathbb{K}}(\mathbb{A}), a \in \mathbb{A}$ *and* $\eta \in \Lambda_{\mathbb{A}}[\mathbb{A} \times \Omega_{\mathbb{K}}(\mathbb{A})]$, *we have*

$$\theta_\varphi(a \cdot \eta) = [\varphi(a) - a \cdot \varphi(1_{\mathbb{A}})] \cdot \eta + a \cdot \theta_\varphi(\eta) \tag{9}$$

$$\theta_{a \cdot \varphi}(\eta) = a \cdot \theta_\varphi(\eta) + [D_{\mathbb{A}/\mathbb{K}}(a) - a \cdot D_{\mathbb{A}/\mathbb{K}}(1_{\mathbb{A}})] \wedge i_\varphi(\eta); \tag{10}$$

and

$$\theta_\varphi[D_{\mathbb{A}/\mathbb{K}}(a)] = D_{\mathbb{A}/\mathbb{K}}[\varphi(a)]. \tag{11}$$

Proof. The result is immediate.

3. Applications in Geometry

In what follows, M denotes a paracompact and connected smooth manifold, $C^\infty(M)$ the algebra of numerical functions of class C^∞ on $M, \mathfrak{X}(M)$ the

$C^{\infty}(M)$-module of vector fields on M, 1 the unit of $C^{\infty}(M)$, $\mathcal{D}(M)$ the $C^{\infty}(M)$-module of differential operators on $C^{\infty}(M)$ and δ the cohomology operator associated with the identity map

$$id : \mathcal{D}(M) \to \mathcal{D}(M).$$

3.1. Differential Operators of $C^{\infty}(M)$

We recall that if $[,]$ is the usual Lie bracket on $\mathcal{D}(M)$ then for all $D, D' \in \mathcal{D}(M)$ and $f, g \in C^{\infty}(M)$, we have

$$[D, f \cdot D'] = [D(f) - f \cdot D(1)] \cdot D' + f \cdot [D, D']. \tag{12}$$

In particular

$$[D, f] = D(f) - f \cdot D(1); \tag{13}$$

$$[D, 1] = 0; \tag{14}$$

$$[f, g] = 0. \tag{15}$$

Proposition 12 [1] [4] [5] *When $D = f + X$ and $D' = g + Y$ are two differential operators of $C^{\infty}(M)$, with f and g elements of $C^{\infty}(M)$, X and Y elements of $\mathfrak{X}(M)$, then the bracket on $\mathcal{D}(M)$ is given by*

$$[D, D'] = X(g) - Y(f) + [X, Y]. \tag{16}$$

Proof. By straightforward calculation, one has

$$\begin{aligned} [D, D'] &= [f + X, g + Y] \\ &= [f, g] + [f, Y] + [X, g] + [X, Y] \\ &= X(g) - Y(f) + [X, Y]. \end{aligned}$$

That ends the proof.

Let N be a smooth manifold and let $f : M \to N$ be a differential map.

The map

$$f^* : C^{\infty}(N) \to C^{\infty}(M), g \mapsto g \circ f$$

is an homomorphism of real algebras.

If

$$f : M \to N$$

is a diffeomorphism and

$$f^{-1} : N \to M$$

its inverse, then

$$f^* : C^{\infty}(N) \to C^{\infty}(M)$$

is an isomorphism of real algebras such that

$$(f^*)^{-1} = (f^{-1})^*. \tag{17}$$

Proposition 13 *If f is a diffeomorphism and D is a differential operator of $C^{\infty}(M)$, then the map*

$$f_*D=\left(f^{-1}\right)^*\circ D\circ f^*:C^\infty(N)\to C^\infty(N)$$

is a differential operator of $C^\infty(N)$ called image of D by *f*.

Theorem 14 *Let* D, D' *be two differential operators of* $C^\infty(M)$, *if f is a diffeomorphism, then*

$$[f_*D, f_*D'] = f_*[D, D']. \tag{18}$$

Proof. Indeed, we verify that

$$[f_*D, f_*D'] = f_*D\circ f_*D' - f_*D'\circ f_*D = f_*[D, D'].$$

That ends the proof.

Proposition 15 *Let D be a differential operator, when f is a diffeomorphism then for any g element of* $C^\infty(M)$, *we have*

$$D(g\circ f)=[(f_*D)(g)]\circ f. \tag{19}$$

Proof. Let $x\in M$, we have

$$\begin{aligned}[D(g\circ f)](x) &= [D[f^*(g)]](x)\\ &=(f_*D)[f(x)](g)\\ &=[[(f_*D)(g)]\circ f](x).\end{aligned}$$

We deduce that $D(g\circ f)=[(f_*D)(g)]\circ f$.

Proposition 16 *When f is a diffeomorphism and* ξ *be a multilinear p-form on* $\mathcal{D}(N)$, *for all* $D_1, D_2, \cdots, D_p$ *differential operators of* $C^\infty(M)$, *we have*

$$\left(f^*\xi\right)\left(D_1, D_2, \cdots, D_p\right)=\left[\xi\left(f_*D_1, f_*D_2, \cdots, f_*D_p\right)\right]\circ f. \tag{20}$$

Theorem 17 *If f is a diffeomorphism then* δ *and* f^* *commute.*

Proof. We have

$$\begin{aligned}&\left[\delta\left(f^*\xi\right)\right]\left(D_1, D_2, \cdots, D_p, D_{p+1}\right)\\ &=\sum_{i=1}^{p+1}(-1)^{i+1}D_i\left[\left(f^*\xi\right)\left(D_1, D_2, \cdots, \widehat{D_i}, \cdots, D_p, D_{p+1}\right)\right]\\ &\quad+\sum_{i<j}(-1)^{i+j}\left(f^*\xi\right)\left(\left[D_i, D_j\right], D_1, D_2, \cdots, \widehat{D_i}, \cdots, \widehat{D_j}, \cdots, D_p, D_{p+1}\right)\\ &=\left[\sum_{i=1}^{p+1}(-1)^{i+1}\left(f_*D_i\right)\left[\xi\left(f_*D_1, f_*D_2, \cdots, f_*\widehat{D_i}, \cdots, f_*D_{p+1}\right)\right]\right]\circ f\\ &\quad+\left[\sum_{i<j}(-1)^{i+j}\xi\left(\left[f_*D_i, f_*D_j\right], f_*D_1, \cdots, f_*\widehat{D_i}, \cdots, f_*\widehat{D_j}, \cdots, f_*D_p, f_*D_{p+1}\right)\right]\circ f.\end{aligned}$$

On the other hand, we get

$$\begin{aligned}&\left[f^*(\delta\xi)\right]\left(D_1, D_2, \cdots, D_p, D_{p+1}\right)\\ &=\left[(\delta\xi)\left(f_*D_1, f_*D_2, \cdots, f_*D_{p+1}\right)\right]\circ f\\ &=\left[\sum_{i=1}^{p+1}(-1)^{i+1}\left(f_*D_i\right)\left[\xi\left(f_*D_1, f_*D_2, \cdots, f_*\widehat{D_i}, \cdots, f_*D_{p+1}\right)\right]\right]\circ f\\ &\quad+\left[\sum_{i<j}(-1)^{i+j}\xi\left(\left[f_*D_i, f_*D_j\right], f_*D_1, \cdots, f_*\widehat{D_i}, \cdots, f_*\widehat{D_j}, \cdots, f_*D_{p+1}\right)\right]\circ f.\end{aligned}$$

That ends the proof.

3.2. Differential Operators of $C_p^\infty(M)$

Let $C_p^\infty(M)$ be the algebra of smooth functions defined in the neighborhood of $p \in M, 1$ the unit of $C_p^\infty(M)$.

A differential operator at $p \in M$ is a linear map

$$D_p : C_p^\infty(M) \to \mathbb{R}$$

such that

$$D_p(f \cdot g) = D_p(f) \cdot g(p) + f(p) \cdot D_p(g) - f(p) \cdot g(p) \cdot D_p(1) \tag{21}$$

for all functions *f* and *g* in $C_p^\infty(M)$.

Example 18 *The Dirac mass at* $p \in M$,

$$d_p : C_p^\infty(M) \to \mathbb{R}, \quad f \mapsto f(p)$$

is a differential operator at *p*.

We verify that the set, $\mathcal{D}_p(M)$, of differential operators at *p* is a real vector space and the map

$$\mathcal{D}_p(M) \to \mathbb{R} \oplus T_pM, \quad \varphi_p \mapsto \varphi_p(1) + \left(\varphi_p - L_{\varphi_p(1)}\right),$$

is an isomorphism of vector spaces. Moreover, for $f \in C_p^\infty(M)$, the map

$$\delta f_{|p} : \mathcal{D}_p(M) \to \mathbb{R}, \quad \varphi_p \mapsto \varphi_p(f),$$

is a linear form on $\mathcal{D}_p(M)$.

Theorem 19 *Let U be an open neighborhood of p in M of coordinate functions* $(x_1, \cdots, x_n)$, *then* $\left(d_p, \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p\right)$ *is the basis of* $\mathcal{D}_p(M)$. *The system*

$$\left(\delta 1/_p, \delta x_1/_p - x_1(p) \cdot \delta 1/_p, \delta x_2/_p - x_2(p) \cdot \delta 1/_p, \cdots, \delta x_n/_p - x_n(p) \cdot \delta 1/_p\right) \tag{22}$$

is the dual basis of the basis of $\mathcal{D}_p(M)$ *and*

$$\delta f/_U = f \cdot \delta 1 + \sum_{i=1}^n \left(\frac{\partial f/_U}{\partial x_i}\right) \cdot \left(\delta x_i - x_i \cdot \delta 1\right). \tag{23}$$

Conclusion 20 *In this paper, after having given the definition of differential operator by a relation of recurrence, one decomposes it and one states some properties from the algebraic viewpoint. We define the Lie derivative with respect to a differential operator. From a geometric viewpoint, some properties on the algebra of smooth functions were given. The Dirac mass at a point was the best example of differential operators at this point.*

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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