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The Global Existence of Smooth Solutions for Timoshenko-Cattaneo System with Two-Sound Waves in Besov Space

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Abstract

This paper is devoted to studying the global existence of smooth solutions for the Timoshenko-Cattaneo system with two sound waves. In the case of equal wave speeds and non-equal wave speeds, the Timoshenko-Cattaneo system exhibits regularity loss in the high-frequency part in order to obtain global well-posedness for the nonlinear Timoshenko-Cattaneo system with the minimum initial value of regularity index. This article applied harmonic analysis tools to establish the global solution for the Timoshenko-Cattaneo system in Besov space with a regularity index $s=\frac{3}{2}$.

Keywords

Timoshenko-Cattaneo System, Regularity-Loss, Global Existence

1. Introduction

In the present work, we investigate the following Timoshenko-Cattaneo system in thermoelasticity of second sound in one-dimensional whole space:

$$\begin{cases} \varphi_{tt}-\left(\varphi_x-\psi\right)_x=0, \\ \psi_{tt}-\left[\sigma\left(\psi_x\right)\right]_x-\left(\varphi_x-\psi\right)+\beta\psi_t+\lambda\theta_x=0, \\ \theta_t+kq_x+\beta\psi_{tx}=0, \\ \tau_0 q_t+\delta q+k\theta_x=0. \end{cases} \tag{1}$$

where $t\in(0,\infty)$ denotes the time variable and $x\in\mathbb{R}$ is the space variable, the functions φ and ψ denote the displacements of the elastic material, the function θ is the temperature difference, q is heat flux, and γ,τ_0,k,λ and β are certain positive constants depending on the material elastic and thermal properties. The smooth function $\sigma(\eta)$ satisfies $\sigma'(\eta)>0$ for any $\eta>0$. In this paper, we focus on the Cauchy problem to (1), assuming initial conditions:

$$\begin{cases}\varphi(.,0)=\varphi_0(x),\varphi_t(.,0)=\varphi_1(x),\psi(.,0)=\psi_0(x),\\ \psi_t(.,0)=\psi_1(x),\theta(.,0)=\theta_0(x),q(.,0)=q_0.\end{cases} \tag{2}$$

The linearized version of (1) reads correspondingly

$$\begin{cases}\varphi_{tt}-(\varphi_x-\psi)_x=0,\\ \psi_{tt}-a^2\psi_{xx}-(\varphi_x-\psi)+\beta\psi_t+\lambda\theta_x=0,\\ \theta_t+kq_x+\beta\psi_{tx}=0,\\ \tau_0 q_t+\delta q+k\theta_x=0.\end{cases} \tag{3}$$

Note that here, we do not need to distinguish between $a=1$ and $a\neq 1$, as in both cases, the decay property of solutions to (3) is of regularity-loss type. In the whole space, introduce the following variables

$$v=\varphi_x-\psi,\ u=\varphi_t,\ z=a\psi_x,\ y=\psi_t,\ w=q.$$

System (3) can be rewritten as the following first-order hyperbolic system

$$\begin{cases}v_t-u_x+y=0,\\ u_t-v_x=0,\\ z_t-ay_x=0,\\ y_t-az_x-v+\lambda y+\beta\theta_x=0,\\ \theta_t+kw_x+\beta y_x=0,\\ \tau_0 w_0+\delta w+k\theta_x=0.\end{cases} \tag{4}$$

And the initial conditions (2) are taken in the form

$$(v,u,z,y,\theta,w)(x,0)=(v_0,u_0,z_0,y_0,\theta_0,w_0), \tag{5}$$

where

$$v_0=\varphi_{0,x}-\psi_0,\ u_0=\varphi_1,\ z_0=a\psi_{0,x},\ y_0=\psi_1,\ w_0=q_0.$$

System (4)-(5) is equivalent to

$$\begin{cases}A^0U_t+AU_x+LU=0,\\ U(x,0)=U_0,\end{cases} \tag{6}$$

where

$$A^0=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&\tau_0\end{pmatrix},\quad A=\begin{pmatrix}0&-1&0&0&0&0\\-1&0&0&0&0&0\\0&0&0&-a&0&0\\0&0&-a&0&\beta&0\\0&0&0&\beta&0&k\\0&0&0&0&k&0\end{pmatrix}, \tag{7}$$

$$L=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix}$$

and $U_0=\left(v_0,u_0,z_0,y_0,\theta_0,w_0\right)^{\mathrm{T}}$. Note that the relaxation matrix L is not symmetric.

Taking the Fourier transform of (6), we obtain the following Cauchy problem for a first-order system

$$\begin{cases} A^0\hat{U}_t+i\xi A\hat{U}+L\hat{U}=0, \\ \hat{U}\left(\xi,0\right)=\hat{U}_0. \end{cases} \tag{8}$$

Solving this equation, it holds

$$\hat{U}\left(\xi,t\right)=\mathrm{e}^{t\hat{\Phi}(i\xi)}\hat{U}_0\left(\xi\right),$$

where

$$\hat{\Phi}\left(i\xi\right)=-\left(A^0\right)^{-1}\left(i\xi A+L\right). \tag{9}$$

The solution of (6) is then given by

$$U\left(x,t\right)=\mathrm{e}^{t\Phi}U_0\left(x\right),$$

where

$$\left(\mathrm{e}^{t\Phi}w\right)=\mathcal{F}^{-1}\left[\mathrm{e}^{t\hat{\Phi}(i\xi)}\hat{w}\left(\xi\right)\right]\left(x\right).$$

The conclusions related to the non-symmetry of relaxation matrices have been studied in [1], which provides the theoretical support for our study.

From [2], we know that under the assumption $U_0\in H^s\left(\mathbb{R}\cap H^1\left(\mathbb{R}\right)\right)$, the following decay properties were shown for $U=\left(v,u,z,y,\theta,w\right)$ to (6):

$$\left\|\partial_x^l U\left(t\right)\right\|_{L^2}\le C\left(1+t\right)^{-\frac{1}{4}-\frac{k}{2}}\left\|U_0\right\|_{L^1}+C\left(1+t\right)^{-\frac{l}{2}}\left\|\partial_x^{k+l}U_0\right\|_{L^2}. \tag{10}$$

Let $\gamma\in\left[0,1\right]$ and assume $U_0\in H^s\left(\mathbb{R}\right)\cap L^{1,\gamma}\left(\mathbb{R}\right)$, the following decay properties were shown for $U=\left(v,u,z,y,\theta,w\right)$ to (6):

$$\begin{aligned}\left\|\partial_x^l U\left(t\right)\right\|_{L^2}\le{}& C\left(1+t\right)^{-\frac{1}{4}-\frac{k}{2}-\frac{\gamma}{2}}\left\|U_0\right\|_{L^1}+C\left(1+t\right)^{-\frac{l}{2}}\left\|\partial_x^{k+l}U_0\right\|_{L^2} \\ &+C\left(1+t\right)^{-\frac{1}{4}-\frac{k}{2}}\left|\int_{\mathbb{R}}U_0\left(x\right)\mathrm{d}x\right|,\end{aligned} \tag{11}$$

where s,k,l are non-negative integers satisfying $k+l\le s$.

Remark 1.1 *Whether equal wave speeds or non-equal wave speeds, the Timoshenko-Cattaneo system model exhibits a loss of regularity in the high-frequency part, and the pure Timoshenko system also produced* $\left(1+t\right)^{-\frac{l}{2}}$ *only under the assumption of additional* l *-th regularity of the initial data.*

Moreover, Under the requirement of high regularity $H^8\cap H^1$ for the initial data, Racke and Said-Houari [3] used an energy method with negative weights to create artificial damping to control the nonlinearity, thus obtaining an overall existence and decay estimate for solutions. [4] is an improvement of [3]. The requirement for initial data is reduced in the literature of [4]. Not only the global existence of solutions on H^2 is obtained without using the weighted energy method but also the decay estimate of solutions on $H^2 \cap H^1$ is obtained by using the L^p - L^q - L^r estimate.

In this paper, we will use suitable variable substitution methods to transform the research problem of the Timoshenko-Cattaneo model into the Cauchy problem for a first-order system. Due to the asymmetry of L, the general theories in [5] [6] cannot be directly applied to the Timoshenko-Cattaneo system (1), which is the motivation of our research. Then the methods in [7] are referred. We hope to establish similar results for the Timoshenko-Cattaneo model (12). The main difference between the models studied in this article and the Timoshenko-Fourier model with respect to the computation of the global existence is the dissipation of θ. While there is no loss of regularity of θ, the L^2 norm on θ itself is missing in the Timoshenko-Cattaneo model.

System (4)-(5) admits the decay property (10)-(11), which is of regularity-loss type at the high frequency, consequently it seems impossible to obtain the optimal decay rate with the relatively lower regularity. In order to overcome it, one can use the methods in [8] [9]. In the case of equal wave speeds, the frequency localized Duhamel's principle is used in [9]. In the case of non-equal wave speeds, the new frequency-localized time decay inequality is used in [8]. In [9], the corresponding Littlewood-Paley pointwise energy inequality is obtained through the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E\left[\hat{U}\right]+c_1\eta_1(\xi)\left|\hat{U}\right|^2 \lesssim \xi^2\left|\hat{g}\right|^2, \text{ with } \eta_1(\xi)=\xi^2\Big/\left(1+\xi^2\right)^2$$

which is given in [4], also the corresponding Grönwall's inequality $\frac{\mathrm{d}}{\mathrm{d}t}E\left[\widehat{\dot{\Delta}_q U}\right]+c_2\eta_1\left|\widehat{\dot{\Delta}_q U}\right|^2 \lesssim \xi^2\left|\widehat{\dot{\Delta}_q g}\right|^2$ holds, and finally the new frequency-localized time decay inequality is used at low and high frequencies to obtain the optimal decay for the Timoshenko-Fourier model in the critical Besov space. The pure Timoshenko model has an important research value for the study of the Timoshenko-Fourier and Timoshenko-Cattaneo models coupled with the heat equation. And it is well known that the pure Timoshenko model plus the damping term a decay rate exhibits an exponential decay (in the case of equal waves), the Timoshenko-Fourier model continues this decay property, while for the Timoshenko-Cattaneo model, even with the addition of the classical damping, this decay property cannot be recovered in any case. The structure of the nonlinear part of the Timoshenko-Cattaneo model with second sound is the same as that of the nonlinear part of the Timoshenko-Fourier model, and the method performed by Xu Jiang, Naofumi Mori and Shuichi Kawashima [9] can be used for the study of the decay rate in this paper.

2. Preliminary

In this section, we mainly show the lemmas and propositions used in this article. For proof of the propositions, see the references [10] [11].

Lemma 2.1 *Let* $0 < R_1 < R_2$ *and* $1 \le a \le b \le \infty$.

i) *Suppose* $\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \le R_1\lambda\}$, *for any* $\alpha \ge 0$, *then*

$$\|\Lambda^\alpha f\|_{L^b} \lesssim \lambda^{\alpha+n\left(\frac{1}{a}-\frac{1}{b}\right)} \|f\|_{L^a},$$

ii) *Suppose* $\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : R_1\lambda \le |\xi| \le R_2\lambda\}$, *for any* $\alpha \in \mathbb{R}$, *then*

$$\|\Lambda^\alpha f\|_{L^b} \approx \lambda^\alpha \|f\|_{L^a}.$$

Lemma 2.2 *Let* $s \in \mathbb{R}$ *and* $1 \le p, r \le \infty$.

1) *If* $s > 0$, *then* $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$.

2) *If* $\tilde{s} \le s$, *then* $B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,r}$. *This inclusion relation is false for the homogeneous Besov spaces.*

3) *If* $1 \le r \le \tilde{r} \le \infty$, *then* $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\tilde{r}}$ *and* $B^s_{p,r} \hookrightarrow B^s_{p,\tilde{r}}$.

4) *If* $1 \le p \le \tilde{p} \le \infty$, *then* $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-n\left(\frac{1}{p}-\frac{1}{\tilde{p}}\right)}_{\tilde{p},r}$ *and* $B^s_{p,r} \hookrightarrow B^{s-n\left(\frac{1}{p}-\frac{1}{\tilde{p}}\right)}_{\tilde{p},r}$.

5) *If* $r \ge \theta$, *then* $\|f\|_{\tilde{L}^\theta_T(B^s_{p,r})} \le \|f\|_{L^\theta_T(B^s_{p,r})}$; *If* $r \le \theta$, *then* $\|f\|_{\tilde{L}^\theta_T(B^s_{p,r})} \ge \|f\|_{L^\theta_T(B^s_{p,r})}$.

6) $\dot{B}^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0$, $\dot{B}^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0 \, (1 \le p < \infty)$,

where $\mathcal{C}_0$ *is the spaces continuous bounded functions which decay at infinity.*

Lemma 2.3 *Suppose that* $\varrho > 0$ *and* $1 \le p < 2$. *It holds that*

$$\|f\|_{\dot{B}^{-\varrho}_{r,\infty}} \lesssim \|f\|_{L^p},$$

with $\frac{1}{p} - \frac{1}{r} = \frac{\varrho}{n}$. *In particular, this holds with* $\varrho = \frac{n}{2}, r = 2, p = 1$.

Global existence depends on the connection between homogeneous Chemin-Lerner spaces and non-homogeneous Chemin-Lerner spaces, which will be briefly illustrated here; see [11] for a detailed proof.

Proposition 2.1 *Let* $s \in \mathbb{R}$ *and* $1 \le \theta, p, r \le \infty$, *for any* $T > 0$

1) *It holds that ss*

$$L^\theta_T(L^p) \cap \tilde{L}^\theta_T(\dot{B}^s_{p,r}) \subset \tilde{L}^\theta_T(B^s_{p,r}).$$

2) *Moreover, as* $s > 0$ *and* $\theta \ge r$, *it holds that*

$$L^\theta_T(L^p) \cap \tilde{L}^\theta_T(\dot{B}^s_{p,r}) = \tilde{L}^\theta_T(B^s_{p,r}).$$

Proposition 2.2 *Let* $s > 0$ *and* $1 \le p, r \le \infty$, *then* $\dot{B}^s_{p,r} \cap L^\infty$ *is an algebra and*

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty} \|f\|_{\dot{B}^s_{p,r}}.$$

Let $s_1, s_2 \le n/p$ *such that* $s_1 + s_2 > n \max\left\{0, \frac{2}{p} - 1\right\}$. *Then one has*

$$\|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}} \|g\|_{\dot{B}^{s_2}_{p,1}}.$$

Proposition 2.3 *Let* $1 < p < \infty, 1 \le \theta \le \infty$ *and* $s \in \left(-\frac{n}{p}-1, \frac{n}{p}\right]$. *Then there exists a constant* $C > 0$ *that depends only on* s, n *such that*

$$\begin{cases} \left\| \left[f, \dot{\Delta}_q \right] g \right\|_{L^p} \le C c_q 2^{-q(s+1)} \| f \|_{\dot{B}^{\frac{n}{p}+1}_{p,1}} \| g \|_{\dot{B}^s_{p,1}}, \\ \left\| \left[f, \dot{\Delta}_q \right] g \right\|_{L^\theta_T(L^p)} \le C c_q 2^{-q(s+1)} \| f \|_{\tilde{L}^{\theta_1}_T\left(\dot{B}^{\frac{n}{p}+1}_{p,1}\right)} \| g \|_{\tilde{L}^{\theta_2}_T\left(\dot{B}^s_{p,1}\right)}, \end{cases}$$

with $\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$, *where the commutators* $[\cdot,\cdot]$ *is defined by* $[f, g] = fg - gf$ *and* $\{c_q\}$ *denotes a sequence such that* $\|c_q\|_{l^1} \le 1$.

3. Main Theorem and Proofs

It is convenient to rewrite (1)-(2) as the following Cauchy problem

$$\begin{cases} A^0 U_t + A U_x + LU = G(U)_x, \\ U(x,0) = U_0(x), \end{cases} \tag{12}$$

where $G(U) = (0,0,0,g(z),0,0)^{\mathrm{T}}(x)$ with $g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0) z/a := O(z^2)$ near $z = 0$.

The main theorem of this article is shown as follows:

Theorem 3.1 *Suppose* $U_0 \in B^{3/2}_{2,1}(\mathbb{R})$. *There exists a positive constant* δ_0 *such that*

$$\|U_0\|_{B^{3/2}_{2,1}} \le \delta_0,$$

then the Cauchy problem (12) *has a unique global classical solution* $U \in \tilde{\mathcal{C}}^1(\mathbb{R}^+ \times \mathbb{R})$ *satisfying*

$$U \in \tilde{\mathcal{C}}\left(B^{3/2}_{2,1}(\mathbb{R})\right) \cap \tilde{\mathcal{C}}^1\left(B^{1/2}_{2,1}(\mathbb{R})\right).$$

In addition, the following energy inequality holds true

$$\|U\|_{\tilde{L}^\infty_T\left(B^{3/2}_{2,1}(\mathbb{R})\right)} + \left(\|(y,w)\|_{\tilde{L}^2_T\left(B^{\frac{3}{2}}_{2,1}\right)} + \|(v, z_x, \theta_x)\|_{\tilde{L}^2_T\left(B^{\frac{1}{2}}_{2,1}\right)} + \|u_x\|_{\tilde{L}^2_T\left(B^{-\frac{1}{2}}_{2,1}\right)} \right) \le C_0 \|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})},$$

C_0 *is arbitrary constant.*

Next, we start with a couple of equations

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - a y_x = 0, \\ y_t - [\sigma(z/a)]_x - v + \lambda y + \beta \theta_x = 0, \\ \theta_t + k w_x + \beta y_x = 0, \\ \tau_0 w_t + \delta w + k \theta_x = 0. \end{cases} \tag{13}$$

The second and third authors in [11] established a local existence theory for general symmetric hyperbolic systems in critical Besov spaces on the basis of the basic theory established by Kato and Majda [11] [12], which can be applied to problem (12). Precisely,

Proposition 3.1 *Assuming* $U_0 \in B^{3/2}_{2,1}$, *there exists a time* $T_0 > 0$ (*dependent only on initial data*) *when there is*

i) (*Existence*) *The system* (12) *has a unique solution* $U(t,x) \in \mathcal{C}^1([0,T_0]\times\mathbb{R})$ *that is satisfying* $U \in \tilde{\mathcal{C}}_{T_0}(B^{3/2}_{2,1}) \cap \tilde{\mathcal{C}}^1_{T_0}(B^{1/2}_{2,1})$;

ii) (*Blow-up Criterion*) *If the maximal time* $T^*(>T_0)$ *existing of such a solution is finite, then*

$$\limsup_{n\to T^*} \left\|U(t,\cdot)\right\|_{B^{3/2}_{2,1}} = \infty,$$

if and only if

$$\int_0^{T^*} \left\|\nabla U(t,\cdot)\right\|_{L^\infty} \mathrm{d}t = \infty.$$

Moreover, to prove that the classical solution in Proposition 3.1 is globally defined, we need to construct a priori estimates based on the dissipation mechanism generated by the Timoshenko-Cattaneo system. To this end, we define energy functional in terms of $N_0(T)$ and the corresponding dissipation functional in terms of $D_0(T)$:

$$N_0(T) := \|U\|_{\tilde{L}^\infty_T(B^{3/2}_{2,1})},$$

$$D_0(T) := \|(y,w)\|_{\tilde{L}^2_T\left(B^{\frac{3}{2}}_{2,1}\right)} + \|(v,z_x,\theta_x)\|_{\tilde{L}^2_T\left(B^{\frac{1}{2}}_{2,1}\right)} + \|u_x\|_{\tilde{L}^2_T\left(B^{-\frac{1}{2}}_{2,1}\right)},$$

for any time $T > 0$.

Next, we will complete the proof of Proposition 3.1 in several steps:

Step 1: The $L^\infty_T(L^2)$ estimate of U and the $L^2_T(L^2)$ one of (y,w).

In the same way as in [5], the equations in (13) are multiplied by $v, u, (\sigma(z/a)-\sigma(0))/a, y, \theta, w$, and then the resulting equations are added together to give

$$\partial_t \mathcal{H}^l - \partial_x \mathcal{F}^l + \lambda y^2 + \delta w^2 = 0, \tag{14}$$

where

$$\mathcal{H}^l := \frac{1}{2}\left(v^2 + u^2 + F(z) + y^2 + \theta^2 + \tau_0 w^2\right),$$
$$\mathcal{F}^l := uv + \left(\sigma(z/a) - \sigma(0)\right)y - \beta\theta y - k\theta w,$$
$$F(z) := 2\int_0^{\frac{z}{a}} \left(\sigma(\eta) - \sigma(0)\right)\mathrm{d}\eta.$$

In addition, we know that $F(z) \approx |z|^2$. Hence, integrating (14) over $[0,t]\times\mathbb{R}$ yields

$$\|U(t)\|^2_{L^2} + \int_0^t 2\left(\lambda\|y(\tau)\|^2_{L^2} + \delta\|w(\tau)\|^2_{L^2}\right)\mathrm{d}\tau \lesssim \|U_0\|^2_{L^2}. \tag{15}$$

Step 2: The dissipation of (y, w).

The dissipation rate of y, w is obtained by frequency localization estimation in homogeneous Chemin-Lerner space. Applying the operator $\dot{\Delta}_q (q \in \mathbb{Z})$ to (13) yields that

$$\begin{cases} \dot{\Delta}_q v_t - \dot{\Delta}_q u_x + \dot{\Delta}_q y = 0 \\ \dot{\Delta}_q u_t - \dot{\Delta}_q v_x = 0 \\ \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0 \\ \dot{\Delta}_q y_t - \sigma'(z/a)\dot{\Delta}_q (z/a)_x - \dot{\Delta}_q v + \lambda \dot{\Delta}_q y + \beta \dot{\Delta}_q \theta_x = \left[\dot{\Delta}_q, \sigma'(z/a)\right](z/a)_x \\ \dot{\Delta}_q \theta_t + k\dot{\Delta}_q w_x + \beta \dot{\Delta}_q y_x = 0, \\ \tau_0 \dot{\Delta}_q w_t + \delta \dot{\Delta}_q w + k\dot{\Delta}_q \theta_x = 0. \end{cases} \tag{16}$$

In the above equation, the commutator is defined as $[f, g] := fg - gf$. Multiplying the first equation in (16) by $\dot{\Delta}_q v$, the second by $\dot{\Delta}_q u$, the third by $\sigma'(z/a)\dot{\Delta}_q (z/a)_x$, the fourth by $\dot{\Delta}_q y$, the fifth by $\dot{\Delta}_q \theta$, and the sixth by $\dot{\Delta}_q w$, respectively. Then adding up the resulting equations together gives

$$\partial_t \mathcal{H}_1^l + \partial_x \mathcal{F}_1^l + \lambda \left(\dot{\Delta}_q y\right)^2 + \delta \left(\dot{\Delta}_q w\right)^2 = \mathcal{R}_1^l, \tag{17}$$

where

$$\mathcal{H}_1^l := \frac{1}{2}\left(\left(\dot{\Delta}_q v\right)^2 + \left(\dot{\Delta}_q u\right)^2 + \sigma'\left(\frac{z}{a}\right)\left(\dot{\Delta}_q \frac{z}{a}\right)^2 + \left(\dot{\Delta}_q y\right)^2 + \left(\dot{\Delta}_q \theta\right)^2 + \tau_0 \left(\dot{\Delta}_q w\right)^2\right),$$

$$\mathcal{F}_1^l := \dot{\Delta}_q u \dot{\Delta}_q v + \sigma'\left(\frac{z}{a}\right)\dot{\Delta}_q \frac{z}{a}\dot{\Delta}_q y + \beta \dot{\Delta}_q \theta \dot{\Delta}_q y + k\dot{\Delta}_q \theta \dot{\Delta}_q w,$$

$$\mathcal{R}_1^l := \frac{1}{2}\sigma'\left(\frac{z}{a}\right)_t \left(\dot{\Delta}_q \frac{z}{a}\right)^2 - \sigma'\left(\frac{z}{a}\right)_x \left(\dot{\Delta}_q \frac{z}{a}\right)\left(\dot{\Delta}_q y\right) + \dot{\Delta}_q y\left[\dot{\Delta}_q, \sigma'\left(\frac{z}{a}\right)\right]\left(\frac{z}{a}\right)_x.$$

Further, integrating (17) over x, with the aid of Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} N_0\left[\dot{\Delta}_q U\right] + \lambda \left\|\dot{\Delta}_q y\right\|_{L^2}^2 + \delta \left\|\dot{\Delta}_q w\right\|_{L^2}^2 \\ &\lesssim \left\|\sigma'(z)_t\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2}^2 + \left\|\sigma'(z)_x\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2} \left\|\dot{\Delta}_q y\right\|_{L^2} + \left\|\left[\dot{\Delta}_q, \sigma'(z)\right] z_x\right\|_{L^2} \left\|\dot{\Delta}_q y\right\|_{L^2}, \end{aligned} \tag{18}$$

where

$$N_0\left[\dot{\Delta}_q U\right] = \left\|\left(\dot{\Delta}_q v, \dot{\Delta}_q u, \dot{\Delta}_q y, \dot{\Delta}_q \theta, \tau_0 \dot{\Delta}_q \theta\right)\right\|_{L^2}^2 + \int_{\mathbb{R}} \sigma'\left(\frac{z}{a}\right)\left|\dot{\Delta}_q \frac{z}{a}\right|^2 \mathrm{d}x \approx \left\|\dot{\Delta}_q U\right\|_{L^2}^2.$$

From the above (12) and the following a priori assumption (39), we get

$$\left\|\sigma'(z)_t\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2}^2 \lesssim \left\|z_t\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2}^2 \lesssim \left\|y_x\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2}^2. \tag{19}$$

Similarly,

$$\left\|\sigma'(z)_x\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2} \left\|\dot{\Delta}_q y\right\|_{L^2} \lesssim \left\|z_x\right\|_{L^\infty} \left\|\dot{\Delta}_q z\right\|_{L^2} \left\|\dot{\Delta}_q y\right\|_{L^2}. \tag{20}$$

Combining (19)-(20), by integrating over $t\in[0,T]$ and applying the Young's inequality, we get

$$
\begin{aligned}
&\sqrt{N_0\left[\dot{\Delta}_q U\right]}+\sqrt{2\lambda}\left\|\dot{\Delta}_q y\right\|_{L_T^2(L^2)}+\sqrt{2\delta}\left\|\dot{\Delta}_q w\right\|_{L_T^2(L^2)}\\
&\lesssim\sqrt{N_0\left[\dot{\Delta}_q U_0\right]}+\sqrt{\left\|(y_x,z_x)\right\|_{L_T^\infty(L^\infty)}}\left(\left\|\dot{\Delta}_q y\right\|_{L_T^2(L^2)}+\left\|\dot{\Delta}_q z\right\|_{L_T^2(L^2)}\right)\\
&\quad+\sqrt{\left\|\left[\dot{\Delta}_q,\sigma'(z)\right]z_x\right\|_{L_T^2(L^2)}\left\|\dot{\Delta}_q y\right\|_{L_T^2(L^2)}}.
\end{aligned}\tag{21}
$$

By the commutator estimate in Proposition 2.3, it holds

$$
\left\|\left[\dot{\Delta}_q,\sigma'(z)\right]z_x\right\|_{L_T^2(L^2)}\lesssim c_q 2^{-\frac{3q}{2}}\|z\|_{\tilde{L}_T^\infty\left(\dot{B}_{2,1}^{3/2}\right)}\|z_x\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{1/2}\right)},\tag{22}
$$

where $\{c_q\}$ denotes a sequence satisfying $\|c_q\|_{l^1}\le 1$. Therefore, we have

$$
\begin{aligned}
&2^{\frac{3q}{2}}\left\|\dot{\Delta}_q U\right\|_{L_T^\infty(L^2)}+\sqrt{2\lambda}\,2^{\frac{3q}{2}}\left\|\dot{\Delta}_q y\right\|_{L_T^2(L^2)}+\sqrt{2\delta}\,2^{\frac{3q}{2}}\left\|\dot{\Delta}_q w\right\|_{L_T^2(L^2)}\\
&\lesssim\left\|\dot{\Delta}_q U_0\right\|_{L^2}+c_q\sqrt{\left\|(y_x,z_x)\right\|_{L_T^\infty\left(\dot{B}_{2,1}^{1/2}\right)}}\left(\|y\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{3/2}\right)}+\|z_x\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{1/2}\right)}\right)\\
&\quad+c_q\sqrt{\|z\|_{\tilde{L}_T^\infty\left(\dot{B}_{2,1}^{3/2}\right)}}\left(\|y\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{3/2}\right)}\|z_x\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{1/2}\right)}\right).
\end{aligned}\tag{23}
$$

Here, we would like to point out that each $\{c_q\}$ may have a different form in (23) or the inequality that emerges after, but the bound of $\|c_q\|_{l^1}\le 1$ is well satisfied. Thus, summing over $q\in\mathbb{Z}$, we can get

$$
\begin{aligned}
&\|U\|_{\tilde{L}_T^\infty\left(\dot{B}_{2,1}^{3/2}\right)}+\sqrt{2\lambda}\|y\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{3/2}\right)}+\sqrt{2\delta}\|w\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{3/2}\right)}\\
&\lesssim\|U_0\|_{\left(\dot{B}_{2,1}^{3/2}\right)}+\sqrt{\|(y,z)\|_{\tilde{L}_T^\infty\left(\dot{B}_{2,1}^{3/2}\right)}}\left(\|y\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{3/2}\right)}+\|z_x\|_{\tilde{L}_T^2\left(\dot{B}_{2,1}^{1/2}\right)}\right).
\end{aligned}\tag{24}
$$

From (15) we have

$$
\|U\|_{L_T^\infty(L^2)}+\sqrt{2\lambda}\|y\|_{L_T^2(L^2)}+\sqrt{2\delta}\|w\|_{L_T^2(L^2)}\lesssim\|U_0\|_{L^2}.\tag{25}
$$

Combining (24)-(25) gives

$$
N_0(T)+\|(y,w)\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)}\lesssim\|U_0\|_{B_{2,1}^{3/2}}+\sqrt{N_0(T)}D_0(T).\tag{26}
$$

Step 3: The dissipation of θ_x.

Multiplying the fifth equation and the sixth equation in (16) by $-\tau_0\dot{\Delta}_q w_x$ and $\dot{\Delta}_q\theta_x$, respectively. Then adding up the results obtained yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}N_1\left[\dot{\Delta}_q U\right]+\left\|\dot{\Delta}_q\theta_x\right\|_{L^2}^2\lesssim\left\|\dot{\Delta}_q w_x\right\|_{L^2}^2+\left\|\dot{\Delta}_q w_x\dot{\Delta}_q y_x\right\|_{L^2}+\left\|\dot{\Delta}_q\theta_x\dot{\Delta}_q w\right\|_{L^2},\tag{27}
$$

where

$$
N_1\left[\dot{\Delta}_q U\right]:=-\tau_0\int_{\mathbb{R}}\dot{\Delta}_q\theta\dot{\Delta}_q w_x\mathrm{d}x.
$$

Integrating (27) on $t\in[0,T]$, we can get

$$\begin{aligned}\|\theta_x\|_{L_T^2(L^2)}^2 &\lesssim \left(\left|N_1\left[\dot{\Delta}_q U\right]\right| + N_1\left[\dot{\Delta}_q U_0\right]\right) + \left\|\dot{\Delta}_q w_x\right\|_{L_T^2(L^2)}^2 \\ &\quad + \left\|\dot{\Delta}_q w_x\right\|_{L_T^2(L^2)} \left\|\dot{\Delta}_q y_x\right\|_{L_T^2(L^2)} + \left\|\dot{\Delta}_q \theta_x\right\|_{L_T^2(L^2)} \left\|\dot{\Delta}_q w\right\|_{L_T^2(L^2)} \\ &\lesssim \left\|\dot{\Delta}_q\right\|_{L_T^\infty(L^2)}^2 + \|U_0\|_{L^2}^2 + \left\|\dot{\Delta}_q w_x\right\|_{L_T^2(L^2)}^2 \\ &\quad + \left\|\dot{\Delta}_q w_x\right\|_{L_T^2(L^2)} \left\|\dot{\Delta}_q y_x\right\|_{L_T^2(L^2)} + \left\|\dot{\Delta}_q \theta_x\right\|_{L_T^2(L^2)} \left\|\dot{\Delta}_q w\right\|_{L_T^2(L^2)}\end{aligned} \tag{28}$$

Furthermore, with the aid of Young's inequality, we obtain

$$\begin{aligned}2^{\frac{q}{2}} \left\|\dot{\Delta}_q \theta_x\right\|_{L_T^2(L^2)} &\lesssim c_q \|U\|_{\tilde{L}_T^\infty\left(\dot{B}_{2,1}^{1/2}\right)} + c_q \|U_0\|_{\dot{B}_{2,1}^{3/2}} + \varepsilon_1 \|w_x\|_{L_T^2\left(\dot{B}_{2,1}^{1/2}\right)} \\ &\quad + C(\varepsilon_1)\|y_x\|_{L_T^2\left(\dot{B}_{2,1}^{1/2}\right)} + C(\varepsilon_2)\|w\|_{L_T^2\left(\dot{B}_{2,1}^{1/2}\right)}.\end{aligned} \tag{29}$$

By multiplying the fifth equation and the sixth equation in (13) by $-\tau_0 w_x$ and θ_x, respectively, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} N_2(U) + \|\theta_x\|_{L^2}^2 \lesssim \|w_x\|_{L^2}^2 + \|w_x y_x\|_{L^2} + \|\theta_x w\|_{L^2}, \tag{30}$$

where

$$N_2(U) := -\tau_0 \int_{\mathbb{R}} \theta w_x \mathrm{d}x.$$

Applying Young's inequality enables us to get

$$\frac{\mathrm{d}}{\mathrm{d}t} N_2(U) + \|\theta_x\|_{L^2}^2 \lesssim \|y\|_{L^2}^2 + \|y_x\|_{L^2}^2 + \|w_x\|_{L^2}^2 + \|w\|_{L^2}^2. \tag{31}$$

Integrating (31), we obtain

$$\begin{aligned}\|\theta_x\|_{L_t^2(L^2)}^2 &\lesssim \left(\left|N_2[U]\right| + N_2[U_0]\right) + \|w_x\|_{L_t^2(L^2)}^2 \\ &\quad + \|w_x\|_{L_t^2(L^2)} \|y_x\|_{L_t^2(L^2)} + \|\theta_x\|_{L_t^2(L^2)} \|w_x\|_{L_t^2(L^2)} \\ &\lesssim N_0^2(t) + \|U_0\|_{L_T^2(L^2)}^2 + \|y\|_{L_T^2(L^2)}^2 + \|w\|_{L_T^2(L^2)}^2.\end{aligned} \tag{32}$$

Then, we use the Young's inequality to give

$$\|\theta_x\|_{L_T^2(L^2)} \lesssim N_0(t) + \|U_0\|_{B_{2,1}^{3/2}} + \|y\|_{L_T^2(L^2)} + \|w\|_{L_T^2(L^2)}. \tag{33}$$

Combining (29) and (33), we get

$$\|\theta_x\|_{L_T^2\left(B_{2,1}^{1/2}\right)} \lesssim \|U_0\|_{B_{2,1}^{3/2}} + \|w\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)} + \|y\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)} + N_0(t). \tag{34}$$

The calculations in Step 4 to Step 7 below are the same as those in [8] [9], with the difference that there is an additional dissipation about θ in our model, but the presence of this item does not affect the overall operation. The θ appearing in the computation for $z_x, v(a=1)$ is handled by using Hölder inequality, and the θ appearing in $v(a \neq 1)$ is handled with the aid of the inequality $2^q \|\Delta_q v\|_{L_T^2(L^2)} \|\Delta_q \theta_x\|_{L_T^2(L^2)} \le \varepsilon_1 \|v\|_{L_T^2\left(B_{2,1}^{1/2}\right)}^2 + C_{\varepsilon_1} \|\theta_x\|_{L_T^2\left(B_{2,1}^{1/2}\right)}^2$, thus the desired dissipation constraints on θ can be obtained. Here the proofs are omitted for the sake of simplicity.

Step 4: The dissipation for z_x**.**

If $U \in \tilde{C}_T\left(B_{2,1}^{3/2}\right) \cap \tilde{C}_T^1\left(B_{2,1}^{1/2}\right)$ is a solution of (12) for any $T>0$, then

$$\begin{aligned}\|z_x\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)} \lesssim\, & N_0(T)+\|U_0\|_{B_{2,1}^{3/2}}+\|y\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)}+\|v\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)} \\ & +\|\theta_x\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)}+\sqrt{N_0(T)}D_0(T).\end{aligned} \tag{35}$$

Step 5: The dissipation for u_x**.**

If $U \in \tilde{C}_T\left(B_{2,1}^{3/2}\right) \cap \tilde{C}_T^1\left(B_{2,1}^{1/2}\right)$ is a solution of (12) for any $T>0$, we have

$$\|u_x\|_{\tilde{L}_T^2\left(B_{2,1}^{-1/2}\right)} \lesssim N_0(T)+\|U_0\|_{B_{2,1}^{3/2}}+\|v\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)}+\|y\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)}. \tag{36}$$

Step 6: The dissipation for $v(a=1)$**.**

If $U \in \tilde{C}_T\left(B_{2,1}^{3/2}\right) \cap \tilde{C}_T^1\left(B_{2,1}^{1/2}\right)$ is a solution of (12) for any $T>0$, it holds

$$\|v\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)} \lesssim N_0(T)+\|U_0\|_{B_{2,1}^{3/2}}+\|y\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)}+\|\theta_x\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)}+\sqrt{N_0(T)}D_0(T). \tag{37}$$

Step 7: The dissipation for $v(a\neq 1)$**.**

If $U \in \tilde{C}_T\left(B_{2,1}^{3/2}\right) \cap \tilde{C}_T^1\left(B_{2,1}^{1/2}\right)$ is a solution of (12) for any $T>0$, then

$$\begin{aligned}\|v\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)} \lesssim\, & N_0(T)+\|U_0\|_{B_{2,1}^{3/2}}+\varepsilon_3\|u_x\|_{\tilde{L}_T^2\left(B_{2,1}^{-1/2}\right)}+\left(1+C_{\varepsilon_3}\right)\|y\|_{\tilde{L}_T^2\left(B_{2,1}^{3/2}\right)} \\ & +C_{\varepsilon_4}\|\theta_x\|_{\tilde{L}_T^2\left(B_{2,1}^{1/2}\right)}+N_0(T)D_0(T).\end{aligned} \tag{38}$$

Combining (26), (34)-(38), the proof of Proposition 3.2 is finished.

Proposition 3.2 *Suppose that for any* $T>0$, $U \in \tilde{C}\left(B_{2,1}^{3/2}(\mathbb{R})\right) \cap \tilde{C}^1\left(B_{2,1}^{1/2}(\mathbb{R})\right)$ *is a solution of* (12), *and there exists* $\delta_1>0$ *such that when*

$$N_0(T) \le \delta_1, \tag{39}$$

the next estimate holds:

$$N_0(T)+D_0(T) \lesssim \|U_0\|_{B_{2,1}^{3/2}}+\left(N_0(T)+\sqrt{N_0(T)}\right)D_0(T). \tag{40}$$

Therefore, the next inequality will hold

$$N_0(T)+D_0(T) \lesssim \|U_0\|_{B_{2,1}^{3/2}}. \tag{41}$$

By using the standard boot-strap argument, the proof of Theorem 3.1 is similar to the process of [8], and we omit the details here for brevity.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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