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Hyper Catastrophe on 4-Dimensional Canards

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Abstract

When discovering the potential of canards flying in 4-dimensional slow-fast system with a bifurcation parameter, the key notion "symmetry" plays an important role. It is of one parameter on slow vector field. Then, it should be determined to introduce parameters to all slow/fast vectors. It is, however, there might be no way to explore for another potential in this system, because the geometrical structure is quite different from the system with one parameter. Even in this system, the "symmetry" is also useful to obtain the potentials classified by R. Thom. In this paper, via the coordinates changing, the possible way to explore for the potential will be shown. As it is analyzed on "hyper finite time line", or done by using "non-standard analysis", it is called "Hyper Catastrophe". In the slow-fast system which includes a very small parameter ϵ, it is difficult to do precise analysis. Thus, it is useful to get the orbits as a singular limit. When trying to do simulations, it is also faced with difficulty due to singularity. Using very small time intervals corresponding small ϵ, we shall overcome the difficulty, because the difference equation on the small time interval adopts the standard differential equation. These small intervals are defined on hyper finite number N, which is nonstandard. As ϵ and the intervals are linked to use $1/N$, the simulation should be done exactly.

Keywords

Canards Flying, 4-Dimensional Slow-Fast System, Hyper Catastrophe

1. Introduction

There exist two kinds of "catastrophe". One is a statical model. Corresponding multi variable scalar functions are classified from the Hessian matrix on non-degenerate critical points. See [1] [2] [3] and [4]. The other one is a dynamical model. It is induced from the slow-fast system with parameters, which has the singular limit orbit. The aim of this paper is to describe the relation between "statical model" and "dynamical model". It becomes clear that the 4-dimensional slow-fast system with co-dimension 2 gives us a new structure. In section 2, we give standard assumptions as a preliminary, and "remark 4" and "remark 5" are important to realize the framework. Inserting a bifurcation parameter "b" newly, see [5], it causes to explore for the potential. In section 3, the slow-fast system having bifurcation parameters for all slow/fast vectors will be described. Then, it will be shown how to construct another potential via projection changing the coordinates R^4 to R^2. In section 4, a neuron system [6] will be given as a concrete system.

2. Preliminary

Now, let us consider the following slow-fast system:

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}t} = h(x, y, \epsilon) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, \epsilon) \end{cases} \tag{1}$$

where ϵ is infinitesimal, and

$$x = (x_1, x_2) \in R^2,\ \ y = (y_1, y_2) \in R^2,$$

$$h = (h_1, h_2): R^4 \to R^2,\ \ g = (g_1, g_2): R^4 \to R^2.$$

Then, assume that the origin is a singular point.

Furthermore, we assume that the system (1) satisfies the following conditions (A1) - (A6), they are the same as describing our previous paper.

(A1) h is of class $\mathbf{C}^1$ and g is of class $\mathbf{C}^2$.

(A2) The slow manifold $S = \{(x, y) \in \mathbf{R}^4 \mid h(x, y, 0) = 0\}$ is a two-dimensional differential manifold and intersects the set

$$T = \left\{(x, y) \in \mathbf{R}^4 \mid \det\left[\frac{\partial h}{\partial x}(x, y, 0)\right] = 0\right\} \tag{2}$$

transversely, where

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial x_2} \\ \dfrac{\partial h_2}{\partial x_1} & \dfrac{\partial h_2}{\partial x_2} \end{bmatrix} \tag{3}$$

Then, the pli set

$$PL = \{(x, y) \in S \cap T\} \tag{4}$$

is a one-dimensional differentiable manifold.

(A3) Either the value of g_1 or that of g_2 is nonzero at any point of PL.

Note that the pli set PL divides the slow manifolds $S\backslash PL$ into three parts depending on the signs of the two eigenvalues of $\dfrac{\partial h}{\partial x}(x, y, 0)$.

First, consider the following reduced system which is obtained from (1) with $\varepsilon = 0$:

$$\begin{cases} 0 = h(x, y, 0) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, \epsilon) \end{cases} \tag{5}$$

By differentiating $h(x, y, 0)$ with respect to *t*, we have

$$\frac{\partial h}{\partial x}(x, y, 0)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial h}{\partial y}(x, y, 0) g(x, y, \epsilon) = 0 \tag{6}$$

Then (4) becomes the following:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1} \dfrac{\partial h}{\partial y}(x, y, 0) g(x, y, \epsilon) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = g(x, y, \epsilon) \end{cases} \tag{7}$$

where $(x, y) \in S \setminus PL$. To avoid degeneracy in (6), we consider the time-scaled-reduced system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \left\{-\det\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1}\right\}\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1} \dfrac{\partial h}{\partial y}(x, y, 0) g(x, y, \epsilon) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \left\{\det\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right]^{-1}\right\} g(x, y, \epsilon) \end{cases} \tag{8}$$

The phase portrait of the system (8) is the same as that of (7) except the region where $\det\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right] = 0$, but only the orientation of the orbit is different.

Definition 1 *A singular point of* (8)*, which is on PL, is called a pseudo singular point of* (1)*. The set of pseudo singular points is denoted by PS.*

(A4) $rank\left[\dfrac{\partial h}{\partial x}(x, y, 0)\right] = 2$, $rank\left[\dfrac{\partial h}{\partial y}(x, y, 0)\right] = 2$ for any $(x, y) \in S \setminus PL$.

From (A4), the implicit function theorem guarantees the existence of a unique function $y = \varphi(x)$ such that $h(x, \varphi(x), 0) = 0$. By using $y = \varphi(x)$, we obtain the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left\{-\det\left[\frac{\partial h}{\partial x}(x, \varphi(x), 0)\right]^{-1}\right\}\left[\frac{\partial h}{\partial x}(x, \varphi(x), 0)\right]^{-1} \frac{\partial h}{\partial y}(x, \varphi(x), 0) g(x, \varphi(x), \epsilon). \tag{9}$$

(A5) All singular points of (8) are non-degenerate, that is, the linearization of (8) at a singular point has two nonzero eigenvalues.

Now, let us introduce a definition of "symmetry". It is a key word through this paper.

Definition 2 *If* $h_1(x_1, x_2, y_1, y_2, \epsilon) = h_2(x_2, x_1, y_2, y_1, \epsilon)$*, and* $g_1(x_1, x_2, y_1, y_2, \epsilon) = g_2(x_2, x_1, y_2, y_1, \epsilon)$*, then the system is "symmetric" for the subspace* $I = \{(x_1, x_2, y_1, y_2) \mid x_1 = x_2, y_1 = y_2\}$.

(A6) *I* intersects PL transversely.

Definition 3 *Let* λ_1, λ_2 *be two eigenvalues of the linearization of* (8) *at a pseudo singular point. The pseudo singular point with real eigenvalues is called a pseudo singular saddle point if* $\lambda_1 < 0 < \lambda_2$ *and a pseudo singular node point if*

$\lambda_1 < \lambda_2 < 0$ *or* $\lambda_1 > \lambda_2 > 0$.

The following Theorems 1 is established in [7], [8] and [9] respectively.

Theorem 1 *Let* (x_0, y_0) *be a pseudo singular saddle or node point. If* $trace\left[\frac{\partial h}{\partial x}(x_0, y_0, 0)\right] < 0$, *then there exists a solution which first follows the attractive part and the repulsive part after crossing PL near the pseudo singular point.*

Remark 1 *The solution in Theorem 1 is called "canard".*

Remark 2 *The condition* $trace\left[\frac{\partial h}{\partial x}(x_0, y_0, 0)\right] < 0$ *implies that one of eigenvalues of* $\left[\frac{\partial h}{\partial x}(x_0, y_0, 0)\right]$ *is equal to zero and the other one is negative. Notice that the system has two kinds of vector fields: one is* 2*-dimensional slow and the other is* 2*-dimensional fast one. The condition provides the state of the fast vector field.*

Remark 3 *The singular solution in Theorem* 1 *is called a canard in* R^4 *with* 2*-dimensional slow manifold. As a result, it causes a delayed jumping. The study of canards requires still more precise topological analysis on the slow vector field.*

Remark 4 *On the subspace I, the following system is established for some b. I is an invariant manifold.*

$$\begin{cases} \varepsilon \dfrac{dx_1}{dt} = h_1(x_1, y_1, \varepsilon) \\ \dfrac{dy_1}{dt} = g_1(x_1, y_1, b) \end{cases} \tag{10}$$

Remark 5 *On the set PL,* $\det\left[\frac{\partial h}{\partial x}\right] = 0$ *is satisfied and at* $(x_0, y_0) \in PS$ *the following equation is established:*

$$\begin{aligned} &\left\{\frac{\partial h_1}{\partial x_1}(x_0, \varphi(x_0), 0)\frac{\partial h_2}{\partial x_2}(x_0, \varphi(x_0), 0)\right. \\ &\left.-\frac{\partial h_1}{\partial x_2}(x_0, \varphi(x_0), 0)\frac{\partial h_2}{\partial x_1}(x_0, \varphi(x_0), 0)\right\} g_1(x_0, \varphi(x_0), b) = 0. \end{aligned} \tag{11}$$

Note that there exists $y = \phi(x)$ because of assuming $rank\left[\frac{\partial h}{\partial y}\right] = 2$.

3. Bifurcation on Slow/Fast Vectors

From now on, let us consider the following system extended having parameters for slow/fast vectors:

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = h(\alpha x, \beta y, \epsilon) \\ \dfrac{dy}{dt} = g(\gamma x, \delta y, \epsilon) \end{cases} \tag{12}$$

where $\alpha = (b_1, b_2)$, $\beta = (b_3, b_4)$, $\gamma = (b_5, b_6)$ and $\delta = (b_7, b_8)$.

On the other hand,

$$\det\left[\frac{\partial h}{\partial x}\right] = b_1 b_2 \left\{\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right\}. \tag{13}$$

In fact,

$$\frac{\partial h_1(b_1x_1, b_2x_2, b_3y_1, b_4y_2, 0)}{\partial x_1} = b_1 \frac{\partial h_1(x_1, x_2, b_3y_1, b_4y_2, 0)}{\partial x_1}, \tag{14}$$

$$\frac{\partial h_1(b_1x_1, b_2x_2, b_3y_1, b_4y_2, 0)}{\partial x_2} = b_2 \frac{\partial h_1(x_1, x_2, b_3y_1, b_4y_2, 0)}{\partial x_2},$$

$$\frac{\partial h_2(b_1x_1, b_2x_2, b_3y_1, b_4y_2, 0)}{\partial x_2} = b_1 \frac{\partial h_2(x_1, x_2, b_3y_1, b_4y_2, 0)}{\partial x_2},$$

$$\frac{\partial h_2(b_1x_1, b_2x_2, b_3y_1, b_4y_2, 0)}{\partial x_2} = b_2 \frac{\partial h_2(x_1, x_2, b_3y_1, b_4y_2, 0)}{\partial x_2}.$$

Lemma 1

$$\det\left[\frac{\partial h}{\partial x}\right] = \frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1} = 0. \tag{15}$$

It does not depend on bifurcation parameter b_1, b_2.

Theorem 2 *If the system having parameters for slow/fast vectors is* "*symmetric*", *then it has a potential classified by R. Thom.*

Proof. Under the rank condition (A4), proceeding a projection (changing the co-ordinates):

$$\begin{pmatrix} X \\ Y \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{16}$$

like as $y_1 + y_2 = Y, x_1 + x_2 = X$, then, the potentials are obtained as "elementary catastrophe" under the following conditions.

At around $(x_0, y_0) \in PS$,

(i) $\dfrac{\partial^3 h_1}{\partial x_1^3} \neq 0, \dfrac{\partial h_1}{\partial x_1} \neq 0$.

(ii) $\dfrac{\partial^4 h_1}{\partial x_1^4} \neq 0, \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0, \dfrac{\partial h_1}{\partial x_1} \neq 0$.

(iii) $\dfrac{\partial^5 h_1}{\partial x_1^5} \neq 0, \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0, \dfrac{\partial h_1}{\partial x_1} \neq 0$ and $\dfrac{\partial^4 h_1}{\partial x_1^4} = 0$.

(iv) $\dfrac{\partial^6 h_1}{\partial x_1^6} \neq 0, \dfrac{\partial^4 h_1}{\partial x_1^4} \neq 0, \dfrac{\partial^3 h_1}{\partial x_1^3} \neq 0, \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0, \dfrac{\partial h_1}{\partial x_1} \neq 0$.

(v) $\dfrac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0, \dfrac{\partial^3 h_2}{\partial x_2^3} \neq 0, \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0, \dfrac{\partial h_2}{\partial x_2} \neq 0$ and $\dfrac{\partial^3 h_1}{\partial x_2^2 \partial x_1} = 0, \dfrac{\partial^2 h_1}{\partial x_1 \partial x_2} = 0$.

(vi) $\dfrac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0, \dfrac{\partial^3 h_2}{\partial x_2^3} \neq 0, \dfrac{\partial^2 h_1}{\partial x_1^2} \neq 0, \dfrac{\partial h_1}{\partial x_2} \neq 0, \dfrac{\partial h_1}{\partial x_1} \neq 0$.

(vii) $\frac{\partial^3 h_1}{\partial x_1^2 \partial x_2} \neq 0, \frac{\partial^4 h_2}{\partial x_2^4} \neq 0, \frac{\partial^2 h_2}{\partial x_2^2} \neq 0, \frac{\partial^2 h_1}{\partial x_2^1} \neq 0, \frac{\partial h_2}{\partial x_2} \neq 0, \frac{\partial h_1}{\partial x_1} \neq 0$.

Remark 6 *As the system is "symmetric", the conditions are described exclusively, for example, the condition* (1) *is as the following,*

$$\frac{\partial^3 h_2}{\partial x_2^3} \neq 0, \quad \frac{\partial h_2}{\partial x_2} \neq 0.$$

Remark 7 *It is called "Hyper Catastrophe", which is composed of the potential reduced from the slow manifold* ($\epsilon = 0$). *Although it is using non-standard analysis, for example* ϵ *is infinitesimal, "Transfer Principle" ensures that it is established in standard analysis. Then, the slow manifold is obtained as the singular limit* (ϵ *tends to zero*). *They are "dynamical catastrophe" but not "statical one". See* [10] [11].

4. Concrete Example

Consider the equation,

$$\begin{cases} \varepsilon b_1 \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = b_2 x_2 + b_3 y_1 + \dfrac{b_1^3 x_1^3}{3} \\ \varepsilon b_2 \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = b_1 x_1 + b_4 y_2 + \dfrac{b_2^3 x_2^3}{3} \\ b_3 \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = -\left(b_1 x_1 + b_3 y_1\right) \\ b_4 \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = -\left(b_2 x_2 + b_4 y_2\right) \end{cases} \tag{17}$$

where $b_i \left(i = 1,2,3,4\right)$ are bifurcation parameters.

Changing the coordinates like as

$$\begin{cases} x_1 + x_2 = X \\ \dfrac{3b_3}{b_1^3 y_1} + \dfrac{3b_4}{b_2^3 y_2} = Y \end{cases} \tag{18}$$

then on the axis *X*, getting the function

$$Y = x_1^3 + x_2^3 - \frac{3b_2}{b_1^3 x_2} - \frac{3b_1}{b_2^3 x_1}. \tag{19}$$

This potential function is "hyperbolic umbilic" classified by R. Thom.

Using variables X, Y, when satisfying $b_1 = b_2$,

$$Y = X^3 - kX + O\left(X^3\right) \text{ and } k = \frac{3}{b_1^2}. \tag{20}$$

Corollary 1 *If* $\frac{\partial^2 h_1}{\partial x_1 \partial x_2}$ *or* $\frac{\partial^2 h_2}{\partial x_2 \partial x_1}$ *takes nearly equal* 0, *there exists "hyperbolic umbilic". Changing the coordinates, it is "fold" catastrophe.*

Remark 8 *In the concrete example, the values corresponding partial derivatives are at around the origin. For the simplicity, it is not at the pseudo singular point.*

5. Conclusion

In general, the advanced system having parameters for slow/fast vectors looks like very complicated aspects geometrically. It is sure on 4-dimensional, however, changing coordinates makes their appearance. Note that "changing coordinates" implies proceeding "projection" to the lower dimension. Remember that the original constrained surface is 2-dimensional manifold in R^4. It is "dynamical" catastrophe but not "statical" one. In the beginning of "bifurcation problem", many people used the word "structural stability" for the original differential equations. Under being the stability, the statical model is applied. In this paper, it is used for the bifurcation parameter on the pseudo-singular point. It might be needed to emphasize again in order to avoid giving somehow confusions. In practice, recently, people need a new tool to analyze economics and to do brain mechanism. Dynamical catastrophe describes fundamental structure through these potentials.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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