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Spherical Functions on Fuzzy Lie Group

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Abstract

Let G be a locally compact Lie group and $\mathfrak{g}$ its Lie algebra. We consider a fuzzy analogue of G, denoted by $\mathfrak{G}_f$ called a fuzzy Lie group. Spherical functions on $\mathfrak{G}_f$ are constructed and a version of the existence result of the Helgason-spherical function on G is then established on $\mathfrak{G}_f$.

Keywords

Fuzzy Spherical Function, Fuzzy Lie Group, Fuzzy Manifolds

1. Introduction

One of the basic tools for classical computation, modeling and reasoning is crisp, which is exact in nature. A crisp is dichogamous, indicating Yes or No type rather than more or less type. In this case, a membership function, is often used to assign binary values to each element of the universal X. Fuzzy set theory gives sufficient mathematical configuration in which vague conceptual facts can be precisely and rigorously examined. This has found application in many fields, including, computer science, biomedical engineering, telecommunication, decision making, differential equations, rings, semirings, group, automation and robotics, networking, discrete mathematics, etc.

This originated from the novel work of Zadeh in 1965 [1] which was introduced to handle the notion of partial truth between "absolute true" and "absolute false". Fuzzy vectors, fuzzy topological spaces were introduced and exhaustively considered in [2] [3] and [4]. I. Kubiak [5] and Sostak [6] considered the key idea of fuzzy topological structures, as an expansion of both crisp and fuzzy topology. A locally convex property of these topologies has been given in [7]. The idea of fuzzy topology on fuzzy sets was presented by Chalarabarty & Ahsanullah [8] as one of the treatments of the issue which might be known as the subspace

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issue in fuzzy topological spaces. The general idea of fuzzy Lie algebras was introduced by Akram in 2018 [9] where fuzzy sets were applied to Lie algebras. Nadja khah, M., *et al.* [10] gave the notion of fuzzy sets applications to Lie groups and concepts relating to them. First, they considered C^1-fuzzy manifolds of a fuzzy transformation group and its fuzzy G-invariant property and then gave suitable conditions for defining fuzzy invariant differential operators on the *G*. Spherical function on general locally compact groups has been sufficiently studied (see [11], [12], [13], [14]). Helgason, S. (1984), [15] investigated the structure of the ring $DG(X)$ of *G*-invariant differential operators on a reductive spherical homogeneous space $X/G = H$ with an over group D_eG. We shall construct a polynomial algebra $D\mathfrak{G}_{\mathfrak{f}}^{\mathfrak{K}}(X)$ which is $\mathfrak{G}_{\mathfrak{f}}$-invariant differential operators on *X* with respect to $\mathfrak{K}$, coming from the centers of the enveloping algebra of $\mathscr{U}(G)$ of $\mathfrak{G}_{\mathfrak{f}}$ and $\mathfrak{K}$ where $\mathfrak{K}$ is a maximal proper subgroup of $\mathfrak{G}_{\mathfrak{f}}$.

2. Preliminaries

In this section, we give some basis definitions that will be needed in the sequel following [9].

Definition 2.1 **[9]** Let *S* is a nonempty set and $s \in S$. A fuzzy set μ in a universe *S* is a mapping $\mu : S \to [0,1] \subset \mathbb{R}$. A fuzzy subset $\tilde{A}$ in *S* is a set $\mu_{\tilde{A}} : S \to [0,1] \subset \mathbb{R}$ also identified with the graph

$$\tilde{A} = \{(s, \mu_{\tilde{A}}(s)) \mid s \in S\},$$

of ordered pairs where $\mu_{\tilde{A}}(s)$ is called the membership function.

The fuzzy empty set is denoted by 0_S and defined as $0_S(s) = 0 \quad \forall s \in S$ while the entire set in a set *S* is denoted 1_S and defined as $1_S(s) = 1$ for all $s \in S$. The basic operations on fuzzy sets and their standard results can be seen in [9] and [14].

Definition 2.2 Let the universe of discourse be *S* and *U* a fuzzy set on a *S*. Given $t \in [0,1]$, we define a *t*-cut set (*t*-level set) of *U* as

$$U_t = \{s \in S \mid U(s) \geq t\}.$$

2.1. Fuzzy Vector Spaces and Topology of Fuzzy sets

Definition 2.3 Let *E* be a vector space and $A_1 \cdots A_n$ be fuzzy sets in *E*. We define $A_1 \times \cdots \times A_n$ to be the fuzzy set *A* in E^n whose membership function is given by

$$\mu_A(s_1, \cdots, s_n) = \min\{\mu_{A_1}(s_1), \cdots, \mu_{A_n}(s_n)\}$$

Let $f : E^n \to E$, $f(s_1, \cdots, s_n) = s_1 + \cdots + s_n$. We define $A_1 + \cdots + A_n = f(A)$.

For $\lambda \in \mathbb{K}$ and *D* a fuzzy set in *E*, we define $\lambda D = g(D)$, where $g : E \to E$, $g(s) = \lambda(s)$.

Definition 2.4 Let $S \neq \varnothing$ and $I = [0,1]$. Define the set I^S by $I^S := \{\gamma \mid \gamma : S \to I\}$. Then, $\gamma \in I^S$ is known as fuzzy subset of *S*. Let $\gamma \in I^S$. A collection σ of fuzzy subsets of γ satisfying the following:

$$\kappa \cap \gamma \in \sigma \quad \forall \kappa \in I \tag{1}$$

$$U_i \in \sigma \, \forall i \in N \Rightarrow \bigcup \{U_i : i \in N\} \in \sigma \tag{2}$$

$$U, V \in \sigma \Rightarrow U \cap V \in \sigma \tag{3}$$

is called a fuzzy topology on γ and the pair (γ, σ) is called a fuzzy topological space. Members of σ are called fuzzy open sets and their complements with respect to γ are known as closed sets of (γ, σ).

If $\mathfrak{B}$ be a collection of fuzzy subset of γ, then the family of arbitrary unions and finite intersections of the member of $\mathfrak{B}$ and the family $\{\gamma \cap \kappa; \kappa \in I\}$ forms a fuzzy topology on γ denoted by $\sigma(\mathfrak{B})$.

Definition 2.5 $\mathfrak{B} \in \sigma$ is referred to as open base of σ if every member of σ can be expressed uniquely as the union of certain members of $\mathfrak{B}$.

Definition 2.6 A fuzzy topological space (γ, σ) is said to be Hausdorff if $\forall x_p, y_p \in \gamma$ $(x \neq y)$, $\exists U, V \in \sigma$ such that $x_p \in U$, $y_p \in V$ and $U \cap V = \varnothing$.

Definition 2.7 A fuzzy topological space (γ, σ) is said to be fuzzy compact if $\forall \beta \in \sigma$ with $\bigcup \{U : U \in \beta\} = \gamma$ and $\forall \varepsilon > 0$, $\exists$ a finite subcollection β_0 of β such that $\bigcup \{U, U \in \beta_0\} \geq \gamma_\varepsilon$ where γ_ε is defined by $\gamma_\varepsilon(x) = \gamma(x) - \varepsilon$ or 0 according as $\gamma(x) > \varepsilon$ or $\gamma(x) \leq \varepsilon$.

Definition 2.8 A fuzzy subset U of γ is called fuzzy separated if $\exists v, \delta \in \sigma$ such that $U = v \cup \delta$, $v \neq \delta$ and $v \cap \delta = \varnothing$.

Definition 2.9 A fuzzy topological space (γ, σ) is said to be connected in the fuzzy sense if for any β_0 fuzzy closed subset of (γ, σ) can be fuzzy separated.

We shall take for granted that all information and ideas needed on fuzzy Lie algebras follow from [9].

Definition 2.10 Let W be a vector space over $\mathbb{K}$. A fuzzy subset U of W satisfying the following conditions

$$U(s+t) \geq \min\{U(s), U(t)\} \quad \text{for all } s, t \in W \tag{1}$$

$$U(\alpha s) \geq U(s) \quad \text{for all } s \in W,\ \alpha \in \mathbb{K} \tag{2}$$

is called a fuzzy subspace of W.

Definition 2.11 A fuzzy set $U : \mathfrak{L} \to [0,1]$ is called a fuzzy Lie subalgebra of $\mathfrak{L}$ over a field $\mathbb{K}$ if it is a fuzzy subspace of $\mathfrak{L}$ such that

$$\text{each non empty } U(L, t) \text{ is a subspace of } \mathfrak{L} \tag{1}$$

$$U([s,t]) \geq \min\{\mathfrak{L}(s), U(t)\}. \tag{2}$$

hold for all $s, t \in \mathfrak{L}$ and $\alpha \in \mathbb{K}$.

Definition 2.12 A fuzzy set $U : \mathfrak{L} \to [0,1]$ is called a fuzzy Lie ideal of $\mathfrak{L}$ if

$$U(s+t) \geq \min\{U(s), U(t)\} \tag{1}$$

$$U(\alpha s) \geq U(s) \tag{2}$$

$$U([s,t]) \geq U(s) \tag{3}$$

hold for all $s,t \in \mathfrak{L}$ and $\alpha \in \mathbb{K}$.

Definition 2.13 Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two Lie algebras and φ a function from $\mathfrak{L}_1$ to $\mathfrak{L}_2$. If U is a fuzzy set in $\mathfrak{L}_2$, then the pre-image of U under φ is the fuzzy set in $\mathfrak{L}_1$ defined by

$$\varphi^{-1}(U)(s) = U(\varphi(s)) \ \forall s \in \mathfrak{L}_1$$

2.2. Fuzzy Topological Groups

We now introduce notion of fuzzy topological group and its corresponding differentiable manifold in what follows.

Definition 2.14 A fuzzy subset F of $X \times Y$ is said to be a fuzzy proper function from λ to U if

$$F(x,y) \le \min\{\lambda(x), U(y)\} \quad \forall (x,y) \in X \times Y \tag{1}$$

$\forall x \in X$, $\exists y_0 \in Y$ such that $F(x, y_0) = \lambda(x)$ and $F(x,y) = 0$ if $y \neq y_0$ (2)

Definition 2.15 A proper function $F:(\lambda,\tau) \to (U,\sigma)$, is said to be

1. Fuzzy continuous is $F^{-1}(U) \in \tau$, $\forall U \in \sigma$;
2. Fuzzy open if $F(\delta) \in \sigma \quad \forall \delta \in \tau$;
3. Fuzzy homeomorphism if F be bijective, fuzzy continuous and open.

Definition 2.16 [10] A fuzzy topology σ on a group G is said to be compatible if the mappings

$$m:(G \times G, \sigma \times \sigma) \to (G,\sigma): m(x,y) \mapsto xy$$

$$i:(G,\sigma) \to (G,\sigma): i(x) \mapsto x$$

are fuzzy continuous. A group G equipped with a compatible fuzzy topology σ on G is called a fuzzy topological group.

Definition 2.17 A fuzzy topological vector space (ftvs) is a vector space Ξ over the field $\mathbb{K}$, Ξ equipped with a fuzzy topology σ and $\mathbb{K}$ equipped with the usual topology κ, such that the two mappings

- $(s,t) \to s+t$ of $(\Xi,\sigma) \times (\Xi,\sigma)$ into (Ξ,σ)
- $(\alpha,s) \to \alpha s$ of $(\mathbb{K},\kappa) \times (\Xi,\sigma)$ into (Ξ,σ)

are fuzzy continuous.

Definition 2.18 [10] Let $\Xi, \digamma$ be two fuzzy topological vector spaces. The mapping $\phi:\Xi \to \digamma$ is said to be tangent at 0 if given a neighbourhood W of 0_δ, $0<\delta<1$ in $\digamma$, there exists a neighbourhood V of 0_ε $0<\varepsilon<\delta$ in Ξ such that

$$\phi[tV] \subset \o(t)W$$

for some function $\o(t)$.

Definition 2.19 Let Ξ and $\digamma$ be two fuzzy topological vector space, each equipped with a fuzzy topology (possibly T_1). Let $\varphi:\Xi \to \digamma$ be a fuzzy continuous mapping. Then, φ is called fuzzy differentiable at $s \in \Xi$ if $\exists$ a continuous linear fuzzy map $U:\Xi \to \digamma$ such that

$$\varphi(s+t) = \varphi(s) + U(t) + \rho(t),\ t \in \Xi.$$

where ρ is tangent to 0. This mapping U is known as the fuzzy derivative of φ at $s_\circ$. The fuzzy derivative of φ at $s_\circ$ is denoted by $\varphi'(s_\circ)$; it is an element of $L(\Xi,F):=\{\varphi \mid \varphi:\Xi\to F, \varphi \text{ is fuzzy linear and continuous}\}$. φ is differentiable in the fuzzy sense if it is differentiable at every point in Ξ in the fuzzy sense.

Definition 2.20 Let Ξ, F be ftvs. A bijection $\varphi:\Xi\to F$ is called a fuzzy diffeomorphism of class C^1 if φ and its inverse φ^{-1} are differentiable in the fuzzy sense and φ' and $\left(\varphi^{-1}\right)'$ are fuzzy continuous.

The whole idea of fuzzy C^1-manifold and atlas is well-known (see [10]).

Definition 2.21 A fuzzy Lie group, $\mathfrak{G}_\mathfrak{f}$ is a C^1-fuzzy manifold $\mathfrak{G}_\mathfrak{f}$ which is also a group, such that the mappings

$$m:\left(\mathfrak{G}_\mathfrak{f}\times\mathfrak{G}_\mathfrak{f},\sigma\times\sigma\right)\to\left(\mathfrak{G}_\mathfrak{f},\sigma\right)$$

$$i:\left(\mathfrak{G}_\mathfrak{f},\sigma\right)\to\left(\mathfrak{G}_\mathfrak{f},\sigma\right)$$

are fuzzy differentiable.

3. Spherical Functions on Fuzzy Lie Groups

In this section, we first introduce the general construction of the popular spherical functions as can be seen in [15] [11] [16] etc.

Definition 3.1 Let $\mathfrak{G}_\mathfrak{f}$ be a fuzzy Lie group and $\mathfrak{K}_\mathfrak{f}$ a closed fuzzy Lie subgroup of $\mathfrak{G}_\mathfrak{f}$. Suppose ρ be a complex-valued function on $\mathfrak{G}_\mathfrak{f}/\mathfrak{K}_\mathfrak{f}$ (where $\mathfrak{G}_\mathfrak{f}$ is a fuzzy Lie group and $\mathfrak{K}_\mathfrak{f}$ is a compact subgroup) of class C^∞ which satisfies $\rho(\pi(e))=1$. Then, ρ is referred as spherical function if

$$\rho^{\sigma(k)}=\rho \quad \forall k\in\mathfrak{K}_\mathfrak{f} \tag{1}$$

$$D_\rho=\lambda_D\rho \quad \text{for each} \quad D\in D\left(\mathfrak{G}_\mathfrak{f}/\mathfrak{K}_\mathfrak{f}\right). \tag{2}$$

Here, λ_D is a complex number and $D\left(\mathfrak{G}_\mathfrak{f}/\mathfrak{K}_\mathfrak{f}\right)$ is the algebra of differential operators on $\mathfrak{G}_\mathfrak{f}/\mathfrak{K}_\mathfrak{f}$ invariant under all the translations

$$\sigma(g):x\mathfrak{K}_\mathfrak{f}\to gx\mathfrak{K}_\mathfrak{f} \quad \text{of} \quad \mathfrak{G}_\mathfrak{f}/\mathfrak{K}_\mathfrak{f} \tag{T}$$

In what follows, we shall construct spherical function on a locally compact fuzzy Lie group $\mathfrak{G}_\mathfrak{f}$ following the [15].

Let M be a C^1-fuzzy manifold and $\rho:M\to M$ a C^1-fuzzy diffeomorphism of M onto itself. We put

$$f^\rho=f\circ\rho^{-1}, \ f\in\varepsilon(M)$$

and if D is a differential operator on M, we define D^ρ by

$$D^\rho:f\to\left(Df^{\rho-1}\right)^\rho=\left(D(f\circ\rho)\right)\circ\rho^{-1}, \ f\in\varepsilon(M)$$

where D^ρ is another differential operator. The operator D is said to be invariant under ρ if $D^\rho=D$ i.e. $D(f\circ\rho)=Df\circ\rho$ for all f. Note that $(Df)^\rho=D^\rho f^\rho$.

If T is a distribution on M, we put T^ρ for the distribution $T^\rho(f)=T\left(f^{\rho-1}\right)$, $f\in D(M)$.

Let $\mathfrak{G}_{\mathfrak{f}}$ be a fuzzy Lie group, $\mathfrak{H}_{\mathfrak{f}} \subset \mathfrak{G}_{\mathfrak{f}}$ a closed fuzzy subgroup of $\mathfrak{G}_{\mathfrak{f}}$. $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{H}_{\mathfrak{f}}$ the fuzzy manifold of left cosets $g\mathfrak{H}_{\mathfrak{f}}$ ($g \in \mathfrak{G}_{\mathfrak{f}}$) and $D(\mathfrak{G}_{\mathfrak{f}}/\mathfrak{H}_{\mathfrak{f}})$ the algebra of all differential operators on $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{H}_{\mathfrak{f}}$ which are invariant under the usual transformations (*T*).

Given a coset space $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{H}_{\mathfrak{f}}$, we intend to define the operator in $D(\mathfrak{G}_{\mathfrak{f}}/\mathfrak{H}_{\mathfrak{f}})$. We consider a case when $\mathfrak{H}_{\mathfrak{f}} = \{e\}$ and put $D(\mathfrak{G}_{\mathfrak{f}})$ for $D(\mathfrak{G}_{\mathfrak{f}}/\{e\})$, the set of left-invariant differential operators on $\mathfrak{G}_{\mathfrak{f}}$.

If ⊔ is a fuzzy vector spaces over $\mathbb{R}$, the symmetric fuzzy algebra $\mathcal{S}$ (⊔) over ⊔ is defined as the fuzzy algebra of complex-valued polynomial functions on the dual space ⊔*. If $X_1,\cdots,X_n$ is the basis for ⊔, then $\mathcal{S}$ (⊔) can be identified with the commutative fuzzy algebra of polynomials $\sum_{(k)} a_{k_1,\cdots,k_n} X_1^{k_1}\cdots X_n^{k_n}$. Let $\mathfrak{g}$ denote the fuzzy Lie algebra of $\mathfrak{G}_{\mathfrak{f}}$ (the tangent space to $\mathfrak{G}_{\mathfrak{f}}$ at *e*) and $\exp : \mathfrak{g} \mapsto \mathfrak{G}_{\mathfrak{f}}$ the exponential mapping which maps a line through 0 in $\mathfrak{g}$ onto a one parameter subgroup $t \to \exp tx$ of $\mathfrak{G}_{\mathfrak{f}}$. If $X \in \mathfrak{g}$, let $\tilde{X}$ denote the fuzzy vector field on $\mathfrak{g}$ given by

$$\left(\tilde{X}f\right)(g) = X\left(f \circ L_g\right) = \left\{\frac{\mathrm{d}}{\mathrm{d}t} f\left(g \exp tX\right)\right\}_{t=0} \quad \text{for} \quad f \in \mathfrak{E}\left(\mathfrak{G}_{\mathfrak{f}}\right) \tag{1}$$

where L_g denote the left translation $x \to gx$ of $\mathfrak{G}_{\mathfrak{f}}$ onto itself. Then $\tilde{X}$ is a differential operator on $\mathfrak{G}_{\mathfrak{f}}$ if $h \in \mathfrak{G}_{\mathfrak{f}}$ then

$$\left(\tilde{X}^{L_h} f\right)(g) = \left(\tilde{X}\left(f \circ L_h\right)\right)\left(h^{-1}g\right) = \left(\tilde{X}f\right)(g)$$

So $\tilde{X} \in D(\mathfrak{G}_{\mathfrak{f}})$. Moreover, the bracket on $\mathfrak{g}$ is by definition given by

$$[X,Y]^{\sim} = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X},\ X,Y \in \mathfrak{g}$$

the multiplication on the right-hand side being composition of operators.

Definition 3.2 [17] The pair $(\mathfrak{G}_{\mathfrak{f}}, \mathfrak{K})$ is called a symmetric pair if there exists a involutive automorphism θ of $\mathfrak{G}_{\mathfrak{f}}$ such that $\mathfrak{G}_{\mathfrak{f}\theta}^{0} \subset \mathfrak{K} \subset \mathfrak{G}_{\mathfrak{f}\theta}$, where $\mathfrak{G}_{\mathfrak{f}}$ is the set of fixed points of θ and $\mathfrak{G}_{\mathfrak{f}\theta}^{0}$ is the identity component. The space $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{K}$ is called a symmetric space.

Theorem 3.3 Let $\mathfrak{G}_{\mathfrak{f}}$ be fuzzy Lie group with fuzzy algebra $\mathfrak{g}$. Let $\mathcal{S}(\mathfrak{g})$ denote the symmetric fuzzy algebra over the fuzzy vector space $\mathfrak{g}$. Then there exists a unique linear bijection

$$\lambda : \mathcal{S}(\mathfrak{g}) \to D(\mathfrak{g})$$

such that

$$\lambda\left(X^m\right) = \tilde{X}^m,\ X \in \mathfrak{g},\ m \in [0,1] \tag{2}$$

If $X_1 \cdots X_n$ is any basis of $\mathfrak{g}$ and $p \in S(\mathfrak{g})$, then

$$\left(\lambda(p) f\right)(g) = \left\{p\left(\partial_1,\cdots,\partial_n\right) f\left(g \exp\left(t_1X_1 + \cdots + t_nX_n\right)\right)\right\}_{t=0}, \tag{3}$$

where $f \in \mathfrak{E}\left(\mathfrak{G}_{\mathfrak{f}}\right)$, $\partial_i = \dfrac{\partial}{\partial t_i}$ and $t = \left(t_1,\cdots,t_n\right)$.

Proof. For any fixed basis $X_i,\cdots,X_n$ of $\mathfrak{g}$. The mapping

$$g \exp\left(t_1X_1 + \cdots + t_nX_n\right) \to \left(t_1,\cdots,t_n\right)$$

is a coordinate system on a neighbourhood of g in $\mathfrak{G}_{\mathfrak{f}}$. By equation (2), define a differential operator $\lambda(P)$ on $\mathfrak{G}_{\mathfrak{f}}$. Clearly $\lambda(p)$ is left invariant, and by (1) $\lambda(X_i) = \tilde{X}_i$, so by linearity $\lambda(X) = \tilde{X}$ for $X \in \mathfrak{g}$.

Also we show that λ is one to one.

Suppose $\lambda(P) = 0$ where $P \neq 0$ with respect to a lexicographic ordering. Let $aX_1^m \cdots X_n^m$ be the leading term in *P*. Let f be a smooth function on a neighbourhood of e in $\mathfrak{G}_{\mathfrak{f}}$ such that

$$f\left(\exp\left(t_1 X_1 + \cdots + t_n X_n\right)\right)^n = t_1^{m_1} \cdots t_n^{m_n}$$

for small *t*, then $(\lambda(p) f)(e) \neq 0$ contradicting $\lambda(P) = 0$.

Finally, λ maps $S(\mathfrak{g})$ onto $D(\mathfrak{g})$. Also if $u \in D(\mathfrak{g})$, there exist a polynomial P such that

$$(uf)(e) = \left\{P(\partial_1, \cdots, \partial_n) + (\exp t_1 X_1 + \cdots + t_n X_n)\right\}.$$

Then by the left invariance of *u*, $u = \lambda(P)$ so λ is surjective.

The mapping λ is usually called symmetrization.

4. The main reuslts

Let $\mathfrak{G}_{\mathfrak{f}}$ be a fuzzy Lie group, and $\mathfrak{K}$ a closed fuzzy subgroup of $\mathfrak{G}_{\mathfrak{f}}$. Let ϖ be the natural mapping of $\mathfrak{G}_{\mathfrak{f}}$ onto $\Xi = \mathfrak{G}_{\mathfrak{f}}/\mathfrak{K}$. Let $0 = \varpi(e)$ and $\tilde{f} = f \circ \varpi$ for f any function on Ξ. We denote by $D(\mathfrak{G}_{\mathfrak{f}})$ the set of all (left-) invariant differential operators on $\mathfrak{G}_{\mathfrak{f}}$, and $D_{\mathfrak{K}}(\mathfrak{G}_{\mathfrak{f}})$ the subspace of all right invariant differential operators under $\mathfrak{K}$, then clearly, $D(\mathfrak{G}_{\mathfrak{f}}/\mathfrak{K})$ denotes the algebra of differential operators on $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{K}$ invariant under the usual translations. We shall prove the existence of the following result in the fuzzy sense.

Theorem 4.1 [15] Let f be a complex-valued continuous function on $\mathfrak{G}_{\mathfrak{f}}$, not identically 0. Then f is a spherical function if and only if

$$\int_K f(xky) = f(x) f(y) \ \forall\, x, y \in \mathfrak{G}_{\mathfrak{f}} \tag{4}$$

Theorem 4.2 [15] Let $C_c^*(\mathfrak{G}_{\mathfrak{f}})$ be the space of nonzero continuous functions on with compact support on $\mathfrak{G}_{\mathfrak{f}}$. Let ρ be a continuous complex-valued function on $\mathfrak{G}_{\mathfrak{f}}$ bi-invariant under $\mathfrak{K}$. Then ρ is a spherical function if and only if the mapping

$$L : f \to \int_{\mathfrak{G}_{\mathfrak{f}}} f(x) \rho(x) \mathrm{d}x$$

is a homomorphism of $C_c^*(\mathfrak{G}_{\mathfrak{f}})$ onto $\mathbb{C}$.

Theorem 4.3 The algebra $D(\mathfrak{G}_{\mathfrak{f}}/\mathfrak{K})$ is commutative.

Proof. Let $\mathfrak{G}_{\mathfrak{f}}$ be a fuzzy Lie group and $\mathfrak{g}$ be a fuzzy Lie algebra of $\mathfrak{G}_{\mathfrak{f}}$. Let $\mathfrak{K}$ be a closed fuzzy subgroup of $\mathfrak{G}_{\mathfrak{f}}$ and the symmetric space $\mathfrak{M} = \mathfrak{G}_{\mathfrak{f}}/\mathfrak{K}$ be a fuzzy manifold of left cosets $gk \quad g \in \mathfrak{G}_{\mathfrak{f}}$ where $D(\mathfrak{G}_{\mathfrak{f}}/\mathfrak{K})$ the algebra of all differential operators on $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{K}$ which are invariant under the usual transformations. Let $f \in C^1(\mathfrak{M})$ and let $x_1, x_2, \cdots, x_n$ be a basis in $\mathfrak{M}$. Let $\tilde{X} \in \mathfrak{g}$ and $\tilde{Y} \in \mathfrak{g}$, we define

$$\tilde{X} = \sum_{j=1}^{n} U_j \frac{\partial}{\partial x_j}$$

and

$$\tilde{Y} = \sum_{i=1}^{n} V_i \frac{\partial}{\partial x_i}$$

we have

$$\begin{aligned}\left(\tilde{X} \circ \tilde{Y}\right) f = \tilde{X}\left(\tilde{Y}f\right) = \tilde{X}\left(\sum_{i=1}^{n} V_i \frac{\partial f}{\partial x_i}\right) \\ = \sum_{i,j=1}^{n} U_j \frac{\partial V_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum_{i,j=1}^{n} U_j V_i \cdot \frac{\partial^2 f}{\partial x_j}.\end{aligned} \tag{5}$$

Also

$$\begin{aligned}\left(\tilde{Y} \circ \tilde{X}\right) f = \tilde{Y}\left(\tilde{X}f\right) = \tilde{Y}\left(\sum_{j=1}^{n} V_j \frac{\partial f}{\partial x_j}\right) \\ = \sum_{i,j=1}^{n} V_i \frac{\partial U_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum_{i,j=1}^{n} V_i U_j \frac{\partial^2 f}{\partial x_j \partial x_i}.\end{aligned} \tag{6}$$

By subtraction, we obtain for $f \in C^2\left(\mathfrak{M}\right)$

$$\begin{aligned}\tilde{X}\left(\tilde{Y}f\right) - \tilde{Y}\left(\tilde{X}f\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(U_j \frac{\partial V_i}{\partial x_j} - V_i \frac{\partial U_j}{\partial x_j}\right)\right)\frac{\partial f}{\partial x_i} \\ \sum_{i,j=1}^{n} U_j \frac{\partial V_i}{\partial x_j}\frac{\partial f}{\partial x_i} - \sum_{i,j=1}^{n} V_i \frac{\partial U_j}{\partial x_j}\frac{\partial f}{\partial x_i} \\ = 0 \quad \left(\text{by the invariance property}\right).\end{aligned} \tag{7}$$

This implies that $\tilde{X}\left(\tilde{Y}f\right) - \tilde{Y}\left(\tilde{X}f\right) = 0$, $\tilde{X}\left(\tilde{Y}f\right) = \tilde{Y}\left(\tilde{X}f\right)$ whenever $U = kV$ for k any scalar.

We are now ready to use (6) and (7) to define spherical function.

Let $\mathfrak{G}_{\mathfrak{f}}$ be a fuzzy Lie group and $\mathfrak{K}$ a maximal compact subgroup. Let $C_c\left(\mathfrak{K} \backslash \mathfrak{G}_{\mathfrak{f}} / \mathfrak{K}\right)$ denote the space of all continuous functions with compact support on $\mathfrak{G}_{\mathfrak{f}}$ which satisfy $f\left(k_1 g k_2\right) = f\left(g\right)$ for all $k_1, k_2 \in \mathfrak{K}$. Such spaces are called spherical or bi-invariant. Then, $C_c\left(\mathfrak{K} \backslash \mathfrak{G}_{\mathfrak{f}} / \mathfrak{K}\right)$ forms a commutative Banach algebra under convolution [16] and we call the pair $\left(\mathfrak{G}_{\mathfrak{f}}, \mathfrak{K}\right)$ a Gelfand pair.

Let $\mathfrak{G}_{\mathfrak{f}}$ be a fuzzy Lie group and c a closed fuzzy subgroup of $\mathfrak{G}_{\mathfrak{f}}$. Let $\mathfrak{M} = \mathfrak{G}_{\mathfrak{f}} / \mathfrak{K}$ be a symmetric space. For any function $\rho \in \mathfrak{G}_{\mathfrak{f}}$ $\exists$ a function $\rho \in C_c\left(\mathfrak{K} \backslash \mathfrak{G}_{\mathfrak{f}} / \mathfrak{K}\right)$ which satisfies $\rho\left(kgk'\right) = \rho\left(g\right)$ and are integrable on $\mathfrak{G}_{\mathfrak{f}}$ for a normed algebra A under the convolution product of $\mathfrak{G}_{\mathfrak{f}}$, $g \in \mathfrak{G}_{\mathfrak{f}}$, $k, k' \in \mathfrak{K}$. The functions ρ are continuous, positive-definite, invariant under $g \to kgk'$ and the linear representation

$$f \to \rho\left(f\right) = \int f\left(g\right)\rho\left(g\right)\mathrm{d}g$$

must be a homomorphism of A onto $\left[0,1\right]$.

We next prove the existence of Helgason-functions theorem in the fuzzy sense.

Proposition 4.4 [15] Let *f* be a complex-valued continuous function on $\mathfrak{G}_{\mathfrak{f}}$, not identically 0. Then *f* is a spherical function if and only if

$$\int_{\mathfrak{K}} f(xky)\,\mathrm{d}k = f(x)f(y)$$

Theorem 4.5 Let $\mathfrak{G}_{\mathfrak{f}}$ be a fuzzy Lie group and $\mathfrak{G}_{\mathfrak{f}}$ a closed, compact fuzzy subgroup of $\mathfrak{G}_{\mathfrak{f}}$. Let $f \in C_c(\mathfrak{K} \backslash \mathfrak{G}_{\mathfrak{f}} / \mathfrak{K})$ then *f* is a spherical function on $\mathfrak{G}_{\mathfrak{f}}$, $f : \mathfrak{G}_{\mathfrak{f}} \to [0,1]$ and *f* satisfies the function

$$f : C_c(\mathfrak{K} \backslash \mathfrak{G}_{\mathfrak{f}} / \mathfrak{K}) \to \mathfrak{G}_{\mathfrak{f}|_{[0,1]}}$$

such that

$$\int_K f(xky)\,d_k = \begin{cases} f(x)f(y), & \text{if } \exists \text{ no } x, y \in \mathfrak{G}_{\mathfrak{f}} \text{ such that } t_1 < f(xy) < t_2 \\ 0, & \text{if otherwise} \end{cases}$$

Proof. Let *f* be a fuzzy set on a fuzzy Lie group $\mathfrak{G}_{\mathfrak{f}}$. Let $t \in [0,1]$. We define the *t*-cut set or *t*-level set of μ, by

$$f_t = \{x \in \mathfrak{G}_{\mathfrak{f}} : f(x) \geq t\}.$$

If $x, y \in \mathfrak{G}_{\mathfrak{f}}$ and $f(x) = t_1$ and $f(y) = t_2$.

Then $x \in f_{t_1}$ and $y \in f_{t_2}$. If $t_2 < t_1$, we have

$$f_{t_1} \subseteq f_{t_2} \Rightarrow x \in f_{t_2}$$

and we have $x, y \in f_{t_2}$ and since f_{t_2} is a subgroup of $\mathfrak{G}_{\mathfrak{f}}$, by definition we have $xy \in f_{t_2}$. Therefore

$$f(xy) \geq t_2$$

and $t_2 = \min(f(x), f(y))$.

Also let $x \in \mathfrak{G}_{\mathfrak{f}}$ and $f(x) = t$, we have $x \in f_t$, since f_t is a subgroup. $x^{-1} \in f_t$. Therefore

$$f(x^{-1}) \geq t.$$

Then $f(x^{-1}) \geq f(x)$. This shows that *f* is a subgroup of $\mathfrak{G}_{\mathfrak{f}}$.

Since *f* is a homomorphism, we have

$$f(xy) = f(x)f(y).$$

Now let $x, y \in \mathfrak{G}_{\mathfrak{f}}$ and $k \in \mathfrak{K}$, let $f(x)f(y) = t_0$ and $\int_{\mathfrak{K}} f(xky)\,\mathrm{d}k = t_1$.

Then, $xy \in f_{t_0}$ and $xky \in f_{t_1}$ for

$$\int_{\mathfrak{K}} f(xky)\,\mathrm{d}k = f(x)f(y).$$

Then $\exists$ no $xy \in \mathfrak{G}_{\mathfrak{f}}$ such that

$$t_0 < f(xy) < t_1.$$

Let

$$\int_K f(xky) = f(x)f(y)$$

and we have

$$f_{t_1} = f_{t_0}$$

If $\exists\, x, y \in G$ such that, $t_0 < f(xy) < t_1$ then $f_{t_1} \subset f_{t_0}$. Since $xy \in f_{t_0}$ but $xy \notin f_{t_1}$ then this contradicts our statement.

Conversely, if $\exists$ no $xy \in G$ such that

$$t_0 < f(xy) < t_1.$$

If $t_0 < t_1$ we get $f_{t_1} \subseteq f_{t_0}$. Let $xy \in f_{t_0}$, we have $f(xy) \ge t_0$ and $f(xy) \ge t_1$. This implies that $f(x, y)$ does not lie between t_1 and t_0. Hence $xy \in f_{t_1}$ and $f_{t_0} \subseteq f_{t_1}$. Hence $f_{t_0} = f_{t_1}$ and

$$\int_K f(xky)\,\mathrm{d}k = f(x) f(y).$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Zadeh, L.A. (1965) Fuzzy Sets. *Fuzzy Information and Control*, **8**, 177-200. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Kastaras, A.K. (1981) Topological Linear Spaces I. *Fuzzy Sets and Systems*, **6**, 85-95. https://doi.org/10.1016/0165-0114(81)90082-8

[3] Kastaras, A.K. (1984) Topological Linear Spaces II. *Fuzzy Sets and Systems*, **12**, 143-154. https://doi.org/10.1016/0165-0114(84)90034-4

[4] Kastaras, A.K. and Liu, D.B. (1977) Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces. *Journal of Mathematical Analysis and Applications*, **58**, 135-146. https://doi.org/10.1016/0022-247X(77)90233-5

[5] Kubiak, T. (1985) On Fuzzy Topologies. Ph. D. Thesis, Adam Mickiewicz, Poznan.

[6] Sostak, A.P. (1985) On a Fuzzy Topological Structure. *Proceedings of the* 13*th Winter School on Abstract Analysis*, Circolo Matematico di Palermo, 89-103.

[7] Egwe, M.E. and Oyewo, R.A. (2021) A Fixed Point Theorem on Fuzzy Locally Convex Spaces. *Annals of Fuzzy Mathematics and Informatics*, **22**, 17-24. https://doi.org/10.20944/preprints202206.0311.v1

[8] Chakraborty, M. and Ahsanullah, T. (1992) Fuzzy Topology on Fuzzy Sets and Tolerance Topology. *Fuzzy Sets and Systems*, **45**, 103-108. https://doi.org/10.1016/0165-0114(92)90096-M

[9] Akram, M. (2018) Fuzzy Lie Algebras. Springer Verlag, Berlin. https://doi.org/10.1007/978-981-13-3221-0

[10] Nadjafikhah, M. and Bakhshandeh-Chamazkoti, R. (2009) Fuzzy Lie Groups. *Mathematical Sciences*, arXiv: 0908.0254. https://arxiv.org/abs/0908.0254v1

[11] Godement, R. (1952) A Theory of Spherical Functions I. *Trans. Transactions of the American Mathematical Society*, **73**, 496-556. https://doi.org/10.1090/S0002-9947-1952-0052444-2

[12] Chandra, H. (1958) Spherical Functions on a Semisimple Lie Group, I. *American Mathematical Society*, **80**, 241-310. https://doi.org/10.2307/2372786

[13] Egwe, M.E. (2020) On Fixed Point Theorem in Non-Archimedean Fuzzy Normed Spaces. *International Journal of Analysis and Applications*, **18**, 99-103.

[14] Yeol, J.C., Rassias, T.M. and Saadati, R. (2018) Fuzzy Operator Theory in Mathematical Analysis. Springer Verlag, Berlin.

[15] Helgason, S. (1984) Groups and Geometric Analysis: Radon Transforms, Invariant

Differential Operators and Spherical Functions: Volume 1. Academic Press, Cambridge.

[16] Egwe, M.E. and Bassey, U.N. (2013) On Isomorphism between Certain Group Algebras on the Heisenberg Group. *Journal of Mathematical Physics*, *Analysis*, *Geometry*, **9**, 150-164.

[17] Hilgert, J. and Olafsson, G. (1997) Causal Symmetric Spaces: Geometry and Harmonic Analysis. Academic Press Inc., Cambridge.