A Class of New Optimal Ternary Cyclic Codes over $\mathbb{F}_{3^m}$ with Minimum Distance 4

Wenwei Qiu

School of Mathematics and Information, China West Normal University, Nanchong, China
Email: 1686789000@qq.com

Abstract

As a branch of applied mathematics, coding theory plays an important role. Among them, cyclic codes have attracted much attention because of their good algebraic structure and easy analysis performance. In this paper, we will study one class of cyclic codes over $\mathbb{F}_3$. Given the length and dimension, we show that it is optimal by proving its minimum distance is equal to 4, according to the Sphere Packing bound.

Keywords

Cyclic Code, Minimal Distance, Sphere Packing Bound

1. Introduction

Information transmission is an important means of human communication, and with the development of technology, coding theory has also been established. In the information age, cyberspace security is a very important issue, and cryptography and encoding play important roles in it. Coding theory is a technique for encoding information. During the process of transmitting information, it is inevitable that information may be distorted due to some reasons. In this process, information cannot correct errors on its own. Therefore, a self-correcting code space has been studied, which is called the error-correcting-codes.

Among error-correction-codes, linear codes are widely studied due to their excellent algebraic structure and other characteristics, and cyclic codes are the most important among them. Due to their excellent algebraic structure and cyclic properties, they can be easily studied and obtained through algebraic methods, and are widely used in various information security systems.

Let $\mathbb{F}_{p^m}$ be a finite field with p^m elements, where p is a prime. A linear code $\mathcal{C}$ with parameters $[n,k,d]$ over $\mathbb{F}_p$ is a linear subspace of $\mathbb{F}_{p^m}^n$, which

has the length n, dimension k and minimum Hamming distance d. We say the linear code $\mathcal{C}$ is a cyclic code if for any codewords $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathcal{C}$, the cyclic shift of the codeword $(c_{n-1}, c_0, \cdots, c_{n-2}) \in \mathcal{C}$. Now we use the polynomial ring $\mathbb{F}_p[x]$ and the quotient ring $\mathbb{F}_p[x]/(x^n - 1)$ to describe the cyclic code. We define a linear code as a cyclic code if for any codeword $f(x) \in \mathcal{C}$, which can be identified with a polynomial

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \in \mathbb{F}_p[x]/(x^n - 1),$$

the codeword $xf(x) \in \mathcal{C}$. So we can easily know that an nonempty set $\mathcal{C}$ in $\mathbb{F}_p^n$ is a cyclic code if and only if $\mathcal{C}$ is a principal ring in $\mathbb{F}_p^n$. We denote any cyclic code $\mathcal{C}$ as $\mathcal{C} = \langle g(x) \rangle$, and $g(x)$ is called the generator polynomial of $\mathcal{C}$.

The study of cyclic codes has been the focus of attention in recent years. Because of its excellent characteristics, it has been widely used in lots of fields. We always hope that a cyclic code has better error correction ability. The error correction ability is closely related to the minimum distance. The larger minimum distance it has, the better error correction ability it gets. Therefore, we are very interested in the minimum distance of a code.

Let $p = 3$, we consider the cyclic code $\mathcal{C}_{(u,v)}$ over the finite field $\mathbb{F}_3$. Ding and Helleseth [1] state the theory of the APN monomials and used some of these to construct many classes of optimal ternary cyclic codes in 2013. In 2019, by giving a new ternary power mapping, Yan and Han [2] considered a related optimal ternary cyclic code which $u = 1, v = (3^m - 3)/4$ in some conditions. Zha and Hu [3] proved some new classes of optimal ternary cyclic codes with minimum distance 4 for some given parameters v, $u = 1$ in 2020. For the given $u = (3^m + 1)/2$, Ding and Zhou [4] studied the cyclic code is optimal when $v = (3^s + 1)/2$ in some conditions. Similarly, Fan, Zhou and Li [5] proved that the cyclic code $\mathcal{C}_{\left(\frac{3^m+1}{2}, 2 \cdot 3^{(m-1)/2} + 1\right)}$ is optimal when m is odd in 2016. They also discussed the weight distribution of the dual of this code. In 2020, Liu, Cao and Lu [6] studied the code $\mathcal{C}_{(2,v)}$, which is constructed by using monomials x^2 and x^v. For $v = (3^m - 1)/2 + 2(3^k + 1)$, $\mathcal{C}_{(2,v)}$ is optimal by choosing suitable m and k. Recently, by choosing proper u and v, Zha, Hu, Liu and Cao [7] show that $\mathcal{C}_{\left(\frac{3^m+1}{2}, \frac{3^m-1}{2}+v\right)}$ and $\mathcal{C}_{(1,v)}$ have the same optimality.

In previous studies, there are not many studies on cyclic codes with parameter $\mathcal{C}_{(u,v)}$, $u = \dfrac{3^m + 1}{2}$. In this paper, we study the cyclic code $\mathcal{C}_{(u,v)}$ with the parameters which is $\mathcal{C}_{\left(\frac{3^m+1}{2}, \frac{3^{m+1}+7}{8}\right)}$. We show that the minimum distance of this cyclic code is equal to 4 for the given $n = 3^m - 1$ and $k = 3^m - 1 - 2m$, according to the Sphere Packing bound. It is optimal. Therefore, in the coding theory, we can obtain a new class of ternary cyclic codes whose minimum distance can reach the theoretical maximum for the given length and dimension. It can achieve the best

error correction effect and ensure that the information is not distorted as much as possible in the transmission process. These cyclic codes will have important applications in radar, satellite communications and other communication fields.

2. Preliminaries

- *The notation we use in this paper*

  (1) p is prime, and is an odd. Let $p = 3$.

  (2) s, r, m, k are positive integers, m is odd.

  (3) Let SQ be the set of square in $\mathbb{F}_{3^m}$, NSQ be the set of the nonsquare in $\mathbb{F}_{3^m}$.

  (4) In $\mathbb{F}_{3^m}$, we have $\alpha^{\frac{3^m+1}{2}} = \alpha$ if $\alpha \in SQ$ and $\alpha^{\frac{3^m+1}{2}} = -\alpha$ if $\alpha \in NSQ$.

- *The p-cyclotomic cosets modulo n, $n = p^m - 1$*

  We define the p-cyclotomic coset modulo n containing j as

  $$C_j = \left\{ j, pj, p^2 j, \cdots, p^{k-1} j \right\}$$

and k is the smallest positive integer such that $p^k j \equiv j \pmod{n}$. In this paper, let $p = 3$. The cyclic code of length $n = 3^m - 1$. The dimension of this code $\mathcal{C}_{(u,v)}$ is determined by k, where $k = |C_v|$. The dimension of $\mathcal{C}_{(u,v)}$ is equal to $n - (m + k)$. We now consider the case that $v \notin C_1$ and $k = m$, so the dimension is equal to $3^m - 1 - 2m$.

Theorem 2.1. [8] (***Sphere Packing Bound***) $A_p(n,d)$ *is the maximum number of codewords in a code over* $\mathbb{F}_p$ *of length n and minimum distance at least d, or we use* p^k *to represent it. Then*

$$A_p(n,d) \leq \frac{p^n}{\sum_{i=0}^{t} \binom{n}{i}(p-1)^i}$$

where $t = \left[(d-1)/2 \right]$.

We can see that by using Sphere Packing Bound, we can get a bound of the minimum distance of a cyclic code. Taking the cyclic code to be studied in this paper as an example, let $p = 3$, and when $n = 3^m - 1$, $k = 3^m - 1 - 2m$, the minimum distance of this cyclic codes can be obtained no more than 4. We obtain the upper bound of the minimum distance of this cyclic code. Therefore, we only need to prove that the minimum distance of this cyclic code can reach this upper bound, and it can be shown that it is optimal.

The distance d between two codewords $c, \bar{c} \in \mathcal{C}$ is defined to be the number of coordinates in which $c, \bar{c}$ are different. The minimum distance of a code $\mathcal{C}$ is the smallest distance between distinct codewords. The weight $wt(c)$ of a codeword c is the number of the nonzero coordinates in c. It has $d(c, \bar{c}) = wt(c - \bar{c})$ [9]. If $\mathcal{C}$ is a linear code, the minimum distance equal to the minimum weight of the nonzero codewords of $\mathcal{C}$. The parity check matrices of a code is a matrices H which satisfied

$$Hc^{\mathrm{T}} = 0$$

$c \in \mathcal{C}$. The parity check matrices of a code are the generator matrices of its dual code. From the definition of dual codes, the parity check matrices of the code $\mathcal{C}_{(u,v)}$ is define as

$$
\begin{pmatrix}
\pi^u & \pi^{2u} & \cdots & \pi^{(3^m-1)u} \\
\pi^v & \pi^{2v} & \cdots & \pi^{(3^m-1)v}
\end{pmatrix}
$$

π is a generator of $\mathbb{F}_{3^m}^*$.

If a linear code $\mathcal{C}$ has minimum distance d, there exist two distinct codewords $c, \bar{c} \in \mathcal{C}$, $Hc^{\mathrm{T}} = 0$, $H\bar{c}^{\mathrm{T}} = 0$, satisfied

$$
\begin{cases}
\tilde{c}_1 x_1^u + \tilde{c}_2 x_2^u + \cdots + \tilde{c}_d x_d^u = 0 \\
\tilde{c}_1 x_1^v + \tilde{c}_2 x_2^v + \cdots + \tilde{c}_d x_d^v = 0
\end{cases}
$$

$c_j, \bar{c}_j$ is the coordinates of the codeword $c, \bar{c}$, respectively. $\tilde{c}_i = c_j - \bar{c}_j$, $c_j \neq \bar{c}_j$, $x_i = \pi^j$, $1 \leq j \leq 3^m - 1$, $i = 1, 2, \cdots, d$.

If the code has minimum distance d, the equations above has solution, if the code has not minimum distance d, the equations above has not solution. So we can discuss the solution of the equations to find if the code has the codeword of weight d. According to the minimum distance $d \leq 4$ given by the sphere packing bound, we can prove that $d = 4$.

Lemma 2.2. *Let* $u = (3^m + 1)/2$, *v be an odd,* $v \notin C_1$, *and* $\ell_v = |C_v| = m$. *Cyclic code* $\mathcal{C}_{(u,v)}$ *has parameters* $[3^m - 1, 3^m - 1 - 2m, 4]$ *if and only if the following equations:*

$$
1 + x^v = \pm(1 + x)^v \tag{1}
$$

$$
1 + x^v = \pm(1 - x)^v \tag{2}
$$

$$
x^v - 1 = \pm(1 - x)^v \tag{3}
$$

and the equation

$$
x^v - 1 = \pm(1 + x)^v \tag{4}
$$

have no solution in $\mathbb{F}_{3^m} \setminus \{0, 1\}$.

Proof. It is clear that the distance of the code cannot be 1. The code $\mathcal{C}_{(u,v)}$ has a codeword of Hamming weight 2 if and only if there exist two elements $c_1, c_2 \in \mathbb{F}_3^*$ and two distinct elements $x_1, x_2 \in \mathbb{F}_{3^m}^*$ such that

$$
\begin{cases}
c_1 x_1^u + c_2 x_2^u = 0 \\
c_1 x_1^v + c_2 x_2^v = 0
\end{cases}
$$

Case 1: $c_1 = c_2 = 1$ If $x_1 \in SQ$, $x_2 \in SQ$, the first equation becomes to $x_1 + x_2 = 0$, which is impossible because x_1, x_2 are all SQ. If $x_1 \in NSQ$, $x_2 \in NSQ$, the first equation becomes to $x_1 = -x_2$, let $x_1 = -a^2$, then we have $-a^2 = -x_2$, $a^2 = x_2$ but x_2 is a NSQ, which is also impossible. If $x_1 \in SQ$, $x_2 \in NSQ$ or $x_1 \in NSQ$, $x_2 \in SQ$, the first equation becomes to $x_1 = x_2$, which is also a contradiction..

Case 2: $c_1 = 1, c_2 = -1$ If $x_1 \in SQ$, $x_2 \in SQ$ or $x_1 \in NSQ$, $x_2 \in NSQ$, the first

equation becomes to $x_1 = x_2$, which is a contradiction. If $x_1 \in SQ$, $x_2 \in NSQ$ or $x_1 \in NSQ$, $x_2 \in SQ$, the first equation becomes to $x_1 = -x_2$. Taking it into the second equation we will get $2x_1 = 0$, which is a contradiction.

Thus it does not have a codeword of Hamming weight 2.

The code $C_{(u,v)}$ has a codeword of Hamming weight 3 if and only if there exist three elements $c_1, c_2, c_3 \in \mathbb{F}_3^*$ and three distinct elements $x_1, x_2, x_3 \in \mathbb{F}_{3^m}^*$ such that

$$\begin{cases} c_1 x_1^u + c_2 x_2^u + c_3 x_3^u = 0 \\ c_1 x_1^v + c_2 x_2^v + c_3 x_3^v = 0 \end{cases} \tag{5}$$

Case 1: $c_1 = c_2 = c_3 = 1$. In this case, let $y_1 = x_2/x_1$, $y_2 = x_3/x_1$. It follows from (5) that

$$\begin{cases} y_1^u + y_2^u + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases} \tag{6}$$

$y_1, y_2 \notin \{0,1\}$. If $y_1, y_2 \in SQ$ or $y_1 \in SQ, y_2 \in NSQ$, (6) becomes to

$$y_1^v + 1 = \pm(1 + y_1)^v$$

If $y_1, y_2 \in NSQ$ or $y_1 \in NSQ, y_2 \in SQ$, (6) becomes to

$$y_1^v + 1 = \pm(1 - y_1)^v$$

Case 2: $c_1 = c_2 = 1, c_3 = -1$. Similarly, we arrive at

$$\begin{cases} y_1^u + y_2^u - 1 = 0 \\ y_1^v + y_2^v - 1 = 0 \end{cases} \tag{7}$$

$y_1, y_2 \notin \{0,1\}$. If $y_1, y_2 \in SQ$ or $y_1 \in SQ, y_2 \in NSQ$, (7) becomes to

$$y_1^v - 1 = \pm(1 - y_1)^v$$

If $y_1, y_2 \in NSQ$ or $y_1 \in NSQ, y_2 \in SQ$, (7) becomes to

$$y_1^v - 1 = \pm(1 + y_1)^v$$

So if the four equations have no solutions in $\mathbb{F}_{3^m}^*$, we get $d \geq 4$, according to the Sphere Packing bound, the minimal distance of any linear code with length $3^m - 1$ and the dimension $3^m - 1 - 2m$ should be less than or equal 4. Hence $d = 4$. $\qquad\square$

The following Lemma will be used in the sequel of the proof.

Lemma 2.3. [10] Let $f(x)$ be a irreducible polynomial with degree r over $\mathbb{F}_p$. If $f(x)$ has a root in $\mathbb{F}_{p^m}$, then $r \mid m$.

3. A Class of Optimal Ternary Cyclic Codes

In this section, we construct a class of optimal ternary cyclic codes $C_{(u,v)}$ with parameters $\left[3^m - 1, 3^m - 1 - 2m, 4\right]$.

Theorem 3.1. *Let m is odd, $m \geq 3$, $u = (3^m + 1)/2$, $v = (3^{m+1} + 7)/8$. $m \equiv 3 \pmod 4$, $9 \nmid m$, $5 \nmid m$. The cyclic code $C_{(u,v)}$ is an optimal ternary cyclic code with parameters $\left[3^m - 1, 3^m - 1 - 2m, 4\right]$.*

Proof. It is easy to prove that the minimal distance of the code $d \geq 2$. By lemma 2.2 we can know that it does not have a codeword of Hamming weight 2, which means $d \geq 3$. Now we prove that the minimal distance of the code $d = 4$.

Now, we prove that the code has no codewords of Hamming weight 3. It has a codeword of Hamming weight 3 if and only if there exist three elements $c_1, c_2, c_3 \in \mathbb{F}_3^*$ and three distinct elements $x_1, x_2, x_3 \in \mathbb{F}_{3^m}^*$ such that

$$\begin{cases} c_1 x_1^u + c_2 x_2^u + c_3 x_3^u = 0 \\ c_1 x_1^v + c_2 x_2^v + c_3 x_3^v = 0 \end{cases} \tag{8}$$

Case 1: $c_1 = c_2 = c_3 = 1$. In this case, let $y_1 = x_2/x_1$, $y_2 = x_3/x_1$ It follows from (8) that

$$\begin{cases} y_1^u + y_2^u + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases} \tag{9}$$

$$y_1, y_2 \notin \{0,1\}.$$

Now we consider the following four cases.

Case 1.1: When $y_1, y_2 \in SQ$, (9) follows to

$$\begin{cases} y_1 + y_2 + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases} \tag{10}$$

The Equation (10) leads to

$$1 + y_1^v = (1 + y_1)^v$$

Let $y_1 = a^2$, Then we have

$$1 + a^{\frac{3^{m+1}+7}{4}} = (1 + a^2)^{\frac{3^{m+1}+7}{8}}$$

By taking the eight power of both sides of the equation, we can get

$$\left(1 + a^{\frac{3^{m+1}+7}{4}}\right)^8 = (1 + a^2)^{3^{m+1}+7}$$

If $a \in SQ$, let $a = t^2$, and if t also $\in SQ$, we have

$$(1 + t^4)^{3^{m+1}+7} = \left(1 + t^{\frac{3^{m+1}+7}{2}}\right)^8$$

It becomes to

$$(1 + t^4)^{10} = (1 + t^5)^8$$

Expand it, we can get

$$t^{36} + t^{35} + 2t^{30} + t^{25} + 2t^{20} + t^{15} + 2t^{10} + t^5 + t^4 = 0$$

Because $t \neq 0$, we have

$$t^{32} + t^{31} + 2t^{26} + t^{21} + 2t^{16} + t^{11} + 2t^6 + t + 1 = 0$$

It can be factorized over $\mathbb{F}_3$ by Magma to

$$(t-1)^2 \left(t^2+1\right)\left(t^5-t^2+t+1\right)\left(t^5+t^4-t^3+1\right)\left(t^9-t^6+t^4+t^3+t-1\right)$$
$$\times\left(t^9-t^8-t^6-t^5+t^3-1\right)=0$$

if $t=1$, it means $y_1=1$, is impossible, and by lemma 2.3, we get it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$.

If $a\in SQ$, let $a=t^2$, and if $t\in NSQ$, we have

$$\left(1+t^4\right)^{10}=\left(1-t^5\right)^8$$

By following the same steps, we get

$$t^{32}-t^{31}-t^{26}-t^{21}-t^{16}-t^{11}-t^6-t+1=0$$

It can be factorized over $\mathbb{F}_3$ by Magma to

$$(t+1)^2\left(t^2+1\right)\left(t^5+t^2+t-1\right)\left(t^5-t^4-t^3-1\right)\left(t^9+t^6-t^4+t^3+t+1\right)$$
$$\times\left(t^9+t^8+t^6-t^5+t^3+1\right)=0$$

By the same reason, it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$.

If $a\in NSQ$, $a=-t^2$ and $t\in SQ$ or $t\in NSQ$, it is similar to the above case, we omit it here.

Case 1.2: When $y_1\in NSQ, y_2\in NSQ$, (8) follows to

$$\begin{cases} y_1+y_2+1=0 \\ y_1^v+y_2^v+1=0 \end{cases} \tag{11}$$

The Equation (11) leads to

$$1+y_1^v=-\left(1-y_1\right)^v$$

Let $y_1=-a^2$, Then we have

$$1-a^{\frac{3^{m+1}+7}{4}}=-\left(1+a^2\right)^{\frac{3^{m+1}+7}{8}}$$

By taking the eight power of both sides of the equation, we can get

$$\left(1-a^{\frac{3^{m+1}+7}{4}}\right)^8=\left(1+a^2\right)^{3^{m+1}+7}$$

If $a\in SQ$, let $a=t^2$, and if t also $\in SQ$, by the similar steps, we directly obtain

$$\left(1+t^4\right)^{10}=\left(1-t^5\right)^8$$

As before, with the help of Magma, we get the same equation

$$(t+1)^2\left(t^2+1\right)\left(t^5+t^2+t-1\right)\left(t^5-t^4-t^3-1\right)\left(t^9+t^6-t^4+t^3+t+1\right)$$
$$\times\left(t^9+t^8+t^6-t^5+t^3+1\right)=0$$

if $t=-1$, it means $y_1=-1$, but $y_1=-1$ is not the solution of $1+y_1^v=-\left(1-y_1\right)^v$, and by lemma 2.3, we get it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$

If $a\in SQ$, let $a=t^2$, and if $t\in NSQ$, we have

$$\left(1+t^4\right)^{10} = \left(1+t^5\right)^8$$

By following the same steps, we get

$$(t-1)^2\left(t^2+1\right)\left(t^5-t^2+t+1\right)\left(t^5+t^4-t^3+1\right)\left(t^9-t^6+t^4+t^3+t-1\right)$$
$$\times\left(t^9-t^8-t^6-t^5+t^3-1\right)=0$$

By the same reason as before, it has no root in $\mathbb{F}_{3^m}\setminus\{0,1\}$.

If $a\in NSQ$, $a=-t^2$ and $t\in SQ$ or $t\in NSQ$, it is similar to the above case, we omit it here.

Case 1.3: When $y_1\in NSQ, y_2\in SQ$. It is similar as before, we omit it here.

Case 1.4: When $y_1\in SQ, y_2\in NSQ$. It is similar as before, we omit it here.

Case 2: $c_1=c_2=1, c_3=-1$. By the similar calculation as Case 1, we can prove that the equation

$$\begin{cases} y_1^u + y_2^u - 1 = 0 \\ y_1^v + y_2^v - 1 = 0 \end{cases} \tag{12}$$

also has no solution in $\mathbb{F}_{3^m}\setminus\{0,1\}$. We omit the details of the proof.

By the Lemma 2.2, we have finished the proof. $\square$

Example

Let $m=3$, $u=\left(3^m+1\right)/2$, $v=\left(3^{m+1}+7\right)/8$. Let α be the generator of $\mathbb{F}_{3^m}^*$ with $\alpha^3+2\alpha+1=0$. Then $\mathcal{C}_{(u,v)}$ is a ternary cyclic code with parameters [26, 20, 4] and generator polynomial $x^6+x^5+x^4+2x^3+2$.

4. Conclusions

In this paper, based on the Sphere Packing Bound, we show that for the fixed length and dimension, with the help of factorization by Magma, by discussing the solutions of some correlative equations on $\mathbb{F}_{3^m}$, the ternary cyclic code $\mathcal{C}_{\left(\frac{3^m+1}{2},\frac{3^{m+1}+7}{8}\right)}$ has the minimum distance 4, according to the Sphere Packing bound. It is optimal.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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