Applied Mathematics, 2023, 14, 580-601
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Conformable and Caputo's Derivatives in Generalized Viscoelastic Models

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How to cite this paper: Fujioka, J., Rodríguez, R.F. and Espinosa-Cerón, Á. (2023) Conformable and Caputo's Derivatives in Generalized Viscoelastic Models. *Applied Mathematics*, **14**, 580-601.
https://doi.org/10.4236/am.2023.149035

Received: July 11, 2023
Accepted: September 12, 2023
Published: September 15, 2023

Abstract

We study two generalized versions of a system of equations which describe the time evolution of the hydrodynamic fluctuations of density and velocity in a linear viscoelastic fluid. In the first of these versions, the time derivatives are replaced by *conformable derivatives*, and in the second version left-handed Caputo's derivatives are used. We show that the solutions obtained with these two types of derivatives exhibit significant similarities, which is an interesting (and somewhat surprising) result, taking into account that the conformable derivatives are local operators, while Caputo's derivatives are nonlocal operators. We also show that the solutions of the generalized systems are similar to the solutions of the original system, if the order α of the new derivatives (conformable or Caputo) is less than one. On the other hand, when α is greater than one, the solutions of the generalized systems are qualitatively different from the solutions of the original system.

Keywords

Conformable Derivatives, Caputo's Derivatives, Fractional Derivatives, Viscoelastic Fluids, Hydrodynamic Fluctuations

1. Introduction

Fractional derivatives (FDs) are interesting and complex operators which, in spite of having an obscure geometric interpretation, have found many applications in engineering and in a broad range of physical, biological and social phenomena. Several applications have been found in acoustics [1]-[5], optical solitons [6] [7], complex viscoelastic media [8]-[13], turbulence [14], anomalous diffusion and random walkers [15] [16], Bose-Einstein condensates [17], and image and signal processing [18] [19] [20] [21]. FDs are also useful to describe complex relaxation processes where a characteristic time scale cannot be defined, and the traditional assumption that time averages and ensemble averages coincide, is no longer valid [22]. Moreover, several authors have proposed that FDs can be used to explain memory effects [23]-[27]. When FDs are used in partial differential equations (PDEs), the FDs calculated with respect to the evolution variable (which in most cases is the time, but it is a distance along an optical fiber in the study of optical solitons) are usually different from the FDs calculated with respect to the other variables. For example, when FDs are used to describe the dispersion of temporal optical solitons, it has been proved that a combination of left and right Grünwald-Letnikov derivatives [6], or a combination of Ortigueira's central FDs of types 1 and 2 [7], provides an excellent description of the dispersion of the optical pulses. On the other hand, when we describe complex viscoelastic fluids, the long-term memory of these systems can be taken into account by introducing a temporal FD (in this case the time is the evolution variable), and in this case it has been shown that a left-handed Caputo's derivative provides an adequate description of the evolution of these systems [8] [9] [11] [13]. These three types of FDs (Grünwald-Letnikov's, Ortigueira's and Caputo's) are *nonlocal* operators, as occurring with most definitions of FDs. However, in 2014 and 2016 two *local* differential operators were also proposed as possible definitions for two new "fractional derivatives". These two local operators were called, respectively, "conformable derivatives" [28] and "beta derivatives" [29]. These new operators have aroused considerable interest in certain areas, but they have also generated various criticisms, and it has been pointed out that they do not deserve to be considered as *fractional derivatives* [30] [31]. In Appendix A we can see some of the systems where these new derivatives have been used, and some of the criticisms they have received. Independently of the controversial issue of whether or not the conformable and beta derivatives might be considered as fractional derivatives, both of them are operators which depend on a continuous parameter (the order of the derivatives), and when this parameter is equal to a positive integer, these two operators reduce to standard, integer-order derivatives. This is a potentially useful property, because if we take a physical model described by a differential equation, and we replace one of the standard derivatives of the model by a conformable or a beta derivative, we would obtain a generalized model which depends on a new continuous degree of freedom: the order of the new derivative. And adjusting this order we might obtain solutions which may be closer to reality than the solutions of the original model. In fact, we already know that replacing integer-order derivatives by "orthodox" fractional derivatives, such as those of Caputo, Grünwald-Letnikov or Ortigueira, we may obtain better descriptions of reality than those provided by standard models which only employ integer-order derivatives. The positive results obtained with orthodox fractional derivatives almost immediately suggest a question: may we also obtain good results if we used a conformable or a beta derivative instead of an orthodox fractional one? The answer is not obvious at all, as the definitions of the conformable and beta derivatives are very different from those of Caputo or Grünwald-Letnikov. In order to contribute to clarify this issue, in the present communication we will consider a system of two coupled PDEs that describe hydrodynamic fluctuations of density and velocity in a linear viscoelastic fluid, and we will consider two generalized versions of this system. In the first version, we will replace the standard first-order temporal derivatives by conformable derivatives, and in the second version, we will replace these derivatives with Caputo's derivatives. Then, we will calculate particular solutions of the three systems (*i.e.*, the original system using first-order derivatives, and those using conformable and Caputo's derivatives). The comparison of these solutions will show to what extent the introduction of the new derivatives modifies the standard solution (*i.e.*, the solution of the system with first-order derivatives), and it will also show that the results obtained with the conformable derivatives are similar to those obtained with Caputo's derivatives. This similarity is an interesting result, considering that the definitions of the conformable and Caputo's derivatives are so different.

The structure of this communication is the following. In Section 2 we briefly explain the physical origin of the system of PDEs studied in this article, we present the definition of the conformable derivative, and a first generalized version of the aforementioned system (with conformable derivatives instead of first-order time derivatives) will be proposed. Then we will obtain analytical and numerical solutions of both, the standard (*i.e.*, involving only integer-order derivatives) and the conformable systems. In Section 3 we will propose a second generalized version of the system of PDEs presented in Section 2, this time replacing the first-order time derivatives with Caputo's derivatives. Then we will show that this fractional system (with Caputo's derivatives) can be analytically solved via the Laplace transformation. And we show solutions for different values of the fractional order of the Caputo's derivatives. Finally, in Section 4 and 5, we enumerate the principal characteristics of the solutions found in the previous sections, and we present the main conclusions of this work.

2. Standard and Conformable Derivatives

Let us consider a model which describes the time retarded density and velocity hydrodynamic fluctuations for a linear viscoelastic fluid in a non-equilibrium state, but near thermodynamic equilibrium. This model was introduced in a previous work [13]. The dynamics of the fluctuations is described in terms of a generalized Langevin equation with a long-time power-law memory kernel with a long-correlation noise. We used a fluctuating hydrodynamic approach that leads to Equations (8) and (9) in [13], which are obtained from the general conservation laws of mass and momentum. Their derivation is discussed in detail in this reference, and here we only describe the general trend of ideas used. It is important to point out that the explicit calculation of these quantities is essential to calculate measurable properties of the system, such as the fractional diffusion coefficient (from the velocity fluctuations correlation function), and the fractional light structure factor (from the density correlations).

If we denote the hydrodynamic fluctuations of density and velocity as f and g, respectively, and we consider (for simplicity) a one-dimensional case, the Equations (8)-(9) of [13] take the forms:

$$\frac{\partial f}{\partial t}=-\left(\rho_0\frac{\partial}{\partial z}+\beta\right)g \tag{1}$$

$$\frac{\partial g}{\partial t}=-\frac{1}{\rho_0^2\kappa}\frac{\partial f}{\partial z}+\frac{1}{\rho_0}\int_0^t M\left(t-t'\right)\frac{\partial^2 g}{\partial z^2}\mathrm{d}t', \tag{2}$$

where we have used the symbol β to represent the concentration gradient which was denoted as α in [13], since in this work we use the symbol α for the order of the conformable and Caputo's derivatives.

The integral on the right side of Equation (2) is the usual way of incorporating memory into this equation, and it follows from the procedure introduced by Wang in [32]. However, in order to obtain a simpler system, in this communication we will consider that the function $M(t)$ is a Dirac delta function. Besides, the left-handed Caputo's derivative already takes into account memory effects, and the conformable derivative also incorporates a time-dependent factor which, in certain cases, can be considered as an approximate way to take into account memory (as we shall see further ahead).Therefore, considering that $M(t)$ is a Dirac delta function, multiplied by a constant M_0, Equations (1)-(2) reduce to:

$$\frac{\partial f}{\partial t}=-\rho_0\frac{\partial g}{\partial z}-\beta g, \tag{1a}$$

$$\frac{\partial g}{\partial t}=-\frac{1}{\rho_0^2\kappa}\frac{\partial f}{\partial z}+\frac{M_0}{\rho_0}\frac{\partial^2 g}{\partial z^2}. \tag{2a}$$

Now let us take the Fourier transform (FT) of Equations (1a)-(2a). If the FT of the functions $f\left(z,t\right)$ and $g\left(z,t\right)$ are denoted as $\hat{f}\left(q,t\right)$ and $\hat{g}\left(q,t\right)$, respectively, we obtain:

$$\frac{\partial\hat{f}}{\partial t}=\left(i\rho_0 q-\beta\right)\hat{g}\equiv A_0\hat{g}, \tag{1b}$$

$$\frac{\partial\hat{g}}{\partial t}=\frac{iq}{\rho_0^2\kappa}\hat{f}-\frac{M_0 q^2}{\rho_0}\hat{g}\equiv B_0\hat{f}-C_0\hat{g}, \tag{2b}$$

where we have defined:

$$A_0=i\rho_0 q-\beta, \tag{3}$$

$$B_0=\frac{iq}{\rho_0^2\kappa}, \tag{4}$$

$$C_0=\frac{M_0 q^2}{\rho_0}. \tag{5}$$

As the principal goal of this communication is to compare the effects of standard derivatives, conformable ones, and Caputo's derivatives, and not in obtaining an accurate description of a particular viscoelastic fluid, we will consider an ideal case (a toy model) where β, A_0, B_0 and C_0 take the following values:

$$\beta = 1,\ A_0 = A \equiv i - 1,\ B_0 = B \equiv i,\ C_0 = C \equiv 1. \tag{6}$$

These values may be obtained with many different values of ρ_0, q, κ and M_0. In particular, the values presented in (6) could be obtained using $\rho_0 = q = \kappa = M_0 = 1$. It should be observed that using these numerical values of the parameters ρ_0, q, κ and M_0 we will be able to obtain numerical solutions of the Equations (1b)-(2b), but the solutions so-obtained will no longer be the complete Fourier transforms $\hat{f}(q,t)$ and $\hat{g}(q,t)$, since we have fixed the value of the wavenumber q. Therefore, we will define the functions:

$$\bar{f}(t) = \hat{f}(q = 1, t),\ \bar{g}(t) = \hat{g}(q = 1, t) \tag{7}$$

and therefore these functions will be the solutions of the system:

$$\frac{\partial \bar{f}}{\partial t} = A\bar{g} \tag{1c}$$

$$\frac{\partial \bar{g}}{\partial t} = B\bar{f} - C\bar{g} \tag{2c}$$

In the following we will consider two generalizations of the System (1c)-(2c). To begin with, let us consider a generalization of the System (1c)-(2c) where the first-order temporal derivatives are replaced by conformable derivatives.

The conformable derivative (CD) of order $0 < \alpha \le 1$ of a function $f : [0, \infty) \to \mathbb{R}$ is defined in the form [28]:

$$(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{8}$$

Moreover, if $f(t)$ is a differentiable function, and $\alpha \in (0,1]$, the principal property of the CDs tells us that [28]:

$$(T_\alpha f)(t) = t^{1-\alpha} \frac{\mathrm{d}f}{\mathrm{d}t}, \tag{9}$$

At first sight the true meaning of this equation is not apparent. However, in Appendix B we will see that the function $t^{1-\alpha}$ can be considered (at least in certain cases) as an approximate way to incorporate a memory effect.

On the other hand, when $\alpha \in (n, n+1]$, the CD of $f(t)$ is defined in the form [33]:

$$(T_\alpha f)(t) = \left(T_\beta f^{(n)}\right)(t), \tag{10}$$

where $\beta = \alpha - n$ and $f^{(n)}$ is the *n-th* derivative of f. Consequently, if $\alpha \in (1,2]$, we will have $\beta = \alpha - 1$, and therefore:

$$(T_\alpha f)(t) = \left(T_\beta f^{(1)}\right)(t) = t^{1-\beta} \frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = t^{2-\alpha} \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}. \tag{11}$$

If we now replace the first-order temporal derivatives in System (1c)-(2c) by CDs we obtain the system:

$$T_\alpha \bar{f} = A\bar{g}, \tag{1d}$$

$$T_\alpha \bar{g} = B\bar{f} - C\bar{g}. \tag{2d}$$

We can obtain the solution of this system by using either Equation (9), if $\alpha \in (0,1]$, or Equation (11), if $\alpha \in (1,2]$. In the first case, when $\alpha \in (0,1]$, the System (1d)-(2d) can be transformed into the system:

$$\frac{\partial \bar{f}}{\partial t} = \frac{1}{t^{1-\alpha}} A\bar{g}, \tag{1e}$$

$$\frac{\partial \bar{g}}{\partial t} = \frac{B}{t^{1-\alpha}}\bar{f} - \frac{C}{t^{1-\alpha}}\bar{g}, \tag{2e}$$

and if we now introduce the change of variables:

$$x = \frac{t^{\alpha}}{\alpha}, \tag{12}$$

and we define:

$$\bar{F}(x) \equiv \bar{f}\left((\alpha x)^{1/\alpha}\right) = \bar{f}(t) \tag{13}$$

$$\bar{G}(x) \equiv \bar{g}\left((\alpha x)^{1/\alpha}\right) = \bar{g}(t), \tag{14}$$

the System (1e)-(2e) is transformed into:

$$\frac{\mathrm{d}\bar{F}}{\mathrm{d}x} = A\bar{G}, \tag{1f}$$

$$\frac{\mathrm{d}\bar{G}}{\mathrm{d}x} = B\bar{F} - C\bar{G}. \tag{2f}$$

If we now use the values of A, B and C given in (6), the solution of this system is the following:

$$\bar{F}(x) = c_1\left(\frac{2}{5} - \frac{i}{5}\right)\mathrm{e}^{(-1-i)x}\left[\mathrm{e}^{2ix} + (1+i)\mathrm{e}^{x}\right] + c_2\left(\frac{3}{5} + \frac{i}{5}\right)\mathrm{e}^{(-1-i)x}\left[\mathrm{e}^{2ix} - \mathrm{e}^{x}\right], \tag{15}$$

$$\bar{G}(x) = c_1\left(\frac{2}{5} - \frac{i}{5}\right)\mathrm{e}^{(-1-i)x}\left[\mathrm{e}^{2ix} - \mathrm{e}^{x}\right] + c_2\left(\frac{2}{5} - \frac{i}{5}\right)\mathrm{e}^{(-1-i)x}\left[(1+i)\mathrm{e}^{2ix} + \mathrm{e}^{x}\right], \tag{16}$$

where c_1 and c_2 are the initial conditions $c_1 = \bar{F}(0)$ and $c_2 = \bar{G}(0)$. And from (15)-(16) we can obtain the solutions $\bar{f}(t)$ and $\bar{g}(t)$ of the System (1e)-(2e) in the form:

$$\bar{f}(t) = \bar{F}\left(t^{\alpha}/\alpha\right) \quad \text{and} \quad \bar{g}(t) = \bar{G}\left(t^{\alpha}/\alpha\right) \tag{17}$$

It should be noticed that we could also obtain the solution of the System (1e)-(2e) numerically. And the numerical solution might be more convenient if we wanted to use initial conditions defined at a time different from zero, because in such a case the expressions for the constants c_1 and c_2 (in terms of the initial conditions) become rather cumbersome.

Now let us consider the second case, when $\alpha \in (1,2]$. In this case the System (1d)-(2d) can be transformed into:

$$\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{A}{t^{2-a}}\bar{g}, \tag{1g}$$

$$\frac{\partial^2 \bar{g}}{\partial t^2} = \frac{B}{t^{2-a}}\bar{f} - \frac{C}{t^{2-a}}\bar{g}. \tag{2g}$$

In this case we cannot find a change of variables [similar to (12)] which permits to transform this system into a new one, with constant coefficients [similar to (1f)-(2f)]. Therefore, in this case, when $\alpha \in (1,2]$, we cannot find the solution of this system in a closed analytical form, and therefore we will obtain numerical solutions (as we shall see below).

We can now obtain solutions of the System (1d)-(2d) in three different cases: when $\alpha = 0.5$ and $\alpha = 1$ [Using (15)-(17), or solving numerically the system], and when $\alpha = 1.5$ [solving numerically (1g)-(2g)]. When $\alpha = 0.5$ and $\alpha = 1$ we will use the initial conditions:

$$\bar{f}(1) = \bar{g}(1) = 1.5 \tag{18}$$

and in the case $\alpha = 1.5$ the following initial conditions will be used:

$$\bar{f}(1) = \bar{g}(1) = 1.5 \quad \text{and} \quad \bar{f}'(1) = \bar{g}'(1) = -0.6 \tag{19}$$

When $\alpha = 1.5$ we use initial conditions defined at $t = 1$ (instead of using $t = 0$) because the values of the derivatives which appear in the System (1g)-(2g) tend to infinity when $t \to 0$, and this might be troublesome when we obtain numerical solutions of this system. And consequently, also when $\alpha = 0.5$ and $\alpha = 1$ we use initial conditions defined at $t = 1$ [as seen in (18)] to solve the System (1e)-(2e), and in this case the solutions of this system will be obtained numerically.

It is interesting to observe that these initial conditions guarantee that $\bar{f}(t) = \bar{g}(t)$ in the three cases: $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.5$. In **Figure 1** we can see the forms of $|\bar{f}(t)|$ corresponding to these three values of *α*. And the real and imaginary parts of $\bar{f}(t)$ are shown in **Figure 2** and **Figure 3**, respectively.

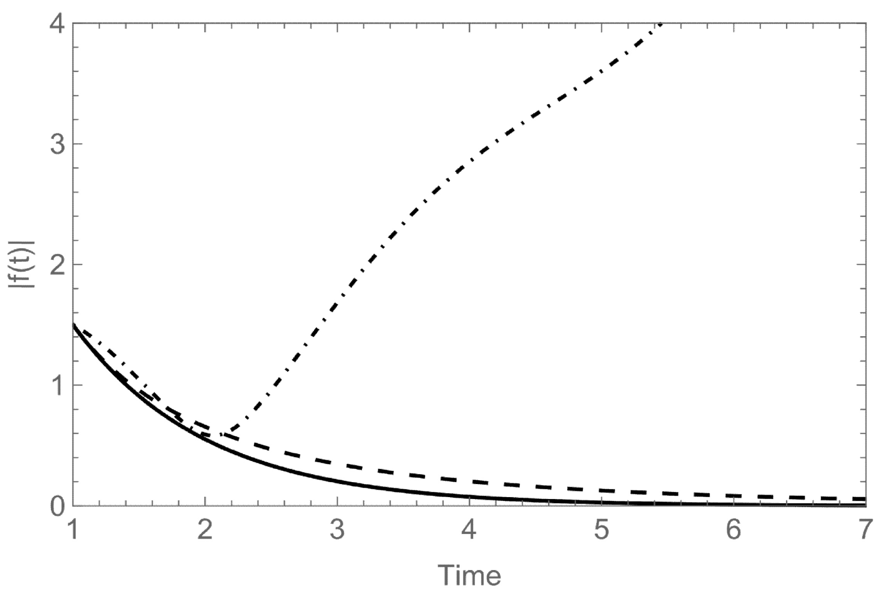

Figure 1. Modulus of the function $\bar{f}(t)$ [solution of the System (1d)-(2d)] when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). The initial conditions used to obtain these three solutions are shown in (18)-(19).

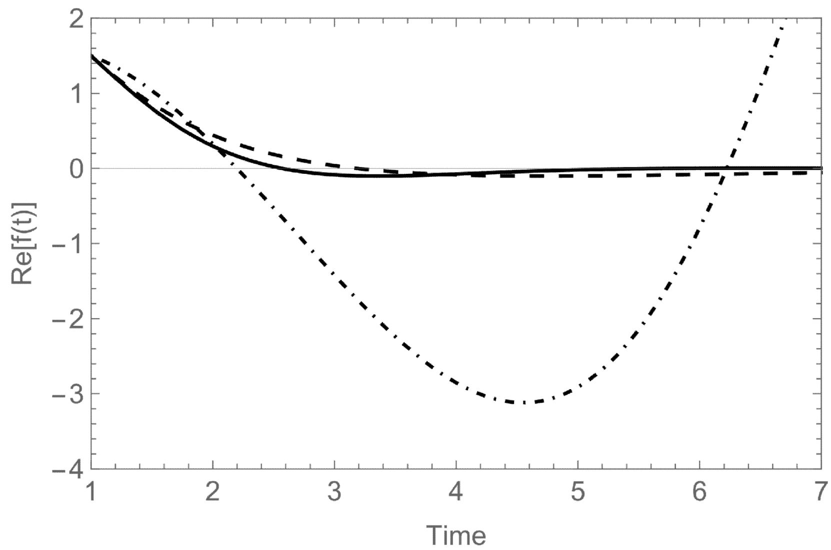

Figure 2. Real part of $\bar{f}(t)$ [solution of the System (1d)-(2d)] when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). The initial conditions used to obtain these three solutions are shown in (18)-(19).

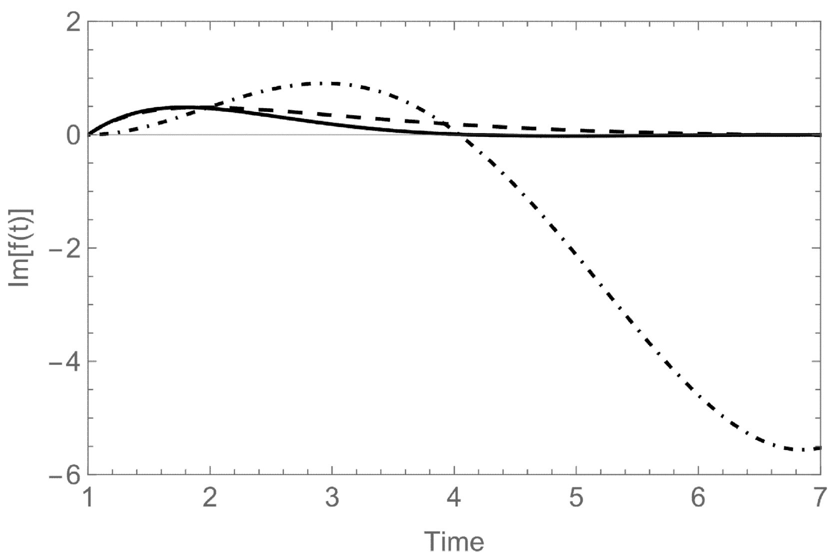

Figure 3. Imaginary part of $\bar{f}(t)$ [solution of the System (1d)-(2d)] when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). The initial conditions used to obtain these three solutions are shown in (18)-(19).

3. Fractional System with Caputo's Derivatives

In this section we consider a fractional generalization of the System (1c)-(2c), where the first-order temporal derivatives are replaced by left-handed Caputo's derivatives.

The left Caputo derivative of a function $f(t)$ is defined in the form [34]:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau. \tag{20} $$

where $\alpha \in [n-1, n)$, and $f^{(n)}$ is the *n-th* derivative of *f*.

Now let us replace the first-order derivatives in (1c)-(2c) by Caputo's derivatives. In this way we obtain the fractional system:

$$ {}_0D_t^\alpha \overline{f} = A\overline{g}, \tag{1h}$$

$$ {}_0D_t^\alpha \overline{g} = B\overline{f} - C\overline{g}. \tag{2h}$$

This system seems difficult to solve. However, it is possible to obtain the Laplace transform (LT) of the Caputo derivative using the equation [34] [35]:

$$\mathcal{L}\left[{}_0D_t^\alpha \overline{f}(t)\right](s) = s^\alpha \overline{\overline{f}}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\overline{f}^{(k)}(0), \tag{21}$$

where $\overline{\overline{f}}(s)$ denotes the LT of $\overline{f}(t)$, $\overline{f}^{(k)}(0)$ is the *k-th* derivative of $\overline{f}(t)$ [evaluated at $t = 0$], $\overline{f}^{(0)}(0) \equiv \overline{f}(0)$, and *n* is the integer such that $\alpha \in (n-1, n]$. It is worth mentioning that the use of this equation implies that we are considering that the values of $\overline{f}^{(k)}(0)$ and $\overline{g}^{(k)}(0)$ are adequate initial conditions to define a particular solution of the fractional System (1h)-(2h). Therefore, in this communication we will not consider other possible ways of initialize a fractional differential equation, since the alternative ways that have been proposed to deal with this problem [36] [37] [38] [39], such as introducing time-varying initializations, or taking into account possible differences between $\overline{f}^{(k)}(0^+)$ and $\overline{f}^{(k)}(0^-)$, are far beyond the scope of this work. In the following we will see that Equation (21) will permit us to obtain the analytical solution of the System (1h)-(2h). Therefore, taking the LT of this system we obtain:

$$\mathcal{L}\left[{}_0D_t^\alpha \overline{f}(t)\right](s) = A\overline{\overline{g}}, \tag{1i}$$

$$\mathcal{L}\left[{}_0D_t^\alpha \overline{g}(t)\right](s) = B\overline{\overline{f}} - C\overline{\overline{g}}, \tag{2i}$$

where $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$ are the LT of $\overline{f}(t)$ and $\overline{g}(t)$, respectively. It is important to observe that the solution of this system changes depending on whether $\alpha \in (0,1]$ or $\alpha \in (1,2]$, because the expression (21) is different if $\alpha < 1$ or $\alpha > 1$. Therefore, we will consider these two cases [$\alpha \in (0,1]$ or $\alpha \in (1,2]$ separately.

In the first case, when $\alpha \in (0,1]$, we have $n = 1$, and Equation (21) reduces to:

$$\mathcal{L}\left[{}_0D_t^\alpha \overline{f}(t)\right](s) = s^\alpha \overline{\overline{f}}(s) - s^{\alpha-1}\overline{\overline{f}}_0 \tag{22}$$

and a similar equation holds for the LT of the Caputo's derivative of $\overline{g}(t)$:

$$\mathcal{L}\left[{}_0D_t^\alpha \overline{g}(t)\right](s) = s^\alpha \overline{\overline{g}}(s) - s^{\alpha-1}\overline{\overline{g}}_0 \tag{23}$$

where we have defined $\overline{\overline{f}}_0 = \overline{\overline{f}}^{(0)}(0) = \overline{\overline{f}}(0)$ and $\overline{\overline{g}}_0 = \overline{\overline{g}}^{(0)}(0) = \overline{\overline{g}}(0)$. If we now substitute Equations (22)-(23) in the System (1i)-(2i), and we use the values of A, B and C given in (6), we obtain a system of two algebraic equations for $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$, whose solution is the following:

$$\overline{\overline{f}}(s) = \frac{s^{\alpha-1}}{s^{2\alpha} + s^\alpha + 1 + i}\left[\left(s^\alpha + 1\right)\overline{\overline{f}}_0 + (i-1)\overline{\overline{g}}_0\right] \tag{24}$$

$$\overline{\overline{g}}(s)=\frac{s^{\alpha-1}}{s^{2\alpha}+s^{\alpha}+1+i}\left[i\overline{\overline{f}}_0+s^{\alpha}\overline{\overline{g}}_0\right] \tag{25}$$

Now it is convenient to rewrite these expressions in a different form, which will be more adequate to obtain the inverse LT of these functions. To clarify the procedure, let us show explicitly how to rewrite the first term of the right-hand-side (rhs) of (24):

$$\frac{s^{2\alpha-1}\overline{\overline{f}}_0}{s^{2\alpha}+s^{\alpha}+1+i}=\frac{\overline{\overline{f}}_0}{s\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)}=\frac{\overline{\overline{f}}_0}{s}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1} \tag{26}$$

Following a similar procedure, we can rewrite the remaining two terms of the rhs of (24), and the two terms of the rhs of (25). In this way we can rewrite $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$ in the following forms:

$$\begin{aligned}\overline{\overline{f}}(s)=&\frac{1}{s}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\overline{\overline{f}}_0+\frac{1}{s^{1+\alpha}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\overline{\overline{f}}_0\\&+\frac{i-1}{s^{1+\alpha}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\overline{\overline{g}}_0,\end{aligned} \tag{27}$$

$$\overline{\overline{g}}(s)=\frac{i}{s^{1+\alpha}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\overline{\overline{f}}_0+\frac{1}{s}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\overline{\overline{g}}_0. \tag{28}$$

The essential point of this transformation is that $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$ are now written in terms of the trinomial:

$$\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1} \tag{29}$$

and this trinomial can be expanded in series using the multinomial expansion (Chap. 24 of [40]):

$$(1+x+y)^{-1}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}(n+m)!}{n!m!}x^n y^m \tag{30}$$

Using this expansion each of the terms which appear in the right-hand-sides of (27) and (28) can be expressed as a double series, and written in this way it is possible to obtain the inverse LT of $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$, as it is possible to obtain the inverse LT of each of the terms in these series. It should be emphasized that this is *precisely* the procedure followed in the Technical Publication of the NASA "Generalized Functions for the Fractional Calculus" [41] to obtain *all the series* which define the seven special functions studied in this reference, and shown in Table 1 of [41].

We can now obtain the inverse LT of $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$. In particular, the inverse LT of (28) is the following function:

$$\begin{aligned}\overline{g}(t)=&i\overline{\overline{f}}_0 t^{\alpha}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{k!}E_{2\alpha,(1-k)\alpha+1}^{(k)}\left[-(1+i)t^{2\alpha}\right]\\&+\overline{g}_0\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{k!}E_{2\alpha,1-k\alpha}^{(k)}\left[-(1+i)t^{2\alpha}\right],\end{aligned} \tag{31}$$

where $E_{\mu,\nu}^{(k)}(z)$ is the *k-th* derivative of the two-parameter Mittag-Leffler function $E_{\mu,\nu}(z)$, and it is given by the equation (see Equation (18) in Ref. [42]):

$$E_{\mu,\nu}^{(k)}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}\frac{z^n}{\Gamma\left[\mu(n+k)+\nu\right]}. \tag{32}$$

Consequently, Equation (31) can be used to obtain the form of $\overline{g}(t)$ when $\alpha\in(0,1]$.

We could proceed in a similar way to obtain $\overline{f}(t)$. However, it is interesting to observe that in the particular case when $\overline{f}_0=\overline{g}_0$, the functions (27) and (28) coincide, and consequently, in this case (when $\overline{\overline{f}}_0=\overline{\overline{g}}_0$), the right-hand-side (rhs) of Equation (31) also gives us the form of $\overline{f}(t)$.

Now let us obtain $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$ in the case when $\alpha\in(1,2]$. Following a procedure similar to that used to obtain the System (27)-(28), in this case the Equations (1i)-(2i) imply that:

$$\begin{aligned}\overline{\overline{f}}(s)=&-\frac{(1+i)\overline{\overline{f}}_0}{s^{2\alpha+1}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}-\frac{(1+i)\overline{\overline{f}}_0'}{s^{2\alpha+2}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\\&+\frac{(i-1)\overline{\overline{g}}_0}{s^{\alpha+1}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}+\frac{(i-1)\overline{\overline{g}}_0'}{s^{\alpha+2}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}+\frac{\overline{\overline{f}}_0}{s}+\frac{\overline{\overline{f}}_0'}{s^2},\end{aligned} \tag{33}$$

$$\begin{aligned}\overline{\overline{g}}(s)=&\frac{i\overline{\overline{f}}_0}{s^{\alpha+1}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}+\frac{i\overline{\overline{f}}_0'}{s^{\alpha+2}}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}\\&+\frac{\overline{\overline{g}}_0}{s}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}+\frac{\overline{\overline{g}}_0'}{s^2}\left(1+\frac{1}{s^{\alpha}}+\frac{1+i}{s^{2\alpha}}\right)^{-1}.\end{aligned} \tag{34}$$

where $\overline{\overline{f}}_0'\equiv\overline{\overline{f}}'(0)$ and $\overline{\overline{g}}_0'\equiv\overline{\overline{g}}'(0)$. It is interesting to observe that also in this case, when $\alpha\in(1,2]$, it has been possible to express $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$ in terms of the trinomial shown in (29). And consequently, also in this case we can use (30) to express $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$ as series. Then we can obtain the inverse LT of each of the terms in these series, and in this form we can obtain the inverse LT of $\overline{\overline{f}}(s)$ and $\overline{\overline{g}}(s)$. In particular, the inverse LT of (34) gives us the following result:

$$\begin{aligned}\overline{g}(t)=&\,i\overline{\overline{f}}_0t^{\alpha}\sum_{n=0}^{\infty}\frac{\left(-t^{\alpha}\right)^n}{n!}E_{2\alpha,(1-n)\alpha+1}^{(n)}\left[-(1+i)t^{2\alpha}\right]\\&+i\overline{\overline{f}}_0't^{\alpha+1}\sum_{n=0}^{\infty}\frac{\left(-t^{\alpha}\right)^n}{n!}E_{2\alpha,(1-n)\alpha+2}^{(n)}\left[-(1+i)t^{2\alpha}\right]\\&+\overline{\overline{g}}_0\sum_{n=0}^{\infty}\frac{\left(-t^{\alpha}\right)^n}{n!}E_{2\alpha,1-n\alpha}^{(n)}\left[-(1+i)t^{2\alpha}\right]\\&+\overline{\overline{g}}_0't\sum_{n=0}^{\infty}\frac{\left(-t^{\alpha}\right)^n}{n!}E_{2\alpha,2-n\alpha}^{(n)}\left[-(1+i)t^{2\alpha}\right].\end{aligned} \tag{35}$$

The form of $\overline{f}(t)$ can be obtained in a similar way. However, if the conditions $\overline{\overline{f}}_0=\overline{\overline{g}}_0$ and $\overline{\overline{f}}_0'=\overline{\overline{g}}_0'$ are satisfied, a bit of algebra shows that the expressions (33) and (34) coincide, and consequently, $\overline{f}(t)$ is also given by the rhs of

Equation (35).

With the expressions (31) and (35) we are now in conditions of calculating the forms of $\overline{f}(t)$ when $\alpha = 0.5, 1.0$ and 1.5. In these three cases we will consider that $\overline{\overline{f}}_0 = \overline{\overline{g}}_0$ and $\overline{\overline{f}}'_0 = \overline{\overline{g}}'_0$, and consequently, in the three cases we will have $\overline{f}(t) = \overline{g}(t)$.

In **Figure 4** we can see the form of $\left|\overline{f}(t)\right|$ when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). And **Figure 5** and **Figure 6** show the forms of $Re\left[\overline{f}(t)\right]$ and $Im\left[\overline{f}(t)\right]$ for these three values of *α*.

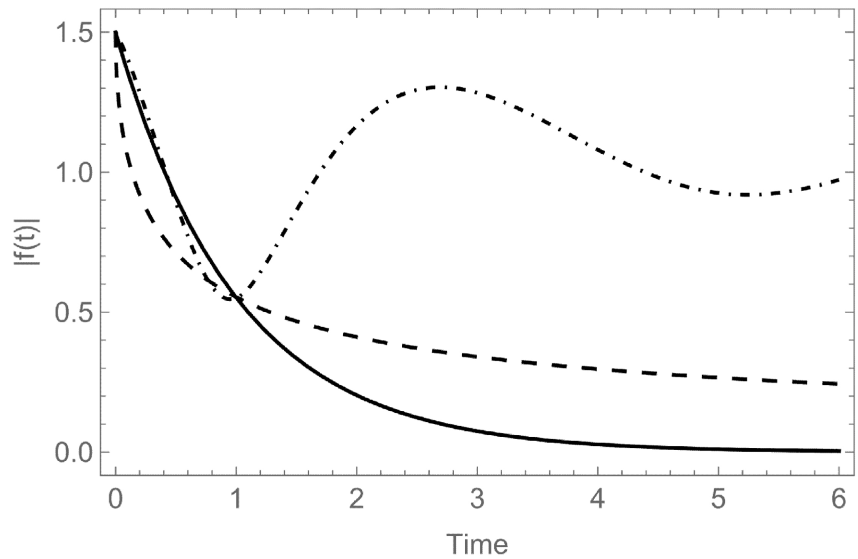

Figure 4. Modulus of the function $\overline{f}(t)$ [solution of the System (1h)-(2h)] when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). The initial conditions used to obtain these three solutions are shown in (36)-(37).

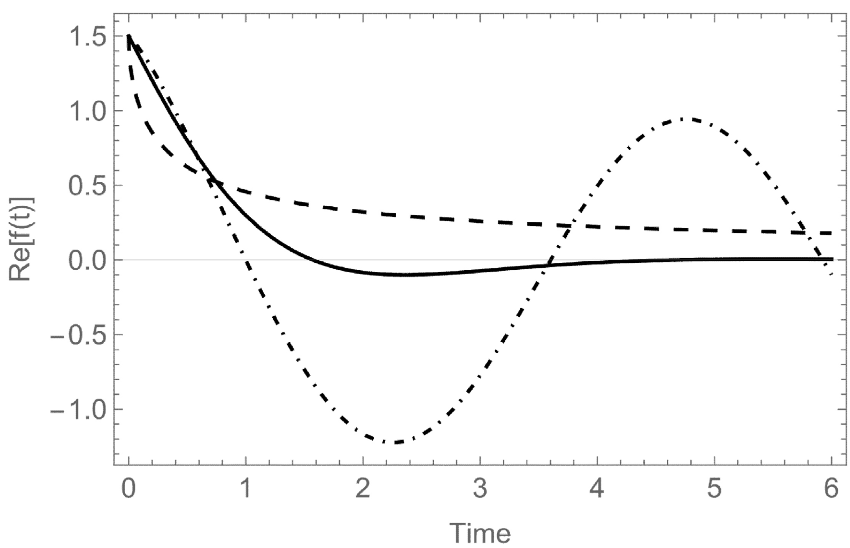

Figure 5. Real part of $\overline{f}(t)$ [solution of the System (1h)-(2h)] when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). The initial conditions used to obtain these three solutions are shown in (36)-(37).

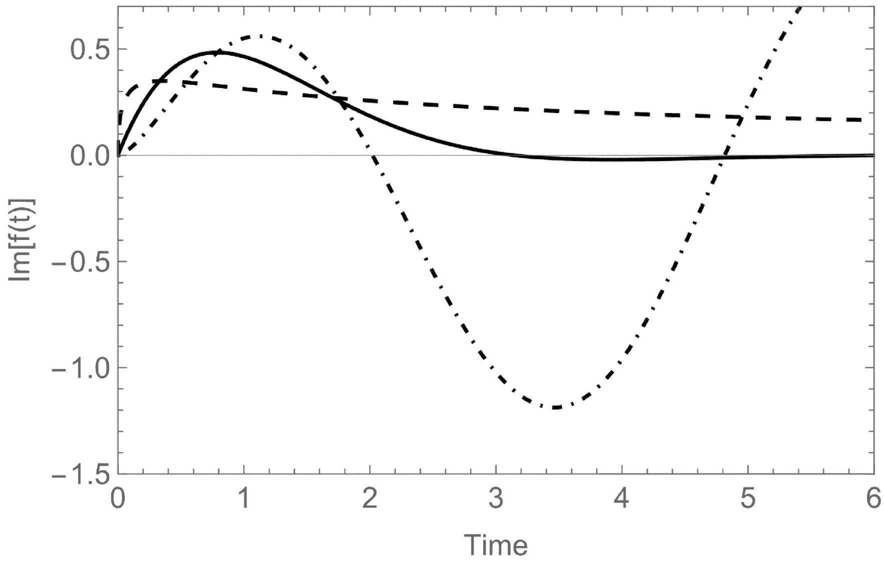

Figure 6. Imaginary part of $\bar{f}(t)$ [solution of the System (1h)-(2h)] when $\alpha = 0.5$ (dashed line), $\alpha = 1$ (solid line), and $\alpha = 1.5$ (dot-dashed). The initial conditions used to obtain these three solutions are shown in (36)-(37).

The solutions corresponding to $\alpha = 0.5$ and $\alpha = 1$ were obtained using the initial conditions:

$$\bar{f}_0 = \bar{g}_0 = 1 \tag{36}$$

and the solution corresponding to $\alpha = 1.5$ was obtained with the initial conditions:

$$\bar{\bar{f}}_0 = \bar{\bar{g}}_0 = 1 \quad \text{and} \quad \bar{\bar{f}}'_0 = \bar{\bar{g}}'_0 = -0.5 \tag{37}$$

4. Discussion

The following 6 observations (OBS) can be drawn from **Figures 1-6**:

Concerning the solutions obtained with conformable derivatives:

OBS-1:

The modulus, as well as the real and imaginary parts, of the solutions of System (1d)-(2d) are extremely similar when $\alpha = 0.5$ and $\alpha = 1$ (see **Figures 1-3**).

OBS-2:

The solution of System (1d)-(2d) corresponding to $\alpha = 1.5$ is significantly different from the solutions obtained with $\alpha = 0.5$ and $\alpha = 1$. In particular, the real and imaginary parts of the solution obtained with $\alpha = 1.5$ exhibit high-amplitude oscillations that do not appear in the solutions obtained with $\alpha = 0.5$ and $\alpha = 1$ (see **Figure 2** and **Figure 3**).

Concerning the solutions obtained with Caputo's derivatives:

OBS-3:

The modulus, as well as the real and imaginary parts of the solutions of System (1h)-(2h) are similar when $\alpha = 0.5$ and $\alpha = 1$ (see **Figures 4-6**). However, these three functions, $\left|\bar{f}(t)\right|$, $Re\left[\bar{f}(t)\right]$ and $Im\left[\bar{f}(t)\right]$, tend to zero at a slower rate when $\alpha = 0.5$, in comparison with the results corresponding to $\alpha = 1$ (see **Figures 4-6**).

OBS-4:

When $\alpha = 1.5$ the modulus, as well as the real and imaginary parts of the solutions of System (1h)-(2h), exhibit high-amplitude oscillations that are not observed when $\alpha = 0.5$ and $\alpha = 1$ (see **Figures 4-6**).

Concerning the comparison between conformable and Caputo's derivatives:

OBS-5:

The behaviours of the real parts of the solutions of System (1d)-(2d) [shown in **Figure 2**] are similar to the behaviours of the real parts of the solutions of System (1h)-(2h) [shown in **Figure 5**].

OBS-6:

The behaviours of the imaginary parts of the solutions of Systems (1d)-(2d) [shown in **Figure 3**], are also similar to the behaviours of the imaginary parts of the solutions of System (1h)-(2h) [shown in **Figure 6**].

From the six observations listed above we can extract three essential conclusions:

FIRST:

In spite of the enormous differences between the definitions of conformable [Equation (8)] and Caputo's derivatives [Equation (20)], the solutions of the System (1d)-(2d) [which contains conformable derivatives] and the System (1h)-(2h) [which contains Caputo's derivatives], exhibit clear (and unexpected) similarities. These similarities constitute a surprising result, since the conformable derivatives are *local operators*, while most of the fractional derivatives (such as those of Caputo, Riemann-Liouville, Grünwald-Letnikov [34], or Ortigueira [43]) are *nonlocal operators*, and consequently it seemed unlikely that these two operators might produce similar results.

SECOND:

The graphs shown in **Figures 1-3** show that we will obtain a small change if we replace the fist-order derivatives which appear in the System (1b)-(2b) by conformable derivatives of an order $\alpha \in (0,1)$. On the other hand, if we replace the first-order derivatives by Caputo's derivatives of an order $\alpha \in (0,1)$, we will obtain solutions which decay in time at a slower rate (as seen in **Figures 4-6**).

THIRD:

The six figures presented in Sections 2 and 3 show that the solutions obtained with conformable or Caputo's derivatives of order $\alpha = 0.5$ are qualitatively similar to those obtained with standard first-order derivatives. On the other hand, conformable and Caputo's derivatives of order $\alpha = 1.5$ produce solutions which are qualitatively different from those of the standard system. The conformable and Caputo's solutions obtained in this case ($\alpha = 1.5$) exhibit high-amplitude oscillations which do not appear when $\alpha = 0.5$. This qualitative difference in the behavior of the solutions obtained with $\alpha = 0.5$, and those obtained with $\alpha = 1.5$, can be understood in both cases, *i.e*., when we use conformable derivatives, and when we use Caputo's derivatives. Let us begin by considering the case of conformable derivatives. When we replace *the first-order derivatives* which appear in System (1c)-(2c) by conformable derivatives of order $\alpha = 0.5$, we obtain *the first-order System* (1e)-(2e). On the other hand, when the first-order derivatives of System (1c)-(2c) are replaced by conformable derivatives of order $\alpha = 1.5$, we obtain *the second order System* (1g)-(2g). Consequently, when we use $\alpha = 0.5$, the order of the system under study *does not change*: initially we had the *first-order* System (1c)-(2c), and after the replacement of the first-order derivatives by conformable derivatives of order $\alpha = 0.5$, we obtain a new *first-order* system: the System (1e)-(2e). This is reason for the similarity of the solutions obtained with $\alpha = 1$ and $\alpha = 0.5$. On the other hand, when we use $\alpha = 1.5$, the original first-order System (1c)-(2c) is transformed into *the second-order System* (1g)-(2g), and this is an important qualitative change (similar to replace the diffusion equation by the wave equation), and this change will necessarily produce important changes in the solutions. This is the reason for the qualitative difference between the solutions obtained with $\alpha = 1$ and $\alpha = 1.5$. Now let us consider the case of Caputo's derivatives. In this case the reason for the qualitative difference between the solutions of the fractional System (1h)-(2h) when $\alpha = 0.5$ and $\alpha = 1.5$ can be found in the definition itself of the Caputo's derivative [Equation (20)]. This definition implies that when the *first-order* derivatives in System (1c)-(2c) are replaced by Caputo's derivatives of order $\alpha = 0.5$, we arrive at a fractional system [System (1h)-(2h)] whose solution requires the values of *the first-order* derivatives $\bar{f}^{(1)}(t)$ and $\bar{g}^{(1)}(t)$ [as the Caputo's derivatives of $\bar{f}(t)$ and $\bar{g}(t)$ when $\alpha = 0.5$ are defined by integrals which contain *first-order* derivatives in the integrands: see Equation (20) with $n = 1$], and these derivatives are precisely the functions defined in the original *first-order* System (1c)-(2c). On the contrary, when the *first-order* derivatives in System (1c)-(2c) are replaced by Caputo's derivatives of order $\alpha = 1.5$, the solution of the fractional System (1h)-(2h) will now require the values of the *second-order* derivatives, $\bar{f}^{(2)}(t)$ and $\bar{g}^{(2)}(t)$ [since the integral in (20) will now contain a *second-order* derivative, because now $n = 2$], and these functions do not appear in the original *first-order* System (1c)-(2c). Therefore, there exists an important qualitative difference in the form of calculating the solutions of the fractional System (1h)-(2h) when we change from $\alpha = 0.5$ to $\alpha = 1.5$, and consequently an important qualitative change in the solutions should be expected.

It is worth observing that the third conclusion mentioned above might be extrapolated as follows.

The solutions of the generalized versions of the System (1c)-(2c) obtained in this paper suggest that whenever we generalize a differential equation (ordinary or partial) by replacing a derivative of *n-th* order by a conformable or a Caputo's derivative of order *α*, the solutions of the generalized equation will be similar to the solutions of the original *n-th* order equation if $\alpha \in (n-1, n)$. However, if $\alpha \in (n, n+1)$, the solutions of the generalized equation will probably be qualitatively different from the solutions of the original equation, and consequently, these generalized solutions may not be physically acceptable. And this qualitative difference could be explained by the same arguments which permitted us to understand the difference in the solutions of the generalized Systems (1d)-(2d) and (1h)-(2h) when we change from $\alpha = 0.5$ to $\alpha = 1.5$.

5. Conclusion

To close this article, we would like to point out that it would be of interest to generalize the analysis presented in this communication in order to apply it to a more realistic system. In particular, the initial conditions and the values of the relevant parameters should be adequately chosen to enable us to calculate measurable transport properties, such as the diffusion coefficient, or the light scattering spectrum of the fluid. This issue will be investigated elsewhere.

Acknowledgements

We thank DGTIC-UNAM (Dirección General de Cómputo y de Tecnologías de Información y Comunicación de la Universidad Nacional Autónoma de México) for granting us access to the computer *Miztli* through Project LANCAD-UNAM-DGTIC-164, in order to carry out this work. We also express our gratitude to Eugenio Ley Koo for his assistance in the calculation of the inverse LTs of Equations (24) and (25).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A

The conformable derivatives have been used in the study of several areas. For example: in optics [44], in chaotic systems [45], in generalizations of Newtonian mechanics [46], and in generalizations of important PDEs of mathematical physics [47] [48]. On the other hand, the beta derivative has also been used in several areas. For example: quantum mechanics [49], optical solitons [50], ferroelectric materials [51], disease propagation [52] and in generalized PDEs of mathematical-physics [53].

The examples mentioned in the previous paragraph show that the conformable and the beta derivatives have aroused interest in many fields, and interesting results have been found using these derivatives. However, a controversial issue concerning these operators is whether they may, or may not, be considered as "fractional derivatives" (FDs). Initially, when they were introduced in [28] [29], they were proposed as FDs. However, in 2015, a few months after the introduction of the conformable derivatives in [28], Ortigueira and Tenreiro Machado proposed a well-defined criterion which permits to decide if an operator may be classified as a FD or not [31], and they found that according to this criterion, the conformable derivatives are not FDs. And afterwards, in 2019, Abdelhakim presented a careful analysis of the conformable derivatives [30], and he also arrived at the conclusion that the conformable derivatives should not be considered as FDs. In spite of these works, conformable and beta derivatives are still being referred to as "fractional derivatives" in some articles [44] [49] [53], although an increasing number of scientists is now regarding the conformable and the beta derivatives as natural extensions of the classical derivative, rather than fractional derivatives [51].

Appendix B

It is well-known that the left-handed fractional derivatives of Caputo, Riemann-Liouville and Grünwald-Letnikov take into account memory effects. In particular, let us consider the left-handed Riemann-Liouville derivative of order $\alpha \in [0,1)$:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \tag{B1} $$

If we now consider the particular case when $f(t) = t$, the integral in (B1) can be evaluated (see Equation (2.12) in [35]), and the following result is obtained:

$$ {}_0D_t^{\alpha} t = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \tag{B2} $$

On the other hand, according to Equation (9), the conformable derivative of $f(t) = t$ is the following:

$$ (T_{\alpha} f)(t) = t^{1-\alpha} \tag{B3} $$

Therefore, in the particular case when $f(t) = t$, the function of time $t^{1-\alpha}$

which appears in the conformable derivative [Equation (9)], reproduces *exactly* the time dependence of the memory contribution given by the Riemann-Liouville fractional derivative shown in (B2). This result shows that the function $t^{1-\alpha}$, which appears in Equation (9), may be considered as the medium used by the conformable derivatives to take into account memory effects (at least in an approximate way).