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Unitariness in Ordered Semigroups

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Abstract

We introduce the concepts of unitary, almost unitary and strongly almost unitary subset of an ordered semigroup. For the notions of almost unitary and strongly almost unitary subset of an ordered semigroup, we use the notion of translational hull of an ordered semigroup. If $(S,\cdot,\le)$ is an ordered semigroup having an element e such that $e \le e^2$ and U is a nonempty subset of S such that $u \le eu$, $u \le ue$ for all $u \in U$, we show that U is almost unitary in S if and only if U is unitary in $(eSe]_{\le}^{S} = \{t \in S : (\exists s \in S)\ t \le ese\}$. Also if $(S,\cdot,\le)$ is an ordered semigroup, $e \notin S$, U is a nonempty subset of S, $S^e := S \cup \{e\}$ and $U^e := U \cup \{e\}$, we give conditions that an ("extension" of S) ordered semigroup $(S^e,*,\preceq)$ and the subset U^e of S^e must satisfy in order for U to be almost unitary or strongly almost unitary in S (in case U is strongly almost unitary in S, then the given conditions are equivalent).

Keywords

Left (Right) Translation of an Ordered Semigroup, Bitranslation of an Ordered Semigroup, Translational Hull of an Ordered Semigroup, Unitary Subset of an Ordered Semigroup, Almost Unitary Subset of an Ordered Semigroup, Strongly almost Unitary Subset of an Ordered Semigroup

1. Introduction: Prerequisites

In a previous paper, the author introduced the concepts of free ordered product and ordered semigroup amalgam showing that an ordered semigroup amalgam is embeddable in an ordered semigroup if and only if it is naturally embedded in its free ordered product. Howie [1] gave sufficient conditions under which a (semigroup) amalgam can be embedded in a semigroup. To do this Howie generalized the concept of unitariness to that of almost unitariness and his basic result is based on almost unitariness of a subset of semigroup. In this paper we study "similar concepts" in case of ordered semigroups applying the (usual for ordered semigroups) following "technique": if $(S,\cdot,\leq)$ is an ordered semigroup, then, instead of an identity element, we consider an element *e* of *S* such that $e \leq e^2$ and also for a nonempty subset *A* of *S* we usually consider the subsets of *S*, $(A]_{\leq}^{S} = \{x \in S : (\exists \alpha \in A) x \leq \alpha\}$ and $[A)_{\leq}^{S} = \{x \in S : (\exists \alpha \in A) \alpha \leq x\}$. In this paper, taking into account Howie's definitions (see [1] [2] (§VIII.3) and ([3] §9.4)) we introduce the concepts of unitary, almost unitary and strongly almost unitary subset of an ordered semigroup *S* in terms of left and right translations of an ordered semigroup *S* (as well as the translational hull of *S*). For the definitions and results presented in this paragraph, we refer to [4] [5]. An ordered semigroup $(S,\cdot,\leq)$ is a semigroup $(S,\cdot)$ with an order relation "$\leq$" which is compatible with the operation "$\cdot$" (*i.e.* for $\alpha,b,c \in S$, $\alpha \leq b$ implies $\alpha \cdot c \leq b \cdot c$ and $c \cdot \alpha \leq c \cdot b$). Now let $(S,\cdot,\leq)$ be an ordered semigroup and $\lambda,\rho : S \to S$.

- λ is called a left translation of S if
  i) $\lambda(xy) = \lambda(x)y$ for all $x,y \in S$.
  ii) For $x,y \in S$, $x \leq y$ implies $\lambda(x) \leq \lambda(y)$ (*i.e.* λ is an isotone mapping).
- ρ is called a right translation of *S* if
  i) $\rho(xy) = x\rho(y)$ for all $x,y \in S$.
  ii) For $x,y \in S$, $x \leq y$ implies $\rho(x) \leq \rho(y)$ (*i.e.* ρ is an isotone mapping).

It is readily to prove that the set $\Lambda(S)$ (resp. $P(S)$) of all left (resp. right) translations of *S* is a semigroup under the usual composition of mappings. On the set:

- $\Lambda(S)$ we define a binary relation

$$f \leq^{\Lambda} g \Leftrightarrow f(x) \leq g(x) \text{ for all } x,y \in S$$

- $P(S)$ we define a binary relation

$$f \leq^{P} g \Leftrightarrow f(x) \leq g(x) \text{ for all } x,y \in S$$

Then it is straightforward to verify that $(\Lambda(S),\circ,\leq^{\Lambda})$, $(P(S),\circ,\leq^{P})$ are ordered semigroups. If $\alpha \in S$, then

- The mapping $\lambda_\alpha : S \to S$, $\lambda_\alpha(x) := \alpha x$ is called inner left translation induced by *α*.
- The mapping $\rho_\alpha : S \to S$, $\rho_\alpha(x) := x\alpha$ is called inner right translation induced by *α*.

It is a matter of routine to prove that λ_α (resp. ρ_α) is a left (resp. right) translation of S and $\lambda_\alpha \circ \rho_\alpha = \rho_\alpha \circ \lambda_\alpha$.

A left translation λ and a right translation ρ are *linked* if for every $x,y \in S$, $x\lambda(y) = \rho(x)y$ and then the pair (λ,ρ) is called a bitranslation of *S*. Clearly (λ_b,ρ_b) is a bitranslation of *S* for every $b \in S$. It is easy to verify that the "product" of two bitranslations of S, say (λ,ρ), (λ',ρ'), defined by

$$(\lambda,\rho)\Diamond(\lambda',\rho') := (\lambda\circ\lambda',\rho'\circ\rho)$$

is again a bitranslation of *S*. Therefore the set $\Omega(S)$ of all bitranslations of *S* under the operation "$\Diamond$" defined above is a semigroup. Also the binary relation "$\leq^{\Omega}$" on $\Omega(S)$ defined by the rule that

$$(\lambda,\rho)\leq^{\Omega}(\lambda',\rho')\Leftrightarrow \lambda\leq^{\Lambda}\lambda',\ \rho\leq^{P}\rho'$$

can be easily shown that it is an order relation on $\Omega(S)$ compatible with the operation "$\Diamond$" and hence $(\Omega(S),\Diamond,\leq^{\Omega})$ is an ordered semigroup called *translational hull* of *S*. The concept of translational hull of an ordered semigroup was introduced by the author in his Doctoral Dissertation.

If *U* is a nonempty subset of *S*, then we denote

- $(U]^{S}_{\leq} := \{x\in S : (\exists\alpha\in U)\, x\leq\alpha\}$
- $[U)^{S}_{\leq} := \{x\in S : (\exists\alpha\in U)\, \alpha\leq x\}$

Let now *T* be a subsemigroup of $(S,\cdot,\leq)$. Then $(T,\cdot_T,\leq_T)$ is an ordered semigroup where "$\cdot_T$" and "$\leq_T$" are the restrictions of the operation "$\cdot$" and order relation "$\leq$" of *S* on *T* respectively, that is,

$$\alpha\cdot_T b := \alpha\cdot b \quad (\alpha,b\in S) \text{ and } \leq_T := \leq\cap(T\times T)$$

It is clear that if *T* is a subsemigroup of S, then $[T)^{S}_{\leq}$ and $(T]^{S}_{\leq}$ are subsemigroups of *S*. In the following, when we have a subsemigroup *T* of *S*, we shall always consider *T* as an ordered semigroup with the previous structure $(T,\cdot_T,\leq_T)$ and hence, if *A* is a nonempty subset of *T*, then

- $(A]^{T}_{\leq_T} = \{x\in T : (\exists\alpha\in A)\, x\leq_T \alpha\}$
- $[A)^{T}_{\leq_T} = \{x\in T : (\exists\alpha\in A)\, \alpha\leq_T x\}$

Obviously

$$(A]^{T}_{\leq_T} = (A]^{S}_{\leq}\cap T \quad\text{and}\quad [A)^{T}_{\leq_T} = [A)^{S}_{\leq}\cap T$$

2. Unitary Subsets of an Ordered Semigroup

Definition 2.1: Let $(S,\cdot,\leq)$ be an ordered semigroup and *U* be a nonempty subset of *S*.

i) *U* is called *left unitary* in *S* if

I) $[U)^{S}_{\leq}\subseteq(U]^{S}_{\leq}$

II) for $w,u\in U$ and $s\in S$ such that $w\leq us$, we have $s\leq v$ for some $v\in U$

ii) *U* is called *right unitary* in *S* if

I) $[U)^{S}_{\leq}\subseteq(U]^{S}_{\leq}$

II) for $w,u\in U$ and $s\in S$ such that $w\leq su$, we have $s\leq v$ for some $v\in U$

iii) *U* is called *unitary* in *S* if it is both left and right unitary. □

Proposition 2.2: Let $(S,\cdot,\leq)$ be an ordered semigroup, *U* be a left (resp. right) unitary subset of *S* and *T* be a subsemigroup of *S* containing *U*. Then *U* is a left (resp. right) unitary subset of *T*.

Proof: As we mentioned above, since *T* is a subsemigroup of *S*, we consider *T* as an ordered semigroup endowed with the operation and order relation defined by

$$\alpha \cdot_T b := \alpha \cdot b,\ \ \alpha, b \in T \ \text{ and } \ \leq_T := \leq \cap (T \times T)$$

and hence, since $U \subseteq T$, by definition we have

$$[U)_{\leq_T}^T = \{t \in T : (\exists w \in U) w \leq_T t\} \ \text{ and } \ (U]_{\leq_T}^T = \{t \in T : (\exists v \in U) t \leq_T v\}$$

To prove that $[U)_{\leq_T}^T \subseteq (U]_{\leq_T}^T$, suppose $y \in [U)_{\leq_T}^T$. Since $[U)_{\leq_T}^T = [U)_{\leq}^S \cap T$, we have $y \in [U)_{\leq}^S \cap T$ and hence $y \in [U)_{\leq}^S$. Since *U* is a left unitary subset of *S* (Definition 2.1 i) I)) then $y \in (U]_{\leq}^S$ and hence $y \in (U]_{\leq}^S \cap T = (U]_{\leq_T}^T$. Next we shall show that the condition i)II) of Definition 2.1 is true. Let $w, u \in U$ and $t \in T$ such that $w \leq_T u \cdot_T t$. Clearly $u \cdot_T t = u \cdot t$ and $w \leq u \cdot t$. Since *U* is a left unitary subset of *S* (Definition 2.1 i) II)) there exists $v \in U$ such that $t \leq v$, *i.e.* $t \leq_T v$. Therefore the conditions I), II) of Definition 2.1 i) hold and so *U* is a left unitary subset of *T*. □

Remark 2.3: By Proposition 2.2 it follows directly that if $(S, \cdot, \leq)$ is an ordered semigroup, *U* is a unitary subset of *S* and *T* is a subsemigroup of *S* containing *U*, then *U* is a unitary subset of *T*. □

Proposition 2.4: Let $(S, \cdot, \leq)$ be an ordered semigroup, *U* be a nonempty subset of *S* and *T* be a subsemigroup of *S* containing *U*. The following are equivalent:

i) *U* is a left (resp. right) unitary subset of *T*.

ii) α) $[U)_{\leq}^S \cap T \subseteq (U]_{\leq}^S$

***β*)** If $w, u \in U$ and $t \in T$ with $w \leq ut$ (resp. $w \leq tu$), then there exists $v \in U$ such that $t \leq v$.

Proof: As we mentioned above, since *T* is a subsemigroup of *S*, we consider *T* as an ordered semigroup $(T, \cdot_T, \leq_T)$ where

$$\alpha \cdot_T b := \alpha \cdot b,\ \ \alpha, b \in T \ \text{ and } \ \leq_T := \leq \cap (T \times T)$$

i) $\Rightarrow$ **ii)** It is clear that *U* is a nonempty subset of *T*.

***α*)** From Definition 2.1 we have $[U)_{\leq_T}^T \subseteq (U]_{\leq_T}^T$ and hence

$$[U)_{\leq}^S \cap T \subseteq (U]_{\leq}^S$$

***β*)** Let $w, u \in U$ and $t \in T$ such that $w \leq ut$. From the definitions of "$\cdot_T$" and "$\leq_T$" we immediately have (since $U \subseteq T$) $w \leq_T u \cdot_T t$ and consequently (since *U* is a left unitary subset of *T*) there exists $v \in U$ such that $t \leq_T v$. Thus (since $\leq_T \subseteq \leq$) $t \leq v$.

ii) $\Rightarrow$ **i)** From α) we obtain directly $[U)_{\leq}^S \cap T \subseteq (U]_{\leq}^S \cap T$, that is $[U)_{\leq_T}^T \subseteq (U]_{\leq_T}^T$. Now let $w, u \in U$ and $t \in T$ such that $w \leq_T u \cdot_T t$. From the definitions of "$\cdot_T$" and "$\leq_T$" we have (since $U \subseteq T$) $w \leq u \cdot t$ and so, by β), there exists $v \in U$ such that $t \leq v$ which clearly means that $t \leq_T v$. Then, from Definition 2.1 i), it follows immediately that *U* is a left unitary subset of *T*. □

By Definition 2.1 and Proposition 2.4 we directly have the following

Proposition 2.5: Let $(S,\cdot,\le)$ be an ordered semigroup, U be a nonempty subset of S and T be a subsemigroup of S containing U. The following are equivalent:

i) U is unitary in T.

ii) *α*) $[U)_{\le}^{S}\cap T\subseteq(U]_{\le}^{S}$

***β*)** If $w,u\in U$ and $t\in T$ with $w\le ut$, then there exists $v\in U$ such that $t\le v$.

***γ*)** If $w,u\in U$ and $t\in T$ with $w\le tu$, then there exists $v\in U$ such that $t\le v$. □

3. Almost Unitary Subsets of an Ordered Semigroup

Proposition 3.1: Let $(S,\cdot,\le)$ be an ordered semigroup, λ be a left translation of S and ρ be a right translation of S such that $\lambda\circ\rho=\rho\circ\lambda$. Then $\lambda(\rho(S))$ is a subsemigroup of S.

Proof: Suppose $\alpha,b\in S$. Then

$$\lambda(\rho(\alpha))\lambda(\rho(b))=\lambda(\rho(\alpha)\lambda(\rho(b)))\quad(\lambda \text{ is a left translation of } S)$$

$$=\lambda(\rho(\alpha)\rho(\lambda(b)))\quad(\lambda\circ\rho=\rho\circ\lambda)$$

$$=\lambda(\rho(\rho(\alpha)\lambda(b)))\quad(\rho \text{ is a right translation of } S)$$

that is, $\lambda(\rho(\alpha))\lambda(\rho(b))\in\lambda(\rho(S))$. Hence $\lambda(\rho(S))\lambda(\rho(S))\subseteq\lambda(\rho(S))$ and so, since clearly $\lambda(\rho(S))$ is a nonempty subset of S, $\lambda(\rho(S))$ is a subsemigroup of S. □

Definition 3.2: Let $(S,\cdot,\le)$ be an ordered semigroup and U be a nonempty subset of S. The subset U is said to be *almost unitary* in S if there exist mappings $\lambda,\rho:S\to S$ (called associated mappings) with the following properties:

i) (λ,ρ) is a bitranslation of S (*i.e.* $(\lambda,\rho)\in\Omega(S)$)

ii) $\lambda\circ\rho=\rho\circ\lambda$ (*i.e.* λ commutes with ρ)

iii) $\lambda(x)\le\lambda(\lambda(x))$ and $\rho(x)\le\rho(\rho(x))$ for all $x\in S$

iv) $u\le\lambda(u)$ and $u\le\rho(u)$ for all $u\in U$

v) U is unitary in $(\lambda(\rho(S))]_{\le}^{S}$. □

Remark 3.3: By Proposition 3.1 we deduce directly that if λ is a left translation of S and ρ is a right translation of S such that $\lambda\circ\rho=\rho\circ\lambda$, then $(\lambda(\rho(S))]_{\le}^{S}$ is a subsemigroup of S and hence the v) of Definition 3.2 is meaningful. □

Theorem 3.4: Let $(S,\cdot,\le)$ be an ordered semigroup, $e\in S$ such that $e\le e^2$ and U be a subset of S such that for all $u\in U$, $u\le eu$ and $u\le ue$. Then

i) $(eSe]_{\le}^{S}$ is a subsemigroup of S

ii) U is almost unitary in S if and only if U is unitary in $(eSe]_{\le}^{S}$.

Proof:

i) Since eSe is clearly a subsemigroup of S then $(eSe]_{\le}^{S}$ is also a subsemigroup of S.

ii) For each $u \in U$ we have

$$u \le eu \le eue \in eUe \subseteq eSe$$

whence it follows immediately that $U \subseteq (eSe]_{\le}^{S}$.

($\Rightarrow$) Let U be almost unitary in S. Then there exist mappings $\lambda, \rho : S \to S$ with the properties i) - v) of Definition 3.2. By condition iv) of Definition 3.2, it follows that $e \le \lambda(e)$ and $e \le \rho(e)$. Hence for every $\alpha \in S$ we have

$$e\alpha e \le \lambda(e)\alpha\rho(e) \quad (\lambda \text{ is a left translation of } S)$$

$$= \lambda(e\alpha\rho(e)) \quad (\rho \text{ is a right translation of } S)$$

$$= \lambda(\rho(e\alpha e))$$

that is, $e\alpha e \le \lambda(\rho(e\alpha e)) \in \lambda(\rho(S))$. Thus $eSe \subseteq (\lambda(\rho(S))]_{\le}^{S}$ and so $(eSe]_{\le}^{S} \subseteq (\lambda(\rho(S))]_{\le}^{S}$. By Proposition 3.1 (since $\lambda \circ \rho = \rho \circ \lambda$, see Definition 3.2 ii)) we have $\lambda(\rho(S))$ is a subsemigroup of S and hence $(\lambda(\rho(S))]_{\le}^{S}$ is a subsemigroup of S. Therefore, since $(\lambda(\rho(S))]_{\le}^{S}$ is a subsemigroup of S containing $(eSe]_{\le}^{S}$ and $(eSe]_{\le}^{S}$ is a subsemigroup of S then we immediately have $(eSe]_{\le}^{S}$ is a subsemigroup of $(\lambda(\rho(S))]_{\le}^{S}$ and so, since $U \subseteq (eSe]_{\le}^{S}$ and (by Definition 3.2 v)) U is unitary in $(\lambda(\rho(S))]_{\le}^{S}$, it follows from Remark 4.3 that U is unitary in $(eSe]_{\le}^{S}$.

($\Leftarrow$) Let U be unitary in $(eSe]_{\le}^{S}$. We take for λ and ρ the inner left and inner right translation λ_e, ρ_e respectively, that is

$$\lambda_e : S \to S, \quad \lambda_e(x) := ex$$

and

$$\rho_e : S \to S, \quad \rho_e(x) := xe$$

We shall show that U is almost unitary in S with λ_e, ρ_e as associated mappings. The conditions i), ii), iv) of Definition 3.2 clearly hold. For the iii) of Definition 3.2, take $x \in S$. Then

$$\lambda_e(x) = ex \le e^2x = e(ex) = \lambda_e(ex) = \lambda_e(\lambda_e(x))$$

that is, $\lambda_e(x) \le \lambda_e(\lambda_e(x))$. Similarly $\rho_e(x) \le \rho_e(\rho_e(x))$. Regarding v) of Definition 3.2, we observe that $\lambda_e(\rho_e(S)) = \lambda_e(Se) = eSe$ whence it follows immediately that $(eSe]_{\le}^{S} = (\lambda_e(\rho_e(S))]_{\le}^{S}$. Consequently, since U is unitary in $(eSe]_{\le}^{S}$, we deduce that U is unitary in $(\lambda_e(\rho_e(S))]_{\le}^{S}$ (*i.e.* the condition v) of Definition 3.2 is true). Therefore, according to Definition 3.2, U is almost unitary in S. □

Remark 3.5: From the proof of Theorem 3.4, we immediately observe that Theorem 3.4 also holds without the condition iii) of Definition 3.2 and the property that λ, ρ are linked. □

For $e \notin S$ and T a nonempty subset of S we denote $T^e := T \cup \{e\}$.

Theorem 3.6: Let $(S, \cdot, \le)$ be an ordered semigroup, U be a nonempty subset of S and $e \notin S$. Also let $(S^e, *, \preceq)$ be an ordered semigroup with the following properties:

i) $\alpha \cdot b = \alpha * b$, $\alpha, b \in S$

ii) $e * S \subseteq S$, $S * e \subseteq S$

iii) $\leq = \preceq \cap (S \times S)$

iv) $\preceq \cap (S \times \{e\}) = \varnothing$

v) $u \preceq e * u$ and $u \preceq u * e$ for every $u \in U^e$

vi) U^e is unitary in $\left(e * S^e * e\right]_{\preceq}^{S^e}$.

Then U is almost unitary in S.

Proof: Set

- $\lambda : S \to S$, $\lambda(x) := e * x$
- $\rho : S \to S$, $\rho(x) := x * e$

Because of ii), it is evident that λ, ρ are well defined. Also for $x, y \in S$ we have

- $\lambda(x \cdot y) = e * (x \cdot y) \underset{\text{i)}}{=} e * (x * y) = (e * x) * y = \lambda(x) * y \underset{\text{i)}}{\overset{\lambda(x)\in S}{=}} \lambda(x) \cdot y$
- $x \leq y \underset{\text{iii)}}{\Rightarrow} x \preceq y \Rightarrow e * x \preceq e * y \Rightarrow \lambda(x) \preceq \lambda(y) \underset{\text{iii)}}{\overset{\lambda(x),\lambda(y)\in S}{\Rightarrow}} \lambda(x) \leq \lambda(y)$

So λ is a left translation of S. Similarly ρ is a right translation of S. Moreover for $\alpha, b \in S$ we have

$$\alpha \cdot \lambda(b) \underset{\text{i)}}{\overset{\lambda(b)\in S}{=}} \alpha * \lambda(b) = \alpha * (e * b) = (\alpha * e) * b = \rho(\alpha) * b \underset{\text{i)}}{\overset{\rho(\alpha)\in S}{=}} \rho(\alpha) \cdot b$$

and hence λ, ρ are linked on S. Thus (λ, ρ) is a bitranslation of S (*i.e.* the condition i) of Definition 3.2 holds). Since for any $\alpha \in S$

$$\lambda(\rho(\alpha)) = \lambda(\alpha * e) = e * (\alpha * e) = (e * \alpha) * e = \rho(e * \alpha) = \rho(\lambda(\alpha))$$

then the condition ii) of Definition 3.2 is true. Moreover by v) for $u = e$ we directly have

$$e \preceq e * e \tag{1}$$

and so for each $x \in S$

$$\lambda(x) = e * x \underset{(1)}{\preceq} (e * e) * x = e * (e * x) = e * \lambda(x) \underset{\lambda(x)\in S}{=} \lambda(\lambda(x))$$

Thus

$$\lambda(x) \preceq \lambda(\lambda(x)) \underset{\text{iii)}}{\overset{\lambda(x),\lambda(\lambda(x))\in S}{\Rightarrow}} \lambda(x) \leq \lambda(\lambda(x))$$

Similarly we show that $\rho(x) \leq \rho(\rho(x))$. Therefore the condition iii) of Definition 3.2 holds. Also, by v), for every $u \in U$ we have

$$u \preceq e * u = \lambda(u) \Rightarrow u \preceq \lambda(u) \underset{\text{i)}}{\overset{\lambda(u)\in S}{=}} u \leq \lambda(u)$$

and in a similar way we show that $u \leq \rho(u)$. Consequently the condition iv) of Definition 3.2 holds.

I) Since for any $\alpha \in S$, $\lambda(\rho(\alpha)) = e * \alpha * e$ then clearly $\lambda(\rho(S)) = e * S * e$.

II) $\left(\lambda(\rho(S))\right]_{\leq}^{S}$ is a subsemigroup of S (see Remark 3.3).

III) $U \subseteq \left(\lambda(\rho(S))\right]_{\leq}^{S}$:

Let $u \in U$. Then by v) we have

$$u \preceq e * u \preceq e * u * e$$

Thus $u \preceq e*u*e$ and since $e*u*e = \lambda(\rho(u))$ then $u \preceq \lambda(\rho(u))$. Obviously $u, \lambda(\rho(u)) \in S$ and hence, by iii), it follows that $u \le \lambda(\rho(u))$. Therefore $u \in (\lambda(\rho(S))]_{\le}^{S}$ and so $U \subseteq (\lambda(\rho(S))]_{\le}^{S}$.

IV) $[U)_{\le}^{S} \cap (\lambda(\rho(S))]_{\le}^{S} \subseteq (U]_{\le}^{S}$:

Let $w \in [U)_{\le}^{S} \cap (\lambda(\rho(S))]_{\le}^{S}$. Since $w \in [U)_{\le}^{S}$ then $u \le w$ for some $u \in U$ and so, by iii), $u \preceq w$ which obviously means that $w \in [U^e)_{\le}^{S^e}$. Also since $w \in (\lambda(\rho(S))]_{\le}^{S}$ then there exists $s \in S$ such that $w \le \lambda(\rho(s)) \underset{\text{I)}}{=} e*s*e$ and so, by iii), $w \preceq e*s*e$. Hence $w \in (e*S^e*e]_{\preceq}^{S^e}$. Consequently

$$w \in [U^e)_{\le}^{S^e} \cap (e*S^e*e]_{\preceq}^{S^e}$$

But U^e is unitary in $(e*S^e*e]_{\preceq}^{S^e}$ and so, by Definition 2.1,

$$[U^e)_{\le}^{S^e} \cap (e*S^e*e]_{\preceq}^{S^e} \subseteq (U^e]_{\le}^{S^e} \cap (e*S^e*e]_{\preceq}^{S^e}$$

Therefore $w \in (U^e]_{\le}^{S^e} \cap (e*S^e*e]_{\preceq}^{S^e}$ and thus there exists $v \in U^e$ such that $w \preceq v$. Since $U^e = U \cup \{e\}$ it follows that $v = e$ or $v \in U$. If $v = e$ then $(w,v) \in \preceq \cap (S \times \{e\})$. Contradiction (see iv)). Therefore $v \in U$ and since $w \preceq v$, we have (see iii)) $w \le v$ and so $w \in (U]_{\le}^{S}$. Consequently

$$[U)_{\le}^{S} \cap (\lambda(\rho(S))]_{\le}^{S} \subseteq (U]_{\le}^{S}$$

V) Let $w, u \in U$ and $t \in (\lambda(\rho(S))]_{\le}^{S}$ such that $w \le u \cdot t$. We shall prove that $t \le \upsilon$, $\upsilon \in U$. Indeed:

Since $t \in (\lambda(\rho(S))]_{\le}^{S}$ we have $t \in S$ and $t \le \lambda(\rho(s)) = e*s*e$ for some $s \in S$. Hence (see iii)) $t \preceq e*s*e$. Therefore $t \in (e*S*e]_{\preceq}^{S^e}$. Also since $w \le t \cdot u$, then (see i), iii)) $w \preceq t*u$. But (see vi) and Definition 2.1 iii)) U^e is a left unitary subset of $(e*S*e]_{\preceq}^{S^e}$. Consequently (see Proposition 2.4 i) $\Rightarrow$ ii)β)) there exists $v \in U^e$ such that $t \preceq v$. Since $U^e = U \cup \{e\}$ it follows that $v = e$ or $v \in U$. If $v = e$ then $(t,v) \in \preceq \cap (S \times \{e\})$. Contradiction (see iv)). Therefore $v \in U$ and since $t \preceq v$, we have (see iii)) $t \le v$. From II) - V) and Proposition 2.4 ii) $\Rightarrow$ i) it follows directly that U is a left unitary subset of $(\lambda(\rho(S))]_{\le}^{S}$. Similarly we show that U is a right unitary subset of $(\lambda(\rho(S))]_{\le}^{S}$. Therefore (see Definition 2.1 iii)) U is a unitary subset of $(\lambda(\rho(S))]_{\le}^{S}$ and so the condition v) of Definition 3.2 holds. This (see Definition 3.2) completes the proof. □

4. Strongly Almost Unitary Subsets of an Ordered Semigroup

<u>**Remark 4.1:**</u> From the proof of the Theorem 3.6 we immediately have that if $e = e*e$ (instead of $e \preceq e*e$), then for every $x \in S$, $\lambda(x) = \lambda(\lambda(x))$ and $\rho(x) = \rho(\rho(x))$ (where λ, ρ are the mappings defined in the proof of the previous Theorem). □

Now due to Remark 4.1 it follows immediately the next

Theorem 4.2: Let $(S,\cdot,\leq)$ be an ordered semigroup, U be a nonempty subset of S and $e \notin S$. Also let $(S^e,*,\preceq)$ be an ordered semigroup with the following properties:

i) $\alpha \cdot b = \alpha * b$, $\alpha, b \in S$

ii) $e * S \subseteq S$, $S * e \subseteq S$

iii) $e = e * e$

iv) $\leq = \preceq \cap (S \times S)$

v) $\preceq \cap (S \times \{e\}) = \varnothing$

vi) $u \preceq e * u$ and $u \preceq u * e$ for every $u \in U$

vii) U^e is unitary in $(e * S^e * e]_{\preceq}^{S^e}$.

Then there exist mappings $\lambda, \rho : S \to S$ such that

α) $\lambda = \lambda \circ \lambda$, $\rho = \rho \circ \rho$

β) U is almost unitary in S with associated mappings λ, ρ. □

Remark 4.3: Observe that, due to Theorem 3.4 ii), we can equivalently replace the conditions vi), vii) of Theorems 3.6, 4.2 with the condition "U^e is almost unitary in S^e". So, if the ordered semigroup $(S^e,*,\preceq)$ satisfies the conditions i) - v) of Theorem 3.6 (resp. the conditions i) - vi) of Theorem 4.2) and also U^e is almost unitary in S^e, then the conclusion of Theorem 3.6 (resp. of Theorem 4.2) remains true. This helps us better understand the connection between "almost unitary in S" and "almost unitary in S^e". □

Definition 4.4: Let $(S,\cdot,\leq)$ be an ordered semigroup and U be a nonempty subset of S. The subset U is said to be strongly almost unitary in S if there exist mappings $\lambda, \rho : S \to S$ (called associated mappings) with the following properties:

i) (λ, ρ) is a bitranslation of S (*i.e.* $(\lambda, \rho) \in \Omega(S)$)

ii) $\lambda \circ \rho = \rho \circ \lambda$ (*i.e.* λ commutes with ρ)

iii) $\lambda(x) = \lambda(\lambda(x))$ and $\rho(x) = \rho(\rho(x))$ for all $x \in S$

iv) $u \leq \lambda(u)$ and $u \leq \rho(u)$ for all $u \in U$

v) U is unitary in $(\lambda(\rho(S))]_{\leq}^{S}$. □

Remark 4.5: Evidently if U is strongly almost unitary in S, then U is almost unitary in S (see Definitions 4.4, 3.2). □

From Theorem 4.2, Remark 4.3 and Definitions 3.2, 4.4 we immediately have the following

Theorem 4.6: Let $(S,\cdot,\leq)$ be an ordered semigroup, U be a nonempty subset of S and $e \notin S$. Also let $(S^e,*,\preceq)$ be an ordered semigroup with the following properties:

i) $\alpha \cdot b = \alpha * b$, $\alpha, b \in S$

ii) $e * S \subseteq S$, $S * e \subseteq S$

iii) $e = e * e$

iv) $\leq = \preceq \cap (S \times S)$

v) $u \preceq e * u$ and $u \preceq u * e$ for every $u \in U^e$

vi) U^e is unitary in $(e * S^e * e]_{\preceq}^{S^e}$ (equivalently: U^e is almost unitary in S^e).

Then U is strongly almost unitary in S. □

The reverse of the previous Theorem also holds:

Theorem 4.7: Let $(S,\cdot,\le)$ be an ordered semigroup, U be a nonempty subset of S such that U is strongly almost unitary in S and $e\notin S$. Then we can define an operation "$*$" and an order relation "$\preceq$" on S^e with the following properties:

i) $\alpha\cdot b=\alpha*b$, $\alpha,b\in S$

ii) $e*S\subseteq S$, $S*e\subseteq S$

iii) $e=e*e$

iv) $\le=\preceq\cap(S\times S)$

v) $(S^e,*,\preceq)$ is an ordered semigroup

vi) $u\preceq e*u$ and $u\preceq u*e$ for every $u\in U$

vii) U^e is unitary in $(e*S^e*e]_{\preceq}^{S^e}$.

Proof: Since U is strongly almost unitary in S then there exist mappings $\lambda,\rho:S\to S$ satisfying the properties of Definition 4.4. On S^e we define

$$\alpha*b:=\begin{cases}\alpha\cdot b, & \alpha,b\in S\\ \rho(\alpha), & \alpha\in S,b=e\\ \lambda(b), & \alpha=e,b\in S\\ e, & \alpha=b=e\end{cases}\quad\text{and}\quad \preceq:=\le\cup\{(e,e)\}$$

Then $(S^e,*)$ is a semigroup (§9.4). Also clearly "$\preceq$" is an order relation on S.

We shall prove that "$\preceq$" is compatible with the operation "$*$":

Let $\alpha,b,c\in S^e$ such that $\alpha\preceq b$. We shall show that $\alpha*c\preceq b*c$ and $c*\alpha\preceq c*b$. Since $\alpha\preceq b$ and $\preceq:=\le\cup\{(e,e)\}$ we distinguish the cases $\alpha\le b$ or $\alpha=b=e$.

A) $\underline{\alpha\le b}$

Then obviously $\alpha,b\in S$. For $c\in S^e$ we have the cases $c\in S$ or $c=e$.

A1) $\underline{c\in S}$

Since $\alpha,b,c\in S$ then $\alpha*c=\alpha\cdot c$ and $b*c=b\cdot c$. Hence

$$\alpha\le b\Rightarrow\alpha\cdot c\le b\cdot c\Rightarrow\alpha*c\le b*c\Rightarrow\alpha*c\preceq b*c$$

A2) $\underline{c=e}$

Since $\alpha,b\in S$ then $\alpha*c=\rho(\alpha)$ and $b*c=\rho(b)$. Hence

$$\alpha\le b\Rightarrow\rho(\alpha)\le\rho(b)\Rightarrow\alpha*c\le b*c\Rightarrow\alpha*c\preceq b*c$$

B) $\underline{\alpha=b=e}$

Since $c\in S^e$ we have the cases $c\in S$ or $c=e$.

B1) $\underline{c\in S}$

Obviously $\alpha*c=\lambda(c)$ and $b*c=\lambda(c)$. Thus

$$\lambda(c)\le\lambda(c)\Rightarrow\alpha*c\le b*c\Rightarrow\alpha*c\preceq b*c$$

B2) $\underline{c=e}$

Since $\alpha=b=c=e$ then $\alpha*c=e$ and $b*c=e$. Thus

$$e\preceq e\Rightarrow\alpha*c\preceq b*c$$

From A), B) we immediately have $\alpha * c \preceq b * c$. Similarly we show that $c * \alpha \preceq c * b$. Therefore $(S^e, *, \preceq)$ is an ordered semigroup. It is evident that the required conditions i) - v) are true. For the condition vi), take $u \in U$. We shall prove that $u \preceq e * u$ and $u \preceq u * e$. By definition of "$*$" we have $e * u = \lambda(u)$. Since U is strongly almost unitary in S, it follows by Definition 4.4 iv) that $u \leq \lambda(u)$ and hence $u \leq e * u$. From iv) we immediately have $u \preceq e * u$. Similarly we show that $u \preceq u * e$ and thus the condition vi) holds. Now it remains to be shown that the condition vii) is also true.

I) Obviously $e * S^e * e$ is a subsemigroup of S^e and so $(e * S * e]_{\preceq}^{S^e}$ is also a subsemigroup of S^e.

II) Since clearly $\lambda(\rho(x)) = e * x * e$ for all $x \in S$ then $\lambda(\rho(S)) = e * S * e$. We shall show that $(\lambda(\rho(S))]_{\leq}^{S} = (e * S * e]_{\preceq}^{S^e}$:

Let $\alpha \in (\lambda(\rho(S))]_{\leq}^{S}$. Then $\alpha \leq \lambda(\rho(b))$, $b \in S$. Since $\lambda(\rho(b)) \in e * S * e$ and, from the definition of "$\preceq$", $\alpha \preceq \lambda(\rho(b))$, then it follows immediately that $\alpha \in (e * S * e]_{\preceq}^{S^e}$. Now take $c \in (e * S * e]_{\preceq}^{S^e}$. Then $c \preceq e * t * e$, $t \in S$. Since $e * t * e \in \lambda(\rho(S))$ and, from the definition of "$\preceq$", $c \leq e * t * e$, then we have $c \in (\lambda(\rho(S))]_{\leq}^{S}$.

III) $U^e \subseteq (e * S^e * e]_{\preceq}^{S^e}$:

First we shall show that $U \subseteq (e * S * e]_{\preceq}^{S^e}$. Let $u \in U$. Then

$$u \underset{\text{vi)}}{\preceq} e * u \underset{\text{vi)}}{\preceq} e * u * e$$

that is, $u \in (e * S * e]_{\preceq}^{S^e}$. Thus $U \subseteq (e * S * e]_{\preceq}^{S^e}$ and since clearly $e = e * e * e \in e * S^e * e$ then it is readily verified that $U^e \subseteq (e * S^e * e]_{\preceq}^{S^e}$.

IV) $[U^e)_{\preceq}^{S^e} \cap (e * S^e * e]_{\preceq}^{S^e} \subseteq (U^e]_{\preceq}^{S^e}$:

Let $\alpha \in [U^e)_{\preceq}^{S^e} \cap (e * S^e * e]_{\preceq}^{S^e}$. Then, since $\alpha \in [U^e)_{\preceq}^{S^e}$, there exists $u \in U^e$ such that $u \preceq \alpha$. Since $u \in U^e = U \cup \{e\}$, we have two cases:

IV1) $\underline{u = e}$

Then $e \preceq \alpha$ and so, by definition of "$\preceq$", we have $\alpha = e$. It is evident now that $\alpha \in (U^e]_{\preceq}^{S^e}$.

IV2) $\underline{u \in U}$

Since $u \in S$ and $u \preceq \alpha$, then, by definition of "$\preceq$", it follows that $\alpha \in S$ and $u \leq \alpha$. Thus $\alpha \in [U)_{\leq}^{S}$. Also, since $\alpha \in (e * S^e * e]_{\preceq}^{S^e}$, there exists $t \in S^e$ such that $\alpha \preceq e * t * e$. But $\alpha \in S$ and so, by definition of "$\preceq$", we have $e * t * e \in S$ and $\alpha \leq e * t * e$. Since $e * t * e \in S$ then, by definition of "$*$", we clearly have $t \in S$ and so, by II), $e * t * e \in \lambda(\rho(S))$. Consequently $\alpha \in (\lambda(\rho(S))]_{\leq}^{S}$ and hence $\alpha \in [U)_{\leq}^{S} \cap (\lambda(\rho(S))]_{\leq}^{S}$. By Definition 4.4 v), U is unitary in $(\lambda(\rho(S))]_{\leq}^{S}$ and so, by Proposition 2.5 i) $\Rightarrow$ ii)α), we have $[U)_{\leq}^{S} \cap (\lambda(\rho(S))]_{\leq}^{S} \subseteq (U]_{\leq}^{S}$. Thus $\alpha \in (U]_{\leq}^{S}$ whence $\alpha \leq w$ for some $w \in U$. Since, by definition of "$\preceq$", $\alpha \preceq w$ then it clear that $\alpha \in (U^e]_{\preceq}^{S^e}$.

V) Let $w,u \in U^e$ and $t \in \left(e * S^e * e\right]_{\preceq}^{S^e}$ such that $w \preceq u * t$. We shall prove that $t \preceq \upsilon$, $\upsilon \in U^e$:

Since $t \in \left(e * S^e * e\right]_{\preceq}^{S^e}$ then $t \in S^e$ and $t \preceq e * s * e$ for some $s \in S^e$. If $t = e$ then obviously $t \preceq e \in U^e$. Suppose $t \neq e$, *i.e.* $t \in S$. Since $t \preceq e * s * e$, then, by definition of "$\preceq$", we have $e * s * e \in S$ and $t \leq e * s * e$. By definition of "$*$", since $e * s * e \in S$, it follows that $s \in S$. Also, since $w \in U^e = U \cup \{e\}$ then $w = e$ or $w \in U$. If $w = e$, then $e \preceq u * t$ and so, by definition of "$\preceq$", we immediately have $u * t = e$ which, by definition of "$*$", means that $t = e$. This is impossible and so $w \in U$. Since $u \in U^e = U \cup \{e\}$ we have the following two cases:

V1) $\underline{u = e}$

Then $w \preceq e * t$. Also

$$e * t \preceq e * e * s * e \underset{\text{iii)}}{=} e * s * e \Rightarrow w \preceq e * s * e \Rightarrow e * s * e \in \left[U^e\right)_{\preceq}^{S^e} \cap \left(e * S^e * e\right]_{\preceq}^{S^e}$$

Then, from IV), we immediately have $e * s * e \in \left(U^e\right]_{\preceq}^{S^e}$ and hence $e * s * e \preceq v$ for some $v \in U^e$. But $t \preceq e * s * e$ and so $t \preceq v \in U^e$.

V2) $\underline{u \in U}$

Since $s \in S$ then $e * s * e \in e * S * e \underset{\text{II)}}{=} \lambda\left(\rho(S)\right)$ and thus $t \in \left(\lambda\left(\rho(S)\right)\right]_{\leq}^{S}$. Also since $w,u,t \in S$ and $w \preceq u * t$ then, by i) and definition of "$\preceq$", we have $w \leq u \cdot t$. Therefore we have $w,u \in U$ and $t \in \left(\lambda\left(\rho(S)\right)\right]_{\leq}^{S}$ such that $w \leq u \cdot t$. But, by Definition 4.4 v), U is unitary in $\left(\lambda\left(\rho(S)\right)\right]_{\leq}^{S}$ and so (see Proposition 2.5 i) $\Rightarrow$ ii)β)) there exists $v \in U$ such that $t \leq v$ which means (see the definition of "$\preceq$") that $t \preceq v \in U^e$.

From I), III) - V) and Proposition 2.4 ii) $\Rightarrow$ i) it follows directly that U^e is a left unitary subset of $\left(e * S^e * e\right]_{\preceq}^{S^e}$. Similarly we show that U^e is a right unitary subset of $\left(e * S^e * e\right]_{\preceq}^{S^e}$. Therefore (see Definition 2.1 iii)) U^e is a unitary subset of $\left(e * S^e * e\right]_{\preceq}^{S^e}$ and so the condition vii) of the Theorem holds. This completes the proof.

□

Writing down Theorems 4.7, 4.6 together, we immediately obtain the following fundamental Theorem which actually summarizes the main results of the paper.

<u>Theorem 4.8:</u> Let $(S,\cdot,\leq)$ be an ordered semigroup, *U* be a nonempty subset of *S* and $e \notin S$. The following are equivalent:

I) *U* is strongly almost unitary in *S*.

II) We can define an operation "$*$" and an order relation "$\preceq$" on S^e with the following properties:

i) $\alpha \cdot b = \alpha * b$, $\alpha, b \in S$

ii) $e * S \subseteq S$, $S * e \subseteq S$

iii) $e = e * e$

iv) $\leq = \preceq \cap (S \times S)$

v) $\left(S^e, *, \preceq\right)$ is an ordered semigroup

vi) $u \preceq e * u$ and $u \preceq u * e$ for every $u \in U$

vii) U^e is unitary in $\left(e * S^e * e\right]_{\preceq}^{S^e}$ (equivalently: U^e is almost unitary in S^e).

□

The results presented in the paper generalize the analogous ones of semigroups without order (cf. [2] §VIII.3 and [3] §9.4) because we can consider that every semigroup without order is an ordered semigroup with order relation being the equality relation. In general, researchers in ordered semigroups should arrive at results that hold in the case of semigroups without order by considering them as ordered semigroups in the previous sense and will therefore have more general results. Note that in Section 1 we mention a technique that can be applied by any researcher studying the topic of the paper.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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