

The New Mixed Generalized Erlang Distribution

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Abstract

In probability theory, the mixture distribution M has a density function $f_M(t) = \sum_{i \in I} w_i f_{X_i}(t)$ for the collection of random variables $X_i, i \in I \subset \mathbb{N}$ and weighted by $w_i \geq 0$ and $\sum_{i \in I} w_i = 1$. These mixed distributions are used in various disciplines and aim to enrich the collection distribution to more parameters. A more general mixture is derived by Kadri and Halat, by proving the existence of such mixture by $w_i \in \mathbb{R}$, and maintaining $\sum_{i \in I} w_i = 1$. Kadri and Halat provided many examples and applications for such new mixed distributions. In this paper, we introduce a new mixed distribution of the Generalized Erlang distribution, which is derived from the Hypoexponential distribution. We characterize this new distribution by deriving simply closed expressions for the related functions of the probability density function, cumulative distribution function, moment generating function, reliability function, hazard function, and moments.

Keywords

Generalized Erlang Distribution, Mixed Distribution, Probability Density Function, Cumulative Distribution Function, Moment Generating Function, Hazard Rate Function, Reliability Function, Moment of Order k

1. Introduction

Mixtures of distributions have been widely used for modeling observed situations whose various characteristics as reflected by the data differ from those that would be anticipated under the simple component distribution. More general families of distributions, well known as mixtures, are usually considered alternative models that offer more flexibility. These are superimpositions of simpler component distributions depending on a parameter, itself being a random variable with some distribution. Mixed Poisson distributions, in particular, have been widely used as claim frequency distributions.

In probability theory, the mixture distribution M has a density function:

$$f_M(t) = \sum_{i \in I} w_i f_{X_i}(t)$$

for the collection of random variables $X_i, i \in I \subset N$ and weighted by $w_i \geq 0$ and $\sum_{i \in I} w_i = 1$. These mixed distributions are used in various disciplines and aim to enrich the collection distribution to more parameters. The basic definitions and properties of continuous random variables with their characteristics that are often used in order to introduce the Mixed distribution presents in [1] and [2]. A more general mixture is derived by Kadri and Halat [3], by proving the existence of such mixture by $w_i \in \mathbb{R}$, and maintaining $\sum_{i \in I} w_i = 1$. Kadri and Amina provided many examples and applications for such new mixed distributions.

The Hypoexponential distribution with different parameters is one of the distributions considered to be a new mixed distribution. Recall that the probability density function of this case $S_n \sim Hypoexp(\alpha)$ where $\alpha \in \mathbb{R}_+^n$ given by Smaili *et al.* [4], as:

$$f_{S_n}(t) = \sum_{i=1}^n \frac{f_{X_i}(t)}{P_i}, t > 0 \tag{1.1}$$

where $X_i \sim Exp(\alpha_i)$ and $P_i = \prod_{j=1, j \neq i}^n \left(1 - \frac{\alpha_i}{\alpha_j}\right)$, $i = 1, 2, \dots, n$. They showed that $\sum_{i=1}^n \frac{1}{P_i} = 1$ having $\frac{1}{P_i} \in \mathbb{R}$. So this staple $Hypoexp(\vec{\alpha})$ to the family of the new defined mixed distributions.

Follow with the work of Kadri and Halat [3], a construction of “New Mixed T-G Distribution” having T to be a new mixed distribution, like the Hypoexponential distribution, and G is function random variable of the parent distribution that exist in the summand of the PDF of T , say X_i . To illustrate, considering “ T ” the Hypoexponential distribution, with PDF presented in 1.1, G will be any function g of the random variable $X_i \sim Exp(\alpha_i)$. The generated distribution will have an expression as:

$$f_{T-G}(t) = \sum_{i=1}^n G_{ij} f_{g(X_i)}(t)$$

Thus many distributions were generated as the Mixed Hypoexponential Weibull distribution, Mixed Hypoexponential Frechet distribution, Mixed Hypoexponential Pareto distribution, Mixed Hypoexponential Power distribution, Mixed Hypoexponential Gumbel distribution, and Mixed Hypoexponential Extreme Value distribution.

Also, we add the importance of generating this new distribution, that these newly generated distributions preserve the nature of being also a mixed distribution. To emphasize in deep the importance, we present it in the following Theorem.

Regarding this special form of pdf, we are able to generalize the cdf, the reliability and hazard rate functions, mgf, and moment of each distribution having

the Generalized Mixed Distribution that is stated in the following Theorem given by Kadri and Halat [3].

Theorem 1. Let N follow a new mixed distribution with pdf then $f_N(t) = \sum_{i \in I} A_i f_{X_i}(t)$ for some minor distribution X_i . The cdf, the reliability, hazard rate functions, mgf, and moments, respectively are:

$$\begin{aligned} F_N(t) &= \sum_{i \in I} A_i F_{X_i}(t) \\ R_N(t) &= \sum_{i \in I} A_i R_{X_i}(t) \\ h_N(t) &= \frac{\sum_{i \in I} A_i f_{X_i}(t)}{\sum_{i \in I} A_i R_{X_i}(t)} \\ \Phi_N(t) &= \sum_{i \in I} A_i \Phi_{X_i}(t) \\ E[N^l] &= \sum_{i \in I} A_i E[X_i^l] \end{aligned}$$

We add to that the coefficients of these sum, $A_i \in \mathbb{R}$ having the property that $\sum_{i \in I} A_i = 1$. It is important of determining a closed simple expression of this coefficient to characterize the new mixed distribution N .

One of the important distributions which is one of the mixed distribution is the Hypoexponential distribution which is considered to be the general case when the parameters are not different. Referring to the work done by Kadri *et al.* [4], if $S_m \sim \text{Hypoexp}(\alpha, \mathbf{k})$ where $\alpha \in \mathbb{R}_+^n$ and $\mathbf{k} \in \mathbb{N}^n$ having $m = \sum_{i=1}^n k_i$ the probability density function is given by

$$f_{S_m}(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} f_{X_{ij}}(t) \quad (1.2)$$

where $X_{ij} \sim \text{Erl}(j, \alpha_i)$ and A_{ij} is the coefficients of the Hypoexponential distribution, where A_{ij} no need to be a positive real number. Thus we imply that:

$$\begin{aligned} F_{S_m}(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} F_{X_{ij}}(t) \\ R_{S_m}(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} R_{X_{ij}}(t) \\ h_{S_m}(t) &= \frac{\sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} f_{X_{ij}}(t)}{\sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} R_{X_{ij}}(t)} \\ \Phi_{S_m}(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} \Phi_{X_{ij}}(t) \\ E[S_m^l] &= \sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} E[X_{ij}^l] \end{aligned}$$

with $\sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} = 1$.

2. The New Mixed Generalized Erlang Distribution

The expression of the probability density function in (1.2) is a generalized mixed

distribution, which opens a type of T-G family having T the general case of Hypoexponential distribution and G is a generated distribution from the minor distribution where the PDF of the Hypoexponential distribution is formed referring to the expression (1.2) this is the Erlang distribution $X_{ij} \sim \text{Erl}(j, \alpha_i)$. So any function of the random variable of X_{ij} will be considered as the G. One of these functions is the Generalized Erlang distribution.

Let $X \sim \text{Erl}(k, \alpha)$ be an Erlang distribution. Whereas, the transformation $G = X^{\frac{1}{c}}, c > 1$ follows the Generalized Erlang with parameters k, β, c that can be written as $G \sim \text{GErl}(k, \beta, c)$ where $\beta^c = \alpha$. The Generalized Erlang distribution is one of the important distributions used in several fields such as queuing theory, telegraphic engineering and stochastic process. The Generalized Erlang random variables has important application and it is well known referring to queuing waiting time context in telecommunication traffic engineering and others, see [5].

In the following proposition, we present the characteristics found for the Generalized Erlang, see [6].

Proposition 1. Let $G \sim \text{GErl}(k, \beta, c)$, $k \in \mathbb{Z}$, $\beta > 0$ and $c > 1$. The expressions of PDF, CDF, Moment of order l and MGF, is given as:

$$f_G(t) = \frac{c e^{-\left(\frac{t}{\beta}\right)^c} t^{-1+ck}}{\beta^{ck} (k-1)!}$$

$$F_G(t) = 1 - \frac{\Gamma\left(k, \frac{t^c}{\beta^c}\right)}{\Gamma(k)}$$

$$R_G(t) = \frac{\Gamma\left(k, \frac{t^c}{\beta^c}\right)}{\Gamma(k)}$$

$$E[G^l] = \frac{\beta^l \Gamma\left(k + \frac{l}{c}\right)}{\Gamma(k)}$$

$$\Phi_G(t) = \sum_{n=0}^{\infty} \frac{(t\beta)^n \Gamma\left(k + \frac{n}{c}\right)}{n! \Gamma(k)}$$

where $\Gamma(a, z)$ is the upper incomplete Gamma function, see [7].

After adopting the general case of the Hypoexponential distribution with PDF

$$f_{S_m}(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} A_{ij} f_{X_{ij}}(t)$$

and by taking $G = G_{ij} = X_{ij}^{\frac{1}{c}}$, where $X_{ij} \sim \text{Erl}(j, \alpha_i)$ then $G_{ij} \sim \text{GErl}(j, \beta_i, c)$ where $\beta_i^c = \alpha_i$, $c > 1$ we generate a new mixed distribution A named the New Mixed Generalized Erlang distribution. Thus the PDF of A is:

$$f_A(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t) \quad (2.1)$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}_+^n$, $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$, we denote where $\beta_i^c = \alpha_i$, $c > 1$. We emphasize that A is a generalized mixed distribution and according to the discussion done by Kadri and Halat [3], the distribution can be easily characterized. Referring to Theorem 1, we conclude for our new distribution $A \sim NMGE(\mathbf{k}, \boldsymbol{\beta}, c)$, the cdf, reliability, hazard and mgf are:

$$\begin{aligned} F_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} F_{G_{ij}}(t) \\ R_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} R_{G_{ij}}(t) \\ h_A(t) &= \frac{\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t)}{\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} R_{G_{ij}}(t)} \\ \Phi_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \Phi_{G_{ij}}(t) \\ E[A^l] &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} E[G_{ij}^l] \end{aligned}$$

and eventually, we have $\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} = 1$. Also we may see these expressions more detailed when we insert the expression of the generalized Erlang distribution G_{ij} given in Proposition 1. Thus we lead to the following expressions:

$$\begin{aligned} f_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{c e^{-\left(\frac{t}{\beta}\right)^c} t^{-1+cj}}{\beta^{cj} (j-1)!} \\ R_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{\Gamma\left(j, \frac{t^c}{\beta^c}\right)}{\Gamma(j)} \\ h_A(t) &= \frac{\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{c e^{-\left(\frac{t}{\beta}\right)^c} t^{-1+cj}}{\beta^{cj} (j-1)!}}{\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{\Gamma\left(j, \frac{t^c}{\beta^c}\right)}{\Gamma(j)}} \\ \Phi_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} \sum_{n=0}^{\infty} Y_{ij} \frac{(t\beta)^n \Gamma\left(j + \frac{n}{c}\right)}{n! \Gamma(j)} \\ E[A^l] &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{\beta^l \Gamma\left(j + \frac{l}{c}\right)}{\Gamma(j)} \end{aligned}$$

$$F_A(t) = 1 - \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{\Gamma\left(j, \frac{t^c}{\beta^c}\right)}{\Gamma(j)}$$

We note out to characterize our new distribution the New Mixed Generalized Erlang distribution, we are left by determining the coefficient Y_{ij} that appears in the expressions. So we aim to find a method to compute Y_{ij} in order to find the pdf, cdf, mgf, moment order l , hazard rate function and reliability function. Different approaches were proposed to determine these coefficients, like recursive algorithms and other by solving a linear matrix equation, see [4] [5] [6] [7] [8].

3. The Matrix Form for Finding PDF of New Mixed Generalized Erlang Distribution

The purpose of this section is to find a linear algebra approach method for finding the coefficients Y_{ij} . The method aims to determine m linear equations for Y_{ij} . Our first equation is given in the previous section having $\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} = 1$.

Next, we find the other $m - 1$ linear equations for Y_{ij} . We start by the following lemma.

Lemma 1. Let $m \geq 2$, $c > 1$, $\mathbf{k} = (k_1, k_2, \dots, k_n)$, and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, where $m = \sum_{i=1}^n k_i$. Then the u^{th} derivative of the PDF of $A \sim NMGE(\mathbf{k}, \boldsymbol{\beta}, c)$ at zero is given by:

$$f_A^u(0) = \begin{cases} 0 & \text{if } 0 \leq u \leq cm - 2 \\ \text{not zero} & \text{if } u = cm - 1 \end{cases}$$

Proof. Starting from the definition of the Hypoexponential distribution $S_m \sim \text{Hypoexp}(\boldsymbol{\alpha}, \mathbf{k})$ which is the sum independent Erlang distribution $E_i \sim E(\alpha_i, k_i)$ which means that the PDF of S_m is the convolution of the PDF of the E_i . Thus:

$$f_{S_n}(t) = f_{E_1} * f_{E_2} * \dots * f_{E_n}(t)$$

then the Laplace transform of $f_{S_n}(t)$ is:

$$\mathcal{L}\{f_{S_n}(t)\}(s) = \prod_{i=1}^n \mathcal{L}\{f_{E_i}(t)\}(s) = \prod_{i=1}^n \left(\frac{1}{1 + s\alpha_i} \right)^{k_i}$$

and $\lim_{s \rightarrow +\infty} \mathcal{L}\{f_{S_n}(t)\}$ is proportional to $\frac{1}{s^{\sum_{i=1}^n k_i}} = \frac{1}{s^m}$. Now, our new distribu-

tion $A \sim NMGE(\mathbf{k}, \boldsymbol{\beta}, c)$ is a transform function of the random variable by $E_i^{\frac{1}{c}}$, and we conclude that the behaviour of the Laplace transform of the PDF of A at infinity is proportional to $\frac{1}{s^{cm}}$ or we may say $\mathcal{L}\{f_A(t)\} \sim \propto \frac{1}{s^{cm}}$. The proof of our result is done by induction. For $u = 0$, and by Initial Value Theorem, we have:

$$f_A(0^+) = \lim_{t \rightarrow 0} f_A(t) = \lim_{s \rightarrow +\infty} s \mathcal{L}\{f_A(t)\} = \lim_{s \rightarrow +\infty} s \left(\propto \frac{1}{s^{cm}} \right) = 0$$

Next step for $u = 1$. We have $L\{f'_A(t)\} = sL\{f_A(t)\} - f_A(0^+) = sL\{f_A(t)\}$.

Again:

$$f_A(0^+) = \lim_{s \rightarrow +\infty} sL\{f'_A(t)\} = \lim_{s \rightarrow +\infty} s^2L\{f_A(t)\} = \infty \lim_{s \rightarrow +\infty} \frac{1}{s^{cm-2}} = 0$$

Next step for $u = 2$. We have $L\{f''_A(t)\} = sL\{f'_A(t)\} - f'_A(0^+) = sL\{f'_A(t)\}$.

Again:

$$f''_A(0^+) = \lim_{s \rightarrow +\infty} sL\{f''_A(t)\} = \lim_{s \rightarrow +\infty} s^3L\{f_A(t)\} = \infty \lim_{s \rightarrow +\infty} \frac{1}{s^{cm-3}} = 0$$

Then continuing in the same manner knowing that

$L\{f_A^{(i)}(t)\} = sL\{f_A^{(i-1)}(t)\} - f_A^{(i-1)}(0^+)$, till reaching the derivative $u = cm - 2$, we get:

$$f_A^{(cm-2)}(0^+) = \lim_{s \rightarrow +\infty} sL\{f_A^{(cm-3)}(t)\} = \lim_{s \rightarrow +\infty} s^{cm-1}L\{f_A(t)\} = \infty \lim_{s \rightarrow +\infty} \frac{s^{cm-1}}{s^{cm}} = 0$$

also we may conclude that:

$$f_A^{(cm-1)}(0^+) = \lim_{s \rightarrow +\infty} sL\{f_A^{(cm-2)}(t)\} = \lim_{s \rightarrow +\infty} s^{cm}L\{f_A(t)\} = \infty \lim_{s \rightarrow +\infty} \frac{s^{cm}}{s^{cm}} = \infty$$

till reaching the derivative $u = cm - 1$.

$$f_A^{(cm-1)}(0^+) = \lim_{s \rightarrow +\infty} sL\{f_A^{(cm-2)}(t)\} = \lim_{s \rightarrow +\infty} s^{cm}L\{f_A(t)\} = \infty \lim_{s \rightarrow +\infty} \frac{s^{cm}}{s^{cm}} = \infty$$

□

In the next definition, we define the YARGHANN Matrix (GH), that is used further to determine the vector $Y = (Y_{ij})_{m \times 1}$. The definition is related to $f_{G_{ij}}^{(u)}(0)$, the u^{th} derivative of the PDF of Generalized Erlang distribution

$G_{ij} \sim GErl(j, \beta_i, c)$ at 0. For simplicity, we write $f_{G_{ij}}^{(u)}(0) = f_{i,j}^{(u)}(0)$.

Next, we find the expression of the u^{th} derivative of $f_{i,j}^{(u)}(0)$.

Proposition 2. Let $G \sim GErl(k, \beta, c)$, and $l \in \mathbb{N}^*$. Then:

$$f_G^{(cl-1)}(0) = \begin{cases} \frac{(-1)^{l-k} c (cl-1)!}{(k-1)!(l-k)! \beta^{cl}} & \text{if } 1 \leq k \leq l \\ 0 & \text{if } k > l \end{cases}$$

Proof. We have the PDF of $G \sim GErl(k, \beta, c)$ is $f_G(x) = \frac{c e^{-\left(\frac{x}{\beta}\right)^c} x^{-1+ck}}{\beta^{ck} (k-1)!}$. But

the series of $e^{-\left(\frac{x}{\beta}\right)^c}$ is $e^{-\left(\frac{x}{\beta}\right)^c} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{\beta}\right)^{cn}$. Then:

$$f_G(x) = \frac{c}{\beta^{ck} (k-1)!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{\beta}\right)^{cn+ck-1}$$

Then:

$$f_G^{(u)}(x) = \frac{c(u!)}{(k-1)!} \sum_{n=u}^{\infty} \frac{(-1)^n}{\beta^{cn+ck} n!} (x)^{cn+ck-(u+1)}$$

for $cn + ck - (u + 1) = 0$, which $cn + ck - (u + 1) = 0$, or $c(n + k) = u + 1$, and to have all integers we must have $c \vee u + 1$ and thus $u + 1 = cl$ where $l = n + k < u + 1$. Therefore, for this case:

$$f_G^{(u)}(0) = \frac{u!c(-1)^n}{(k-1)!\beta^{c(n+k)}n!} = \frac{u!c(-1)^{l-k}}{(k-1)!\beta^{cl}(l-k)!}$$

However, the case $cn + ck - (u + 1) > 0$, $f_G^{(cl-1)}(0) = 0$. In this case $cn + ck - cl > 0$ or $n + k - l > 0$ having $n = 0, 1, 2, \dots$, gives $l - k < 0$. □

Definition 1. Let $f_{i,j}^{(u)}(0)$ be the u^{th} derivative of the PDF of $G_{ij} \sim \text{G Erl}(j, \beta_i, c)$ at 0, where $1 \leq i \leq n$ and $1 \leq j \leq k_i$. We define the GH matrix for $\mathbf{k} = (k_1, k_2, \dots, k_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, $u = cl - 1$ for $l = 1, \dots, m - 1$, and $m = \sum_{i=1}^n k_i$ of size $m \times m$ as:

$$GH = \begin{bmatrix} 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ f_{1,1}^{c-1}(0) & \dots & f_{1,k_1}^{c-1}(0) & \dots & f_{n,1}^{c-1}(0) & \dots & f_{n,k_n}^{c-1}(0) \\ f_{1,1}^{2c-1}(0) & \dots & f_{1,k_1}^{2c-1}(0) & \dots & f_{n,1}^{2c-1}(0) & \dots & f_{n,k_n}^{2c-1}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1,1}^{cl-1}(0) & \dots & f_{1,k_1}^{cl-1}(0) & \dots & f_{n,1}^{cl-1}(0) & \dots & f_{n,k_n}^{cl-1}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1,1}^{c(m-1)-1}(0) & \dots & f_{1,k_1}^{c(m-1)-1}(0) & \dots & f_{n,1}^{c(m-1)-1}(0) & \dots & f_{n,k_n}^{c(m-1)-1}(0) \end{bmatrix} \quad (3.1)$$

Next, the linear matrix equation for Y_{ij} is formed, having the GH matrix, its coefficient matrix.

Theorem 2. Let $f_A(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t)$, where $G_{ij} \sim \text{G Erl}(j, \beta_i, c)$, and $Y = (Y_{ij})_{I_m}$, $I_m = \{(i, j) / 1 \leq i \leq n, 1 \leq j \leq k_i \text{ and } m = \sum_{i=1}^n k_i, \mathbf{k} = (k_1, k_2, \dots, k_n), \boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)\}$. Then $(GH)Y = e_1^T$ where $e_1 = (1 \ 0 \ \dots \ 0)$.

Proof. We have previously started with our first equation from the definition of the mixed distribution that, $\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} = 1$, the first linear equation for Y_{ij} . To find the other equations we take the PDF of, $f_A(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t)$. Then the u^{th} derivative of $f_A(t)$ is $f_A^{(u)}(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}^{(u)}(t)$. But from Proposition 1, we have $f_A^{(u)}(0) = 0$, whenever $0 \leq u \leq cm - 2$. We conclude that $f_A^{(u)}(0) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}^{(u)}(0) = 0$ for $0 \leq u \leq cm - 2$. To not considering any trivial equation, we have from Proposition 2, $f_G^{(cl-1)}(0) \neq 0$, when $1 \leq k_i \leq l$, $l \in \mathbb{N}^*$, so by taking $l = 1, \dots, m - 1$, we get $u = c - 1, \dots, c(m - 1) - 2 < cm - 2$. So $f_A^{(cl-1)}(0) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}^{(cl-1)}(0) = 0$, $l = 1, 2, \dots, m - 1$, and these will be the rest $m - 1$ equations. Thus we obtain m equations that are equivalent to the matrix equation $(GH)Y = e_1^T$ where GH is the $m \times m$ GH matrix defined in Definition 1 and $e_1 = (1 \ 0 \ \dots \ 0)$. □

Next, we illustrate in two special cases of this distribution two $NMGE(\mathbf{k}, \boldsymbol{\beta}, 2)$ and $NMGE(\mathbf{k}, \boldsymbol{\beta}, 3)$.

Corollary 1. Let $A \sim NMGE(\mathbf{k}, \boldsymbol{\beta}, 2)$ and $G_{ij} \sim \text{G Erl}(j, \beta_i, 2)$, $1 \leq i \leq n$ and $1 \leq j \leq k_i$ and $m = \sum_{i=1}^n k_i$. where $u = 2r - 1$ for $r = 1, \dots, m - 1$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, $\mathbf{k} = (k_1, k_2, \dots, k_n)$. Then we define the GH matrix of size $m \times m$ as:

$$GH = \begin{bmatrix} 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ f'_{1,1}(0) & \dots & f'_{1,k_1}(0) & \dots & f'_{n,1}(0) & \dots & f'_{n,k_n}(0) \\ f^{(3)}_{1,1}(0) & \dots & f^{(3)}_{1,k_1}(0) & \dots & f^{(3)}_{n,1}(0) & \dots & f^{(3)}_{n,k_n}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f^{(2r-1)}_{1,1}(0) & \dots & f^{(2r-1)}_{1,k_1}(0) & \dots & f^{(2r-1)}_{n,1}(0) & \dots & f^{(2r-1)}_{n,k_n}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f^{(2m-3)}_{1,1}(0) & \dots & f^{(2m-3)}_{1,k_1}(0) & \dots & f^{(2m-3)}_{n,1}(0) & \dots & f^{(2m-3)}_{n,k_n}(0) \end{bmatrix}$$

Moreover, finding the GH matrix in a simpler way, an expression of the u^{th} derivative of PDF of Generalized Erlang Distribution is derived for $c = 2$ as:

$$f_G^{(cl-1)}(0) = \begin{cases} \frac{2(-1)^{l-j}(2l-1)!}{(j-1)!(l-j)!\beta_i^{2l}} & \text{if } j \leq l \\ 0 & \text{if } j > l \end{cases}$$

where $l = 1, 2, \dots, m-1$.

Corollary 2. Let $G_{ij} \sim GErl(j, \beta_i, 3)$, $m \geq 2$, $1 \leq i \leq n$, $1 \leq j \leq k_i$ and $m = \sum_{i=1}^n k_i$ where $u = 3r-1$, for $r = 1, \dots, m-1$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ and $k = (k_1, k_2, \dots, k_n)$. Then we define the GH matrix as:

$$GH = \begin{bmatrix} 1 & \dots & 1 & \dots & 1 & \dots & 1 \\ f''_{1,1}(0) & \dots & f''_{1,k_1}(0) & \dots & f''_{n,1}(0) & \dots & f''_{n,k_n}(0) \\ f^{(5)}_{1,1}(0) & \dots & f^{(5)}_{1,k_1}(0) & \dots & f^{(5)}_{n,1}(0) & \dots & f^{(5)}_{n,k_n}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f^{(3r-1)}_{1,1}(0) & \dots & f^{(3r-1)}_{1,k_1}(0) & \dots & f^{(3r-1)}_{n,1}(0) & \dots & f^{(3r-1)}_{n,k_n}(0) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f^{(3m-4)}_{1,1}(0) & \dots & f^{(3m-4)}_{1,k_1}(0) & \dots & f^{(3m-4)}_{n,1}(0) & \dots & f^{(3m-4)}_{n,k_n}(0) \end{bmatrix}$$

Moreover for finding GH in a simpler way, an expression of the u^{th} derivative of PDF Generalized Erlang Distribution is derived for $c = 3$ as:

$$f_{i,j}^{(3r-1)}(0) = f_{i,j}^{(cr+c-1)}(0) = \begin{cases} \frac{(-1)^{r-j} c(3r-1)!}{(j-1)!(r-j)!\beta_i^{cr}} & \text{if } j \leq r \\ 0 & \text{if } j > r \end{cases}$$

4. Application

In this section, we illustrate an example of our method in finding the PDF of the Generalized Erlang Distribution using the GH Matrix.

We suppose that $A \sim NMGE(k, \beta, c)$, and $G_{ij} \sim GErl(j, \beta_i, c)$, for $c > 1$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ and $k = (k_1, k_2, \dots, k_n)$, $m = \sum_{i=1}^n k_i$. Then the PDF of A is $f_A(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t)$ where $Y = (GH)^{-1} \cdot e_1^T$.

Example 1. We consider the distribution $A \sim NMGE(k, \beta, 2)$, with $\beta = (\sqrt{3}, \sqrt{4}, \sqrt{5})$ and let $k = (2, 3, 1)$, then $m = \sum_{i=1}^3 k_i = 6$. Then the PDF of A is $f_A(t) = \sum_{i=1}^3 \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t)$ where $Y = (Y_{11}, Y_{12}, Y_{21}, Y_{22}, Y_{23}, Y_{31})$. We have:

$$f_{G_{ij}}(t) = \frac{2e^{-\left(\frac{x}{\beta_i}\right)^2} x^{-1+2j}}{\beta_i^{2j}(j-1)!}, \text{ then } f_{G_{11}}(t) = \frac{2}{3}e^{-\frac{t^2}{3}}t, \quad f_{G_{12}}(t) = \frac{2}{9}e^{-\frac{t^2}{3}}t^3,$$

$$f_{G_{21}}(t) = \frac{1}{2}e^{-\frac{t^2}{4}}t, \quad f_{G_{22}}(t) = \frac{1}{8}e^{-\frac{t^2}{4}}t^3, \quad f_{G_{23}}(t) = \frac{1}{64}e^{-\frac{t^2}{4}}t^5, \quad f_{G_{31}}(t) = \frac{2}{5}e^{-\frac{t^2}{5}}t$$

To find the entries of the GH matrix we must use Corollary 1 to get GH as follows:

$$GH = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ f'_{1,1}(0) & f'_{1,2}(0) & f'_{2,1}(0) & f'_{2,2}(0) & f'_{2,3}(0) & f'_{3,1}(0) \\ f^{(3)}_{1,1}(0) & f^{(3)}_{1,2}(0) & f^{(3)}_{2,1}(0) & f^{(3)}_{2,2}(0) & f^{(3)}_{2,3}(0) & f^{(3)}_{3,1}(0) \\ f^{(5)}_{1,1}(0) & f^{(5)}_{1,2}(0) & f^{(5)}_{2,1}(0) & f^{(5)}_{2,2}(0) & f^{(5)}_{2,3}(0) & f^{(5)}_{3,1}(0) \\ f^{(7)}_{1,1}(0) & f^{(7)}_{1,2}(0) & f^{(7)}_{2,1}(0) & f^{(7)}_{2,2}(0) & f^{(7)}_{2,3}(0) & f^{(7)}_{3,1}(0) \\ f^{(9)}_{1,1}(0) & f^{(9)}_{1,2}(0) & f^{(9)}_{2,1}(0) & f^{(9)}_{2,2}(0) & f^{(9)}_{2,3}(0) & f^{(9)}_{3,1}(0) \end{bmatrix}$$

and by Corollary 1,

$$f_{i,j}^{(2r-1)}(0) = \begin{cases} \frac{(-1)^{r-j} 2(2r-1)!}{(j-1)!(r-j)! \beta_i^{2r}} & \text{if } 1 \leq j \leq r \\ 0 & \text{if } j > r \end{cases}$$

for $r = 1, 2, 3, 4, 5 = m - 1$,

For $r = 1$, if $1 \leq j \leq r$, $j = 1$, we conclude that:

$$f_{i,j}^{(1)}(0) f(x) = \begin{cases} \frac{2}{\beta_i^2}, & \text{if } j = 1 \\ 0, & \text{else} \end{cases}$$

and for $r = 2$, if $1 \leq j \leq r$, $j = 1, 2$, then:

$$f_{i,j}^{(2)}(0) = \begin{cases} -\frac{12}{\beta_i^4}, & \text{if } j = 1 \\ \frac{12}{\beta_i^4}, & \text{if } j = 2 \\ 0, & \text{else} \end{cases}$$

and for $r = 3$, if $1 \leq j \leq r$, $j = 1, 2, 3$, then:

$$f_{i,j}^{(3)}(0) = \begin{cases} \frac{120}{2\beta_i^6}, & \text{if } j = 1 \\ -\frac{240}{\beta_i^6}, & \text{if } j = 2 \\ \frac{120}{\beta_i^6}, & \text{if } j = 3 \\ 0, & \text{else} \end{cases}$$

and for $r = 4$, if $1 \leq j \leq r$, $j = 1, 2, 3$, then:

$$f_{i,j}^{(4)}(0) = \begin{cases} -\frac{1680}{\beta_i^8}, & \text{if } j=1 \\ -\frac{5040}{\beta_i^8}, & \text{if } j=2 \\ \frac{5040}{\beta_i^8}, & \text{if } j=3 \\ 0, & \text{else} \end{cases}$$

and for $r=5$, if $1 \leq j \leq r$, $j=1,2,3$, then:

$$f_{i,j}^{(5)}(0) = \begin{cases} \frac{30240}{\beta_i^{10}}, & \text{if } j=1 \\ -\frac{120960}{\beta_i^{10}}, & \text{if } j=2 \\ \frac{181440}{\beta_i^{10}}, & \text{if } j=3 \\ 0, & \text{else} \end{cases}$$

then we get:

$$GH = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{2}{3} & 0 & \frac{1}{2} & 0 & 0 & \frac{2}{5} \\ -\frac{4}{3} & \frac{4}{3} & -\frac{3}{4} & \frac{3}{4} & 0 & -\frac{12}{25} \\ \frac{40}{9} & -\frac{80}{9} & \frac{15}{8} & -\frac{15}{4} & \frac{15}{8} & \frac{24}{25} \\ -\frac{560}{27} & \frac{560}{9} & -\frac{105}{16} & \frac{315}{16} & -\frac{315}{16} & -\frac{336}{125} \\ \frac{1120}{9} & -\frac{4480}{9} & \frac{945}{32} & -\frac{945}{8} & \frac{2835}{16} & \frac{6048}{625} \end{bmatrix}$$

then the inverse of (GH) is given by:

$$(GH)^{-1} = \begin{bmatrix} \frac{2349}{4} & -\frac{23247}{4} & -\frac{121743}{16} & -\frac{118557}{40} & -\frac{57213}{140} & -\frac{243}{14} \\ \frac{81}{2} & -405 & -\frac{4293}{8} & -\frac{4239}{20} & -\frac{297}{10} & -\frac{9}{7} \\ -1408 & 13376 & 16864 & \frac{95216}{15} & \frac{29696}{35} & \frac{736}{21} \\ 64 & -736 & -\frac{3248}{3} & -\frac{1384}{3} & -\frac{2384}{35} & -\frac{64}{21} \\ -64 & 608 & \frac{2288}{3} & \frac{4264}{15} & \frac{1312}{35} & \frac{32}{21} \\ \frac{3125}{4} & -\frac{28125}{4} & -\frac{134375}{16} & -\frac{71875}{24} & -\frac{10625}{28} & -\frac{625}{42} \end{bmatrix}$$

By applying the Theorem 2, $Y = (GH)^{-1} \cdot e_1^T$, where $e_1 = (1 \ 0 \ \dots \ 0)$.
Next,

$$Y = (GH)^{-1} \cdot e_1^T = \begin{bmatrix} \frac{2349}{4} \\ \frac{81}{2} \\ -1408 \\ 64 \\ -64 \\ \frac{3125}{4} \end{bmatrix}.$$

Then we get $Y_{11} = \frac{2349}{4}$, $Y_{12} = \frac{81}{2}$, $Y_{21} = -1408$, $Y_{22} = 64$, $Y_{23} = -64$, $Y_{31} = \frac{3125}{4}$.

We insure that the sum of entries of Y is given by

$$Y_{11} + Y_{12} + Y_{21} + Y_{22} + Y_{23} + Y_{31} = 1 \text{ as we introduced before.}$$

Finally, the PDF of A (Figure 1) that follows New Mixed Generalized Distribution is:

$$\begin{aligned} f_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} f_{G_{ij}}(t) \\ &= \frac{783}{2} e^{-\frac{t^2}{3}t} + 9e^{-\frac{t^2}{3}t^3} - 704e^{-\frac{t^2}{4}t} + 8e^{-\frac{t^2}{4}t^3} - e^{-\frac{t^2}{4}t^5} + \frac{625}{2} e^{-\frac{t^2}{5}} \end{aligned}$$

where the CDF of A (Figure 2) that follows New Mixed Generalized Erlang distribution is:

$$\begin{aligned} F_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} F_{G_{ij}}(t) \\ &= 1 - \frac{3125}{4} e^{-\frac{t^2}{5}} - \frac{27}{4} e^{-\frac{t^2}{3}} (93 + 2t^2) + 2e^{-\frac{t^2}{4}} (704 + t^4) \end{aligned}$$

However the Reliability of A (Figure 3) that follows New Mixed Generalized Erlang distribution is:

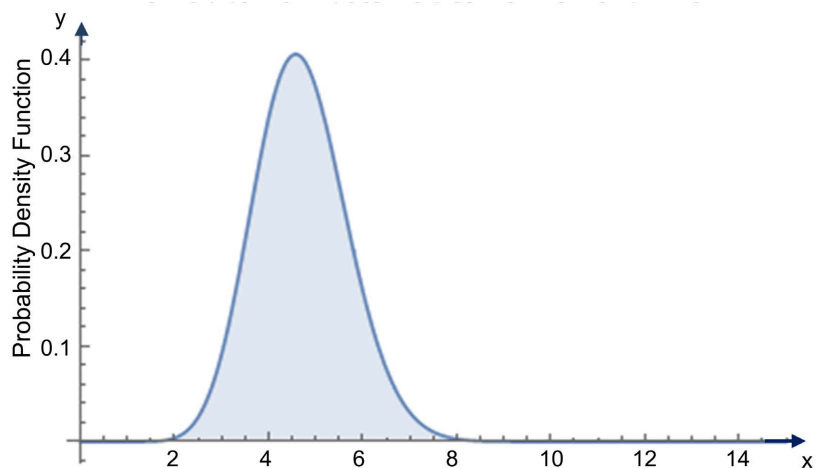


Figure 1. PDF of $A \sim NMGE(k, \beta, 2)$.

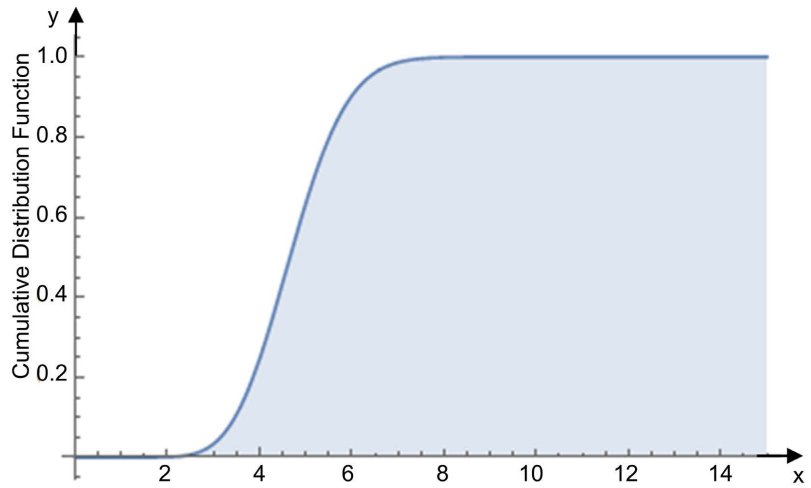


Figure 2. CDF of $A \sim NMGE(k, \beta, 2)$.

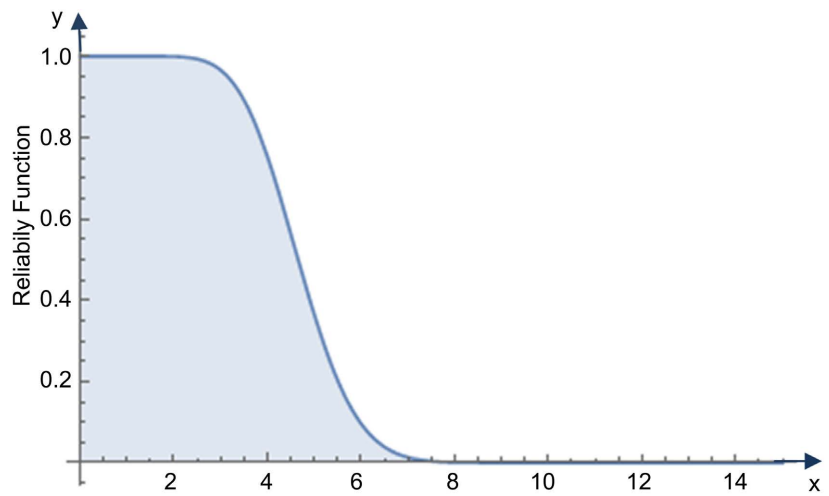


Figure 3. Reliability of $A \sim NMGE(k, \beta, 2)$.

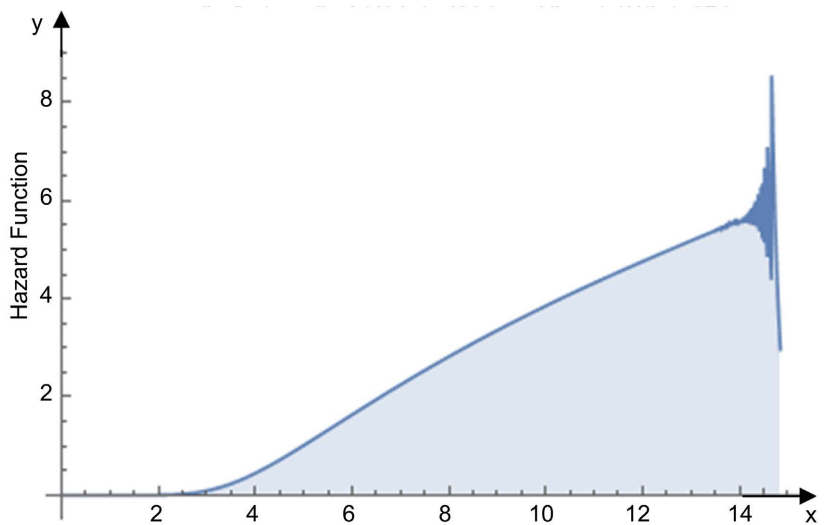


Figure 4. Hazard of $A \sim NMGE(k, \beta, 2)$.

$$\begin{aligned}
 R_A(t) &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} R_{G_{ij}}(t) \\
 &= \frac{1125}{4} e^{-\frac{t^2}{5}} + \frac{27}{4} e^{-\frac{t^2}{3}} (93 + 2t^2) - 2e^{-\frac{t^2}{4}} (704 + t^4)
 \end{aligned}$$

But the hazard rate function of A (Figure 4) is:

$$\begin{aligned}
 h_A(t) &= \frac{\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{c e^{-\left(\frac{t}{\beta}\right)^c} t^{-1+cj}}{\beta^{cj} (j-1)!}}{\sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} \frac{\Gamma\left(j, \frac{t^c}{\beta^c}\right)}{\Gamma(j)}} \\
 &= \frac{\frac{783}{2} e^{-\frac{t^2}{3}} t + 9e^{-\frac{t^2}{3}} t^3 - 704e^{-\frac{t^2}{4}} t + 8e^{-\frac{t^2}{4}} t^3 - e^{-\frac{t^2}{4}} t^5 + \frac{625}{2} e^{-\frac{t^2}{5}} t}{\frac{3125}{4} e^{-\frac{t^2}{5}} + \frac{27}{4} e^{-\frac{t^2}{3}} (93 + 2t^2) - 2e^{-\frac{t^2}{4}} (704 + t^4)}
 \end{aligned}$$

While the moment generating function of A is:

$$\begin{aligned}
 \Phi_A(t) &= \sum_{i=1}^3 \sum_{j=1}^{k_i} Y_{ij} \Phi_{G_{ij}}(t) \\
 &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{2349 \Gamma\left(1 + \frac{n}{2}\right)}{4} + \frac{81(\sqrt{3})^3 \Gamma\left(2 + \frac{n}{2}\right)}{2} - 1408(\sqrt{4})^n \Gamma\left(1 + \frac{n}{2}\right) \right. \\
 &\quad \left. + 64(\sqrt{4})^n \Gamma\left(2 + \frac{n}{2}\right) - 32(\sqrt{4})^n \Gamma\left(3 + \frac{n}{2}\right) + \frac{3125(\sqrt{5})^n \Gamma\left(1 + \frac{n}{2}\right)}{4} \right)
 \end{aligned}$$

Finally, the moment of order n of A is:

$$\begin{aligned}
 E[A^n] &= \sum_{i=1}^n \sum_{j=1}^{k_i} Y_{ij} E[G_{ij}^n] \\
 &= \frac{2349}{4} \left(3^{\frac{n}{2}} 1(1+1) \cdots \left(1 + \frac{n}{2} - 1\right) \right) + \frac{81}{2} \left(3^{\frac{n}{2}} 2(2+1) \cdots \left(2 + \frac{n}{2} - 1\right) \right) \\
 &\quad - 1408 \left(4^{\frac{n}{2}} 1(1+1) \cdots \left(1 + \frac{n}{2} - 1\right) \right) + 64 \left(4^{\frac{n}{2}} 2(2+1) \cdots \left(2 + \frac{n}{2} - 1\right) \right) \\
 &\quad - 64 \left(4^{\frac{n}{2}} 3(3+1) \cdots \left(3 + \frac{n}{2} - 1\right) \right) + \frac{3125}{4} \left(5^{\frac{n}{2}} 1(1+1) \cdots \left(1 + \frac{n}{2} - 1\right) \right)
 \end{aligned}$$

5. Conclusion

A modified form of pdf, cdf, mgf, reliability function, hazard function and moments for the New Mixed Generalized Erlang distribution were established. The proof has been done by writing the PDF of the New Mixed Generalized Erlang distribution as a linear combination of the PDF of the Generalized Erlang distribution. Eventually, we are concerned about finding Y_{ij} whenever you find it

that's an open problem may be discussed in the coming article.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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