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# Obtaining Simply Explicit Form and New Properties of Euler Polynomials by Differential Calculus

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## Abstract

Utilization of the shift operator to represent Euler polynomials as polynomials of Appell type leads directly to its algebraic properties, its relations with powers sums; may be all its relations with Bernoulli polynomials, Bernoulli numbers; its recurrence formulae and a very simple formula for calculating simultaneously Euler numbers and Euler polynomials. The expansions of Euler polynomials into Fourier series are also obtained; the formulae for obtaining all $\pi^m$ as series on $k^{-m}$ and for expanding functions into series of Euler polynomials.

## Keywords

Obtaining Appell Type Euler Numbers and Polynomials, Relations Euler-Bernoulli Polynomials, Sums over $k^m$, Series on $k^{-m}$, Euler Series of Functions

## 1. Introduction

Euler is widely regarded as one of the greatest and most prolific mathematicians of all time [1] as everyone can contemplate when reading the list of his publications concerning plenty of mathematical notions in [2]. From this list we know that he has introduced among others the notion of function $f(x)$ of a variable *x*, the exponential notation $e^x$, the symbol *i* such that $i^2 = -1$, the trigonometric definition of $\cos x$ and $\sin x$ by the formula

$$e^{ix} = \cos x + i \sin x \text{, etc.}$$

In the same period, Leibniz and Newton introduced the notions of infinitesimals d*x*, d*y*, the derivative $f'(x) = \frac{d}{dx} f(x)$ of a function and so on [3] [4].

Later, in 1766, Lagrange succeeded Euler at the Prussian Academy of Sciences and invented the shift operator $\mathrm{e}^{t\partial_x}$ having the property $\mathrm{e}^{t\partial_x} f(x) = f(x+t)$ [5].

After the introduction of shift or translation operators, it was introduced about in 1970 the notion of hyper-differential operator, noticeably by Wolf of Universidad Nacional Autónoma de México [6] who obtained that the Fourier transformation is representable by the operators $FT \equiv \mathrm{e}^{i\frac{\pi}{4}} \mathrm{e}^{-\frac{i}{2}\hat{X}^2} \mathrm{e}^{\frac{i}{2}D_x^2} \mathrm{e}^{-\frac{i}{2}\hat{X}^2}$ where $\hat{X}$ designates the Eckaert "multiply with *x*" operator analogous with the position operator in quantum mechanics.

At the same period, many authors such as Moshinsky, Quesne [7] and Treves [8] introduced the notion of linear canonical transforms.

Now, this work aims to utilize the Lagrange shift operator $\mathrm{e}^{t\partial_z}$ to express $E_m(z)$ as the transform of the monomial $z^m$ by a hyper differential operator then to obtain by differential calculus, its known and new found properties, its recursion relations; maybe all of its relations with the Bernoulli polynomials; its Fourier series which give $\pi^m$ from series in $k^{-m}$ as so as expansions of functions into Euler series; etc.

## 2. Definition

The Euler polynomial is defined from the generating function [9] [10]

$$\sum_{m=0}^{\infty} E_m(z) \frac{t^m}{m!} = \frac{2}{\mathrm{e}^t + 1} \mathrm{e}^{zt} \tag{1}$$

Utilizing the Lagrange translation operator $\mathrm{e}^{a\partial_z}$

$$\mathrm{e}^{a\partial_z} f(z) = f(z+a) \tag{2}$$

that haves the properties

$$\mathrm{e}^{\partial_{az}} = \mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}(az)}} = \mathrm{e}^{\frac{1}{a}\partial_z} \tag{3}$$

$$\begin{aligned} \mathrm{e}^{\partial_{az}} f(az) &= \mathrm{e}^{\frac{\mathrm{d}}{\mathrm{d}(az)}} f(az) = \mathrm{e}^{\frac{1}{a}\partial_z} f(az) \\ &= f\left(a\left(z + \frac{1}{a}\right)\right) = f(az+1) \end{aligned} \tag{4}$$

$$\mathrm{e}^{\partial_z} \mathrm{e}^{tz} = \mathrm{e}^{t(z+1)} = \mathrm{e}^t \mathrm{e}^{tz} \tag{5}$$

$$\frac{2}{\mathrm{e}^{\partial_z} + 1} \mathrm{e}^{tz} = \frac{2}{\mathrm{e}^t + 1} \mathrm{e}^{tz} \tag{6}$$

we may write $E_m(z)$ as the transform of the monomial $z^m$ under a differential operator

$$E_m(z) = \frac{2}{\mathrm{e}^{\partial_z} + 1} z^m \tag{7}$$

*i.e.*, that $E_m(z)$ is an Appell type polynomial [11].

## 3. Properties of $E_m(z)$

### 3.1. From (7) We Get Immediately

$$E'_m(z) = \frac{2m}{e^{\partial_z}+1} z^{m-1} = mE_{m-1}(z) \tag{8}$$

and the interesting property

$$2z^m = \left(e^{\partial_z}+1\right)E_m(z) = E_m(z+1) + E_m(z) \tag{9}$$

saying that

- $z^m$ is equal to the mean value of $E_m(z)$ and $E_m(z+1)$.

### 3.2. Power and Alternating Powers Sums

From the property (9) one gets immediately by addition lines-lines

$$2(z+n-1)^m = E_m(z+n) + E_m(z+n-1)$$

$$-2(z+n-2)^m = -E_m(z+n-1) - E_m(z+n-2)$$

$$\vdots$$

$$2(z+n-n)^m = -E_m(z+1) - E_m(z)$$

the interesting formulae on alternating sums of powers from $z^m$ to $(z+n-1)^m$

$$z^m - (z+1)^m + \cdots + (-1)^{n-1}(z+n-1)^m = \frac{1}{2}\left(E_m(z) + (-1)^{n+1}E_m(z+n)\right) \tag{10}$$

and on sums on powers from $z^m$ to $(z+n-1)^m$

$$\sum_{k=0}^{n-1}(z+k)^m = \frac{1}{2}E_m(z+n) + \sum_{k=1}^{n-1}E_m(z+n-k) + \frac{1}{2}E_m(z) \tag{11}$$

As simple examples one has the remarkable identities

$$1^m - 2^m + 3^m + \cdots + (-1)^{n-1}n^m = \frac{1}{2}\left(E_m(1)\right) + (-1)^{n+1}\left(E_m(n+1)\right) \tag{12}$$

from which we get the formula

$$S_m(n) = \sum_{k=0}^{n-1}k^m = \frac{1}{2}E_m(n) + \sum_{k=1}^{n-1}E_m(n-k) + \frac{1}{2}E_m(0) \tag{13}$$

that one may find in [12] and the followings

$$2\left(0^m + 2^m + \cdots + (2n-2)^m\right) = E_m(0) + E_m(1) + \cdots + E_m(2n-1)$$

$$2\left(1^m + 3^m + \cdots + (2n-1)^m\right) = E_m(1) + E_m(2) + \cdots + E_m(2n)$$

$$2\left(1^m - 2^m + 3^m - \cdots - (2n-1)^m\right) = E_m(2n) - E_m(0) \tag{14}$$

### 3.3. Values of $E_m((2n+1)z)$

From the differential form (7) of $E_m(z)$ and the property

$$E_m\left(\frac{z}{n}\right)=\frac{2}{\exp\left(\partial_{\left(\frac{z}{n}\right)}\right)+1}\left(\frac{z}{n}\right)^m=\frac{2}{e^{\dot{n}\partial_z}+1}\left(\frac{z}{n}\right)^m \quad (15)$$

as so as the formula for a sum of terms of a geometric progression with common ratio identical to $-e^{\partial_z}$

$$1-e^{\partial_z}+\cdots+(-1)^{n-1}e^{(n-1)\partial_z}=\frac{(-1)^n e^{\dot{n}\partial_z}-1}{-e^{\partial_z}-1}=\frac{(-1)^{n+1}e^{\dot{n}\partial_z}+1}{e^{\partial_z}+1} \quad (16)$$

or, replacing $n$ with $2n+1$,

$$e^{(2\dot{n}+1)\partial_z}+1=\left(e^{\partial_z}+1\right)\left(1-e^{\partial_z}+e^{2\partial_z}+\cdots-e^{(2n-1)\partial_z}+e^{2n\partial_z}\right)$$

one gets from (7)

$$E_m\left(\frac{z}{2n+1}\right)=\frac{2}{e^{(2\dot{n}+1)\partial_z}+1}\left(\frac{z}{2n+1}\right)^m \quad (17)$$

*i.e.*,

$$\left(1-e^{\partial_z}+e^{2\partial_z}+\cdots+e^{2n\partial_z}\right)E_m\left(\frac{z}{2n+1}\right)=(2n+1)^{-m}E_m(z)$$

a new property of Euler polynomials

$$(2n+1)^{-m}E_m(z)=\sum_{k=0}^{2n}(-1)^k E_m\left(\frac{z+k}{2n+1}\right) \quad (18)$$

For example

$$3^{-m}E_m(z)=E_m\left(\frac{z}{3}\right)-E_m\left(\frac{z+1}{3}\right)+E_m\left(\frac{z+2}{3}\right)$$

Formula (18) means that

- $E_m(z)$ is $(2n+1)^m$ times the alternating sum $(-1)^k E_m\left(\frac{z+k}{2n+1}\right), 0\le k\le 2n$

or, equivalently,

$E_m((2n+1)z)$ is $(2n+1)^m$ times the alternating sum $(-1)^k E_m\left(z+\frac{k}{2n+1}\right)$

For examples

$$3^{-m}E_m(3z)=E_m(z)-E_m\left(z+\frac{1}{3}\right)+E_m\left(z+\frac{2}{3}\right)$$

$$3^{-m}E_m(z)=E_m\left(\frac{z}{3}\right)-E_m\left(\frac{z+1}{3}\right)+E_m\left(\frac{z+2}{3}\right)$$

$$5^{-m}E_m(5z)=E_m(z)-E_m\left(z+\frac{1}{5}\right)+\cdots+E_m\left(z+\frac{4}{5}\right)$$

### 3.4. Symmetry of Euler Polynomials

From (7) we may write

$$E_m(-z)=\frac{2}{e^{-\partial_z}+1}(-z)^m=\frac{2e^{\partial_z}}{1+e^{\partial_z}}(-z)^m=(-1)^m e^{\partial_z}E_m(z)=(-1)^m E_m(z+1) \quad (19)$$

so that

$$E_m\left(-z+\frac{1}{2}\right)=(-1)^m E_m\left(z+\frac{1}{2}\right) \tag{20}$$

*i.e.*,

- The graph of $E_m(z)$ is symmetric with respect to the straight line $z=\frac{1}{2}$ for *m* even and to the point $\left(\frac{1}{2},0\right)$ for *m* odd.

In particular

$$E_m(0)=(-1)^m E_m(1) \tag{21}$$

$$E_m\left(\frac{1}{2}\right)=(-1)^m E_m\left(\frac{1}{2}\right) \tag{22}$$

$$E_{2m+1}\equiv 2^{2m+1}E_{2m+1}\left(\frac{1}{2}\right)=0 \tag{23}$$

where $E_n$ designated the Euler number of order *n*.

### 3.5. Symbolic Formula of Euler Polynomials

Now from (7) we may the write down symbolic formula

$$E_m(z+a)=\frac{2}{e^{\partial_{(z+a)}}+1}(z+a)^m=\frac{2}{e^{\partial_z}+1}(z+a)^m$$
$$=\frac{2}{e^{\partial_z}+1}\sum_{k=0}^{m}\binom{m}{k}a^{m-k}z^k=\sum_{k=0}^{m}\binom{m}{k}a^{m-k}E_k(z)$$

$$E_m(z+a)=:(E(z)+a)^m \tag{24}$$

where undefined notation $E^k(z)$ is to be replaced with well defined $E_k(z)$.

Permuting *z* and *a* in (24) we get the complementary symbolic formula

$$E_m(z+a)=:(E(a)+z)^m \tag{25}$$

which for $a=0$ leads to the formula

$$E_m(z)=:(E(0)+z)^m \tag{26}$$

similar to the Lucas formula for calculating Bernoulli polynomials [12].

Another way for obtaining symbolic formula of Euler polynomials is by remarking that as

$$e^{\partial_{z+y}}f(z+y)=e^{\partial_z}f(z+y)=e^{\partial_y}f(z+y) \tag{27}$$

we may write

$$E_m(z+y)=\frac{2}{e^{\partial_{(z+y)}}+1}(z+y)^m=\frac{2}{e^{\partial_z}+1}(z+y)^m$$
$$=\frac{2}{e^{\partial_z}+1}\sum_{k=0}^{m}\binom{m}{k}z^k y^m=\sum_{k=0}^{m}\binom{m}{k}E_k(z)y^m$$

or, symbolically, the symbolic relations

$$E_m(z+y) =: (E(z)+y)^m =: (E(y)+z)^m \tag{28}$$

$$E_m(z+E(y)) =: (E(z)+E(y))^m =: (E(E(y))+z)^m \tag{29}$$

The above results are resumed in Table 1.

## 4. Relations between Euler and Bernoulli Polynomials

### 4.1. The First Relation

From the known property of the Bernoulli polynomials [12]

$$B_m(z) = \frac{\partial_z}{\mathrm{e}^{\partial_z}-1} z^m \tag{30}$$

which makes them also of Appell type we get

$$\partial_z E_m(z) = \frac{2\partial_z}{\mathrm{e}^{\partial_z}+1} z^m = \frac{\mathrm{e}^{\partial_z}-1}{\mathrm{e}^{\partial_z}+1} \frac{2\partial_z}{\mathrm{e}^{\partial_z}-1} z^m = 2^m\left(\mathrm{e}^{\partial_z}-1\right) \frac{2\partial_z}{\mathrm{e}^{2\partial_z}-1}\left(\frac{z}{2}\right)^m$$

*i.e.*, according to (7) and the property of the shift operator $\mathrm{e}^{\partial_z}$, the known property one may find in Ref. [13]

$$mE_{m-1}(z) = 2^m\left(\mathrm{e}^{\partial_z}-1\right) B_m\left(\frac{z}{2}\right)$$

$$(m+1)E_m(z) = 2^{m+1}\left(B_{m+1}\left(\frac{z+1}{2}\right) - B_{m+1}\left(\frac{z}{2}\right)\right) \tag{31}$$

### 4.2. The Second Relation

Now, thanks to the formula obtainable from (30).

**Table 1.** Simple properties of Euler polynomials.

| | |
|---|---|
| $E_m(-z)$ | $=(-1)^m E_m(z+1)$ |
| $E_m\left(z+\frac{1}{2}\right)$ | $=(-1)^m E_m\left(-z+\frac{1}{2}\right)$ |
| $2z^m$ | $=E_m(z)+E_m(z+1)$ |
| $\sum_{k=0}^{n-1}(z+k)^m$ | $=\frac{1}{2}E_m(z+n)+\sum_{k=1}^{n-1}E_m(z+n-k)+\frac{1}{2}E_m(z)$ |
| $S_m(n)=\sum_{k=0}^{n-1}k^m$ | $=\frac{1}{2}E_m(n)+\sum_{k=1}^{n-1}E_m(n-k)+\frac{1}{2}E_m(0)$ |
| $\sum_{k=1}^{n}(-1)^{k+1}k^m$ | $=\frac{1}{2}\left(E_m(1)+(-1)^{n+1}E_m(n+1)\right)$ |
| $E_m((2n+1)z)$ | $=(2n+1)\sum_{k=0}^{2n}(-1)^k E_m\left(z+\frac{k}{2n+1}\right)$ |
| $E_m(z+a)$ | $=:(E(z)+a)^m =: (E(a)+z)^m$ |
| $E_m(z+y)$ | $=:(E(z)+E(y))^m$ |

$$B_m\left(\frac{z}{2}\right)=\frac{\partial_{z/2}}{e^{\partial_{z/2}}-1}\left(\frac{z}{2}\right)^m=\frac{2\partial_z}{e^{2\partial_z}-1}\left(\frac{z}{2}\right)^m=\frac{1}{e^{\partial_z}+1}\frac{2\partial_z}{e^{\partial_z}-1}\cdot 2^{-m}z^m$$

$$\left(1+e^{\partial_z}\right)B_m\left(\frac{z}{2}\right)=B_m\left(\frac{z}{2}\right)+B_m\left(\frac{z+1}{2}\right)=2^{1-m}B_m(z) \tag{32}$$

we may put the formula (31) under the more useful form

$$mE_{m-1}(z)=2B_m(z)-2^{m+1}B_m\left(\frac{z}{2}\right) \tag{33}$$

We note that formulae (31) and (33) are proven by Roman [3] by another method.

As consequence of (33) we have the very important relation linking $E_m(0)$ with $B_m$

$$(m+1)E_m(0)=\left(2-2^{m+2}\right)B_{m+1} \tag{34}$$

from which one can obtain easily $E_m(0)$ knowing $B_m$ and vice-versa.

For example, with $B_8=\frac{-1}{30}$

$$E_7(0)=\frac{1}{8}2\left(1-2^8\right)\frac{-1}{30}=\frac{1}{4}\frac{255}{30}=\frac{17}{8}$$

### 4.3. The Third Relation

Searching for other relations between Euler and Bernoulli polynomials we get from (7) the relations

$$\frac{\partial_z}{e^{\partial_z}-1}E_m(z)=\frac{2\partial_z}{e^{2\partial_z}-1}z^m=2^m\frac{\partial_{z/2}}{e^{\partial_{z/2}}-1}\left(\frac{z}{2}\right)^m=2^mB_m\left(\frac{z}{2}\right) \tag{35}$$

$$E_m\left(B(z)\right)=:2^mB_m\left(\frac{z}{2}\right) \tag{36}$$

and

$$mE_{m-1}(z)=\left(e^{\partial_z}-1\right)2^mB_m\left(\frac{z}{2}\right)$$

$$mE_{m-1}(z)=2^m\left(B_m\left(\frac{z+1}{2}\right)-B_m\left(\frac{z}{2}\right)\right) \tag{37}$$

For examples:

$$E_1\left(B(z)\right):=B_1(z)-\frac{1}{2}=z-1=2^1B_1\left(\frac{z}{2}\right)\Rightarrow B_1=-\frac{1}{2}$$

$$3E_2(z):=8\left(B_3\left(\frac{z+1}{2}\right)-B_3\left(\frac{z}{2}\right)\right)$$

$$E_2\left(B(z)\right):=B_2(z)-B_1(z)=z^2-2z+\frac{4}{6}=2^2B_2\left(\frac{z}{2}\right)$$

### 4.4. The Fourth Relation

Moreover we find from the differential representations of Euler and Bernoulli polynomials that

$$\frac{\partial_z}{e^{\partial_z}-1}E_m(z)=\frac{\partial_z}{e^{\partial_z}-1}\frac{2}{e^{\partial_z}+1}z^m=\frac{2\partial_z}{e^{2\partial_z}-1}z^m=\frac{\partial_{z/2}}{e^{\partial_{z/2}}-1}z^m=2^m B_m\left(\frac{z}{2}\right) \quad (38)$$

*i.e.*, another relation between Euler and Bernoulli polynomials

$$E_m(B(z)):=2^m B_m\left(\frac{z}{2}\right)=:2^m\left(B(0)+\frac{z}{2}\right)^m \quad (39)$$

For examples

$$E_1(B):=B_1(z)-\frac{1}{2}=2^1 B_1\left(\frac{z}{2}\right)=:2^1\left(\frac{z}{2}+B\right)^1=z-1$$

$$\Rightarrow B_1(z)=\left(z-\frac{1}{2}\right)$$

$$E_2(B):=B_2(z)-B_1(z)=2^2 B_2\left(\frac{z}{2}\right)=:2^2\left(B+\frac{z}{2}\right)^2$$

$$\Rightarrow B_2(z)=:(B+z)^2=z^2+2B_1z+B_2$$

## 4.5. The Fifth Relation

More curiously concerning the interrelation between Euler and Bernoulli polynomials we discover from the relation

$$\frac{\partial_z}{e^{2\partial_z}-1}\frac{2}{e^{\partial_z}+1}z^m=\frac{2}{e^{\partial_z}+1}\frac{\partial_z}{e^{2\partial_z}-1}z^m \quad (40)$$

the symbolic relation

$$E_m(B(z))=:B_m(E(z)) \quad (41)$$

For example

$$E_2(B(z)):=B_2(z)-B_1(z)=z^2-z+\frac{1}{6}-z+\frac{1}{2}$$

$$=:B_2(E(z))=E_2(z)-E_1(z)+\frac{1}{6}=z^2-z-z+\frac{1}{2}+\frac{1}{6}$$

Formula (41) suggests the great theorem which may be very useful.

- The property $E_m(B(z))=:B_m(E(z))$ holds for any two Appell type polynomials.

## 4.6. The Sixth Relation

Now, if we write

$$E_m(z)=\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(0)z^k=:(z+E(0))^m \quad (42)$$

then apply the operator (30) on both members of it and utilize (39) we get

$$E_m(B(z))=:(B(z)+E(0))^m:=2^m B_m\left(\frac{z}{2}\right) \quad (43)$$

and another relation between $E_m(B)$ and the Bernoulli numbers

$$E_m(B) =: (B + E(0))^m := 2^m B_m \tag{44}$$

For examples

$$E_2(B) := B_2 + 2B_1E_1(0) + E_2(0) = \frac{1}{6} + 2\frac{-1}{2}\frac{-1}{2} + 0 = 2^2 B_2 = \frac{4}{6}$$

The relations between Euler and Bernoulli polynomials are summarized in Table 2.

## 5. Obtaining Euler Polynomials

### 5.1. From Values at Origin $E_m(0)$

From the symbolic formula (42)

$$E_m(z) =: (z + E(0))^m$$

We see that $E_m(z)$ may be obtained from the values $E_k(0), k = 1, \cdots, m$.

Moreover, because of (34)

$$(m+1)E_m(0) = (2 - 2^{m+2})B_{m+1}$$

they are also obtainable from Bernoulli numbers.

Finally, in a recent work on Bernoulli polynomials [12] we have obtained the famous formula giving easily all values $B_m$, says

$$(1-m)B_m =: (B - B)^m \tag{45}$$

Combining (31) with (45) we may calculate very simply all $B_m = B_m(0)$ and all $E_m(0)$.

For examples

$$-5B_4 = 6B_2B_2 = \frac{1}{42}$$

$$-7B_6 = 2\binom{6}{2}B_2B_4 = 30\frac{1}{6}\left(\frac{-1}{30}\right) = \frac{-1}{6}$$

**Table 2.** Relations Euler-Bernoulli polynomials.

| | |
|---|---|
| $(m+1)E_m(z)$ | $= 2^{m+1}\left(B_{m+1}\left(\frac{z+1}{2}\right) - B_{m+1}\left(\frac{z}{2}\right)\right)$ |
| $mE_{m-1}(z)$ | $= 2B_m(z) - 2^{m+1}B_m\left(\frac{z}{2}\right)$ |
| $(m+1)E_m(0)$ | $= (2 - 2^{m+2})B_{m+1}$ |
| $mE_{m-1}(z)$ | $= 2^m\left(B_m\left(\frac{z+1}{2}\right) - B_m\left(\frac{z}{2}\right)\right)$ |
| $E_m(B(z))$ | $=: 2^m B_m\left(\frac{z}{2}\right) =: 2^m\left(B(0) + \frac{z}{2}\right)^m$ |
| $E_m(B(z))$ | $=: B_m(E(z))$ |

$$6E_5(0)=\left(2-2^{5+2}\right)B_6=-126B_6$$

## 5.2. From Recursion Relation on Euler Polynomials

From the following identity of operators that we characterize fundamental [14]

$$f(\partial_z)g(z)\equiv g(z)f(\partial_z)+\frac{1}{1!}g'(z)f'(\partial_z)+\frac{1}{2!}g''(z)f''(\partial_z)+\cdots \tag{46}$$

obtainable by recurrence from the property

$$\partial_z zg(z)=g(z)+zg'(z) \tag{47}$$

we obtain the new symbolic recurrence relation on Euler polynomials

$$E_{m+1}(z)=\frac{2}{e^{\partial_z}+1}zz^m=z\frac{2}{e^{\partial_z}+1}z^m-\frac{2e^{\partial_z}}{\left(e^{\partial_z}+1\right)^2}z^m \tag{48}$$

*i.e.*,

$$E_{m+1}(z)=zE_m(z)-\frac{e^{\partial_z}}{e^{\partial_z}+1}E_m(z)$$

$$E_{m+1}(z)=:zE_m(z)-\frac{1}{2}E_m\left(E(z+1)\right) \tag{49}$$

For example, because $E_2(z)=z^2-z$,

$$E_3(z)=zE_2(z)-\frac{1}{2}\left(E_2(z+1)-E_1(z+1)\right)$$
$$=\left(z^3-z^2\right)-\frac{1}{2}\left((z+1)^2-2(z+1)+\frac{1}{2}\right)=z^3-\frac{3}{2}z^2+\frac{1}{4}$$

## 5.3. From Euler Polynomials of Sums of Arguments

Another way for obtaining recursion relation between $E_m(z)$ comes from (49) and the property

$$e^{\partial_{z+y}}f(z+y)=e^{\partial_z}f(z+y)=e^{\partial_y}f(z+y) \tag{50}$$

which leads to

$$E_{m+1}(z+y)=(z+y)E_m(z+y)-\frac{2e^{\partial_{z+y}}}{\left(e^{\partial_{z+y}}+1\right)^2}(z+y)^m$$
$$=(z+y)E_m(z+y)-\frac{e^{\partial_y}}{e^{\partial_y}+1}\left(y+E(z)\right)^m$$
$$=(z+y)E_m(z+y)-\frac{1}{2}\left(E(y+1)+E(z)\right)^m$$

$$\frac{1}{2}\left(E(y)+E(z)\right)^m=(z+y-1)E_m(z+y)-E_{m+1}(z+y-1) \tag{51}$$

and

$$E_{m+1}(z)=:zE_m(z)-\frac{1}{2}\left(E(1)+E(z)\right)^m$$

*i.e.*, because $E_{2m+1}(1) = -E_{2m+1}(0)$, $E_{2m}(1) = E_{2m}(0) = \delta_{m0}$,

$$E_{m+1}(z) =: (z-1)E_m(z) + \frac{1}{2}(E(0)+E(z))^m \tag{52}$$

In particular

$$E_{m+1}(1) =: \frac{1}{2}(E(0)+E(1))^m$$

For examples

$$E_2(z) = (z-1)\left(z-\frac{1}{2}\right) + \frac{1}{2}\left(-\frac{1}{2}+z-\frac{1}{2}\right) = z^2 - z$$

$$\begin{aligned} E_3(z) &=: (z-1)E_2(z) + \frac{1}{2}(E(z)+E(0))^2 \\ &= (z-1)(z^2-z) + \frac{1}{2}\left((z^2-z) - \left(z-\frac{1}{2}\right)+0\right) \\ &= z^3 - \frac{3}{2}z^2 + \frac{1}{4} \end{aligned}$$

$$E_3(1) =: \frac{1}{2}(E(0)+E(1))^2 = \frac{1}{2}(2E_1(0)E_1(1)) = -\frac{1}{4}$$

We note that (52) is equivalent with the formula about Bernoulli polynomials found by the similar method cited in Ref. [10]

$$(m-1)B_m(z) = mzB_{m-1}(z) - (B(z)+B(1))^m$$

It gives rise also to the second symbolic formula concerning $E_m(0)$

$$E_{m+1}(0) + E_m(0) =: \frac{1}{2}(E(0)+E(0))^m \tag{53}$$

to be compared with the marvellous formula (45) concerning Bernoulli numbers

$$(1-m)B_m(0) =: (B(0)-B(0))^m$$

and the mixed formula coming from (44)

$$E_m(B) =: (E(0)+B)^m := 2^m B_m \tag{54}$$

$$E_2(B) =: (E(0)+B)^m = E_2(0) + 2E_1(0)B_1 + B_2 := 2^2 B_2$$

$$E_4(B) =: (E(0)+B)^m = 4E_3(0)B_1 + B_4 := 2^4 B_4$$

Some examples concerning $E_m(0)$ given by (53) where $E_{2m}(0) = \delta_{m0}$

$$E_1(0) + 1 = \frac{1}{2}$$

$$E_3(0) =: \frac{1}{2}(E(0)+E(0))^2 = \frac{1}{2}(2E_1(0)E_1(0)) = \frac{1}{4}$$

$$E_5(0) =: \frac{1}{2}(E(0)+E(0))^4 = \frac{1}{2}(4E_1(0)E_3(0) + 4E_3(0)E_1(0)) = -\frac{1}{2}$$

$$E_7(0) =: \frac{1}{2}(E(0)+E(0))^6 = \frac{1}{2}\left(2\binom{6}{1}E_1(0)E_5(0) + \binom{6}{3}E_3(0)E_3(0)\right) = \frac{17}{8}$$

**Table 3.** Recursion relations on Euler polynomials.

| | |
|---|---|
| $E_m(z)$ | $=:(z+E(0))^m$ |
| $E_{m+1}(z)$ | $=: zE_m(z)-\frac{1}{2}E_m(E(z+1))$ |
| $E_{m+1}(z+y-1)$ | $=(z+y-1)E_m(z+y)-\frac{1}{2}(E(y)+E(z))^m$ |
| $E_{m+1}(z)$ | $=:(z-1)E_m(z)+\frac{1}{2}(E(0)+E(z))^m$ |
| $E_{m+1}(0)+E_m(0)$ | $=:\frac{1}{2}(E(0)+E(0))^m$ |

Hereafter we summarize the recursion relations between $E_{m+1}(z)$ and $E_m(z)$ of lower orders in **Table 3**.

Although (53) is convenient for calculating $E_m(0)$ we would like to expose hereafter a formula for calculating them not by recurrence but individually.

### 5.4. Obtaining $E_m(z)$ from $E_{m-1}(z)$

From the property (8) one may write

$$E'_m(z)=\frac{2m}{e^{\partial_z}+1}z^{m-1}=mE_{m-1}(z)$$

and by taking primitives of both sides

$$E_m(z)=m\int E_{m-1}(z)+E_m(0) \tag{55}$$

From (55) we get

$$E_{2m+1}(z)=(2m+1)\int E_{2m}(z)+E_{2m+1}(0)$$

But according to (20)

$$E_{2m+1}(z)=-E_{2m+1}(1-z)$$

so that from

$$E_{2m+1}(1)-E_{2m+1}(0)=(2m+1)\left(\int E_{2m}(z)\right)_{z=1}$$

one gets after all the formula giving $E_{2m+1}(0)$ from an integral of $E_{2m}(z)$

$$E_{2m+1}(0)=-\frac{1}{2}(2m+1)\int_0^1 E_{2m}(z)\mathrm{d}z \tag{56}$$

This new algorithm may be utilized to calculate them as shown in **Table 4**.

- We remark that because $B_{2m}(z)=B_{2m}(1-z)$ a similar formula as (56) for Bernoulli numbers and polynomials doesn't exist.

For examples, noting that $E_{2m}(0)=\delta_{m0}$,

$$E_1(0)=-\frac{1}{2}\int_0^1 E_0(z)\mathrm{d}z=-\frac{1}{2}$$

**Table 4.** $E_m(0)$ and Euler polynomials $E_m(z)$.

| $m$ | $E_m(0)=-\frac{m}{2}\int_0^1 E_{m-1}(z)$ | $E_m(z)=m\int_0^1 E_{m-1}(z)\mathrm{d}z+E_m(0)$ | $E_m(z)$ |
|---|---|---|---|
| 0 | $E_0(0)\equiv 1$ | $E_0(z)=E_0(0)$ | $E_0(z)=1$ |
| 1 | $E_1(0)=-\frac{1}{2}\int_0^1 E_0(z)\mathrm{d}z=-\frac{1}{2}$ | $E_1(z)=1\int_0^1 E_0(z)\mathrm{d}z+E_1(0)$ | $E_1(z)=z-\frac{1}{2}$ |
| 2 | $E_2(0)=0$ | $E_2(z)=2\left(\frac{z^2}{2}-\frac{z}{2}\right)$ | $E_2(z)=z^2-z$ |
| 3 | $E_3(0)=-\frac{3}{2}\int_0^1\left(z^2-z\right)\mathrm{d}z=\frac{1}{4}$ | $E_3(z)=3\int_0^1 E_2(z)\mathrm{d}z+E_3(0)$ | $E_3(z)=z^3-\frac{3}{2}z^2+\frac{1}{4}$ |
| 4 | $E_4(0)=0$ | $E_4(z)=z^4-2z^3+z$ | $E_4(z)=z^4-2z^3+z$ |
| 5 | $E_5(0)=-\frac{5}{2}\int_0^1\left(z^4-2z^3+z\right)\mathrm{d}z=-\frac{1}{2}$ | $E_5(z)=5\int_0^1 E_4(z)\mathrm{d}z+E_5(0)$ | $E_5(z)=z^5-\frac{5z^4}{2}+\frac{5z^2}{2}-\frac{1}{2}$ |
| 6 | $E_6(0)=0$ | $E_6(z)=z^6-3z^5+5z^3-3z+E_6(0)$ | $E_6(z)=z^6-3z^5+5z^3-3z$ |
| 7 | $E_7(0)=-\frac{7}{2}\int_0^1 E_6(z)\mathrm{d}z=\frac{17}{8}$ | $E_7(z)=z^7-\frac{7z^6}{2}+\frac{35}{4}z^4-\frac{21}{2}z^2+E_7(0)$ | $E_7(z)=z^7-\frac{7z^6}{2}+\frac{35}{4}z^4-\frac{21}{2}z^2+\frac{17}{8}$ |
| 8 | $E_8(0)=0$ | $E_8(z)=z^8-4z^7+14z^5-28z^3+17z+E_8(0)$ | $E_8(z)=z^8-4z^7+14z^5-28z^3+17z$ |
| 9 | $E_9(0)=-\frac{9}{2}\int_0^1\left(z^8-4z^7+14z^5-28z^3+17z\right)\mathrm{d}z=-\frac{31}{2}$ | $E_9(z)=z^9-\frac{9}{2}z^8+21z^6-63z^4+\frac{153}{2}z^2+E_9(0)$ | $E_9(z)=z^9-\frac{9}{2}z^8+21z^6-63z^4+\frac{153}{2}z^2-\frac{31}{2}$ |
| 10 | $E_{10}(0)=0$ | $E_{10}(z)=z^{10}-5z^9+30z^7-126z^5+255z^3-155z+E_{10}(0)$ | $E_{10}(z)=z^{10}-5z^9+30z^7-126z^5+255z^3-155z$ |

$$E_1(z)=1\int E_0(z)\mathrm{d}z+E_1(0)=z-\frac{1}{2}$$

$$E_2(z)=2\int E_1(z)\mathrm{d}z+E_2(0)=2\left(\frac{z^2}{2}-\frac{z}{2}\right)=z^2-z$$

- Another algorithm for obtaining $E_m(z)$ is:

  Let $E_1(0)=-\frac{1}{2}$ and $E_m^-(z)$ be the primitive of $mE_{m-1}(z)$ we see that

$$E_1(z)=z-\frac{1}{2}\to\to E_2^-(z)=z^2-z\to\to E_2(0)=1-1=0$$

$$E_2(z)=z^2-z\to\to E_3^-(z)=z^3-\frac{3}{2}z^2\to\to E_3(0)=E_3^-(1)=-\frac{1}{2}$$

$$E_8(z)=z^8-4z^7+14z^5-28z^3+17z\to\to$$

$$E_9^-(z)=z^9-\frac{9}{2}z^8+21z^6-63z^4+9\left(\frac{17}{2}z^2\right)\to\to E_9(0)=E_9^-(1)=-\frac{31}{2}$$

$$\vdots$$

## 6. Integral of Product of Euler Polynomials

Thanks to the property (8) one may perform successive integrations by parts on products of Euler polynomials and get

$$\int_0^1 E_n(z)E_m(z)\mathrm{d}z = \frac{1}{m+1}\int_0^1 E_n(z)E'_{m+1}(z)\mathrm{d}z$$

$$= \frac{1}{m+1}\left(E_n(z)E_{m+1}(z)\right)\Big|_0^1 - \frac{n(n-1)}{(m+1)(m+2)}\left(E_{n-1}(z)E_{m+2}(z)\right)\Big|_0^1$$

$$+\cdots+(-1)^{n-1}\frac{n!m!}{(m+n)!}\left(E_1(z)E_{m+n}(z)\right)\Big|_0^1$$

$$+(-1)^n\frac{n!m!}{(m+n)!}\int_0^1 E_0(z)E_{m+n}(z)\mathrm{d}z$$

But $E_{2n}(1)=E_{2n}(0)=0$ and $E_{2n+1}(1)=-E_{2n+1}(0)$ so that we finally get

$$\int_0^1 E_n(z)E_m(z)\mathrm{d}z = (-1)^n\frac{n!m!}{(m+n+1)!}E_{m+n+1}(z)\Big|_0^1 \tag{57}$$

In particular for $m+n=2p$

$$\int_0^1 E_n(z)E_m(z)\mathrm{d}z = (-1)^{n+1}\frac{n!m!}{(m+n+1)!}2E_{2p+1}(0) \tag{58}$$

Thanks to the relation (34) we see that the formula (58) is conformed with the result given in [13] says

$$\int_0^1 E_n(z)E_m(z)\mathrm{d}z = (-1)^n\frac{n!m!}{(m+n+2)!}4\left(2^{m+n+2}-1\right)B_{2m+2} \tag{59}$$

We observe that by integration by parts one may also calculate $\int_a^b E_n(z)E_m(z)\mathrm{d}z$.

## 7. Fourier Series of Euler Polynomials

From the famous relation (33) between Euler and Bernoulli polynomials

$$mE_{m-1}(z) = 2B_m(z) - 2^{m+1}B_m\left(\frac{z}{2}\right)$$

and the Hurwitz formula on Bernoulli polynomials [12]

$$B_m(z) = -\frac{m!}{(2i\pi)^m}\sum_{\substack{k\neq 0\\ k\in Z}}\frac{1}{k^m}\mathrm{e}^{i2\pi kz},\ 0\le z\le 1$$

we get directly the Fourier series expansions of Euler polynomials

$$E_{m-1}(z) = -\frac{m!}{(2i\pi)^m}2\left(\sum_{\substack{k\in Z\\ k\neq 0}}\frac{1}{k^m}\mathrm{e}^{i2\pi kz} - 2^m\sum_{\substack{k\in Z\\ k\neq 0}}\frac{1}{k^m}\mathrm{e}^{i\pi kz}\right)$$

$$= -\frac{m!}{(2i\pi)^m} 2 \left( \sum_{\substack{k \in Z \\ k \neq 0}} \frac{1}{k^m} e^{i2\pi kz} - \sum_{\substack{k \in Z \\ k \neq 0}} \frac{1}{\left(\frac{k}{2}\right)^m} e^{i\pi kz} \right)$$

$$= -\frac{m!}{(2i\pi)^m} 2 \sum_{kodd}^{\infty} \frac{1}{\left(\frac{k}{2}\right)^m} e^{i\pi kz} \tag{60}$$

$$= -\frac{m!}{(i\pi)^m} 2 \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^m} e^{i\pi(2k+1)z}$$

*i.e.*,

$$E_{2n-1}(z) = (-1)^n \frac{(2n-1)!}{\pi^{2n}} 4 \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi z}{(2k+1)^{2n}} \tag{61}$$

and

$$E_{2n}(z) = (-1)^n \frac{(2n)!}{\pi^{2n+1}} 4 \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi z}{(2k+1)^{2n+1}} \tag{62}$$

For example

$$E_1(z) = -\frac{1}{\pi^2} 4 \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi z}{(2k+1)^2}$$

The formulae (61), (62) were known for example in [10]. They do not depend On $B_m$ or $E_m(0)$. On the contrary they show the apparition of $\pi^n$ and permit to calculate them by summations of infinite series, for examples

$$E_{2n-1}(0) = \frac{1}{(2n-1)!4} \pi^{2n} = (-1)^n \left( \frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \cdots \right) \tag{63}$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$E_2(z) = -\frac{2}{\pi^3} 4 \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi z}{(2k+1)^3}$$

$$E_{2n}(z) = (-1)^n \frac{(2n)!}{\pi^{2n+1}} 4 \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi z}{(2k+1)^{2n+1}}$$

$$E_{2n}\left(\frac{1}{2}\right) \frac{\pi^{2n+1}}{4(2n)!} = (-1)^n \left( \frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} + \cdots \right) \tag{64}$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} + \cdots$$

## 8. Euler Polynomials and Euler Zeta Functions

From the Hurwitz formula on Bernoulli polynomials [13]

$$B_m(z) = -\frac{m!}{(2i\pi)^m} \sum_{\substack{k \neq 0 \\ k \in Z}} \frac{1}{k^m} e^{i2\pi kz}, 0 \leq z \leq 1$$

we get

$$B_m(0) = -\frac{m!}{(2i\pi)^m} \sum_{\substack{k \neq 0 \\ k \in Z}} \frac{1}{k^m}$$

$$B_{2m}(0) = -\frac{(2m!)2}{(2i\pi)^{2m}} \sum_{\substack{k>0 \\ k \in Z}} \frac{1}{k^{2m}} \tag{65}$$

Formula (65) leads to the relation of the Euler-Riemann zeta function [15] with Bernoulli numbers

$$\zeta(2m) = \sum_{k>0} \frac{1}{k^{2m}} = \frac{(-1)^{m-1}}{2(2m)!}(2\pi)^{2m} B_{2m} \tag{66}$$

and, thanks to (34), with $E_{2m-1}(0)$

$$\zeta(2m) = (2\pi)^{2m} \frac{m}{1-2^{2m}} \frac{(-1)^{m-1}}{2(2m)!} E_{2m-1}(0) \tag{67}$$

For example

$$\zeta(2) = \frac{1}{1-2^2} \frac{1}{2(2)!}(2\pi)^2 E_1(0) = \frac{4}{3 \times 4 \times 2}\pi^2 = \frac{\pi^2}{6} \tag{68}$$

a result that Euler had proven in 1734 by a laborious method described in [15].

More generally, by putting $z = \frac{1}{n}$ in (65) we get the general formulae permitting to calculate the values of $\pi^m$ from the values of $\sin\left(\frac{2\pi k}{n}\right)$ and $\cos\left(\frac{2\pi k}{n}\right)$

$$(2\pi)^m \frac{1}{m!} B_m\left(\frac{1}{n}\right) = -\sum_{\substack{k \neq 0 \\ k \in Z}} \frac{1}{(ik)^m} e^{i2\pi\frac{k}{n}} \tag{69}$$

$$\pi^m 2^{m-1} \frac{1}{(m-1)!} E_{m-1}\left(\frac{1}{n}\right) = -\sum_{\substack{k \neq 0 \\ k \in Z}} \frac{1}{(ik)^m}\left(e^{i2\pi\frac{k}{n}} - 2^m e^{i\pi\frac{k}{n}}\right) \tag{70}$$

Explicitly, from (69) we get **Table 5**.

Regarding these results one may say that

- There have five types of infinite sums over $\frac{1}{k^m}$ for calculating each $\pi^m B_m\left(\frac{1}{n}\right), n \leq 12$.

Besides, from (70) we get $\pi^m E_{m-1}\left(\frac{1}{n}\right)$ from summation of series on $k^{-m}$

$$\pi^m 2^{m-1} \frac{1}{(m-1)!} E_{m-1}\left(\frac{1}{n}\right) = -\sum_{\substack{k \neq 0 \\ k \in Z}} \frac{1}{(ik)^m}\left(e^{i2\pi\frac{k}{n}} - 2^m e^{i\pi\frac{k}{n}}\right) \tag{71}$$

$$\pi^m 2^{m-1} \frac{1}{(m-1)!} E_{m-1}\left(\frac{1}{1}\right) = -\sum_{\substack{k \neq 0 \\ k \in Z}} \frac{1}{(ik)^m}\left(e^{i2\pi k} - 2^m e^{i\pi k}\right)$$

**Table 5.** Series of Infinite Sums over $\frac{1}{k^m}$.

| $m$ | $n$ | $-\pi B_1\left(\frac{1}{n}\right)=\sum_{k>0}\frac{1}{k}\sin\left(\frac{2\pi k}{n}\right)$ | |
|---|---|---|---|
| 1 | 1 | $-\pi B_1(1)-\frac{1}{2}\pi=\left(\frac{0}{1}+\frac{0}{2}+\frac{0}{3}+\frac{0}{4}+\cdots\right)$ | |
| | 2 | $0=\left(\frac{0}{1}-\frac{0}{2}+\frac{0}{3}-\frac{0}{4}+\cdots\right)$ | |
| | 3 | $-\pi B_1\left(\frac{1}{3}\right)=\frac{\sqrt{3}}{2}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{4}+\cdots\right)$ | |
| | 4 | $-\pi B_1\left(\frac{1}{4}\right)=\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$ | Gregory-Leibnitz formula |
| | 6 | $-\pi B_1\left(\frac{1}{6}\right)=\frac{\sqrt{3}}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\frac{1}{7}+\cdots\right)$ | |
| | 8 | $\left(\frac{\sqrt{2}}{2}\frac{1}{1}+\frac{1}{2}+\frac{\sqrt{2}}{2}\frac{1}{3}-\frac{\sqrt{2}}{2}\frac{1}{5}-\frac{1}{6}-\frac{\sqrt{2}}{2}\frac{1}{7}+\frac{1}{9}\frac{\sqrt{2}}{2}+\cdots\right)$ | Newton-Euler formula |
| | 12 | $-\pi B_1\left(\frac{1}{12}\right)=\left(\frac{1}{2}\frac{1}{1}+\frac{\sqrt{3}}{2}\frac{1}{2}+\frac{1}{3}+\frac{\sqrt{3}}{2}\frac{1}{4}-\frac{1}{2}\frac{1}{5}\right)$<br>$-\frac{1}{2}\frac{1}{7}-\frac{\sqrt{3}}{2}\frac{1}{8}-\frac{1}{9}-\frac{\sqrt{3}}{2}\frac{1}{10}-\frac{1}{2}\frac{1}{11}+\cdots$ | |
| $m$ | $n$ | $-\pi B_2\left(\frac{1}{n}\right)=\sum_{k>0}\frac{1}{k^2}\sin\left(\frac{2\pi k}{n}\right)$ | |
| 2 | 1 | $\pi^2\frac{1}{6}=\left(\frac{0}{1^2}+\frac{0}{2^2}+\frac{0}{3^2}+\cdots\right)$ | |
| | 2 | $\pi^2\frac{1}{12}=\left(\frac{0}{1^2}-\frac{0}{2^2}+\frac{0}{3^2}-\frac{0}{4^2}+\cdots\right)$ | |
| | 3 | $\pi B_2\left(\frac{1}{3}\right)=\frac{\sqrt{3}}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)$ | |
| | 4 | $\pi B_2\left(\frac{1}{4}\right)=\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right)$ | |
| | 6 | $\pi B_2\left(\frac{1}{6}\right)=\frac{\sqrt{3}}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{6^2}+\frac{1}{7^2}+\cdots\right)$ | |
| | 8 | $\pi B_2\left(\frac{1}{8}\right)=\left(\frac{\sqrt{2}}{2}\frac{1}{1^2}+\frac{1}{2^2}+\frac{\sqrt{2}}{2}\frac{1}{3^2}-\frac{\sqrt{2}}{2}\frac{1}{5^2}-\frac{1}{6^2}\right.$<br>$\left.-\frac{\sqrt{2}}{2}\frac{1}{7^2}+\frac{1}{9^2}\frac{\sqrt{2}}{2}+\cdots\right)$ | |
| | 12 | $\frac{1}{2}\left(\frac{1}{1^2}+\sqrt{3}\frac{1}{2^2}+2\frac{1}{3^2}+\sqrt{3}\frac{1}{4^2}+\frac{1}{5^2}\right)$<br>$\frac{1}{2}\left(-\sqrt{3}\frac{1}{7^2}-\frac{1}{8^2}-2\frac{1}{9^2}-\sqrt{3}\frac{1}{10^2}-\frac{1}{11^2}\right)+\cdots$ | |

$$2\pi^2 E_1\left(\frac{1}{2}\right)=0=\left(-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots\right)-4\left(-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{6^2}+\cdots\right)$$

$$2\pi^2 E_1\left(\frac{1}{4}\right)=\left(-\frac{1}{2^2}+\frac{1}{4^2}+\cdots\right)-2\left(\frac{\sqrt{2}}{1^2}-\frac{\sqrt{2}}{3^2}-2\frac{1}{4^2}-\frac{\sqrt{2}}{5^2}+\frac{\sqrt{2}}{7^2}+2\frac{1}{8^2}+\cdots\right)$$

$$\pi^3 2E_2\left(\frac{1}{2}\right)=-\frac{1}{2}\pi^3=-16\sum_{k>0}\frac{1}{k^3}\sin\left(\frac{k\pi}{2}\right)=-16\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots\right)$$

and so all.

## 9. Euler Series of Functions

We already know the formulae on Fourier series (60)

$$B_m(z)=-\frac{m!}{(2i\pi)^m}\sum_{\substack{k\neq 0\\ k\in Z}}\frac{1}{k^m}\mathrm{e}^{i2\pi kz}$$

$$mE_{m-1}(z)=-\frac{m!}{(i\pi)^m}2\sum_{k=-\infty}^{\infty}\frac{1}{(2k+1)^m}\mathrm{e}^{i\pi(2k+1)z}$$

and that entire functions may be expanded into series of Bernoulli polynomials

$$f(z)=\int_0^1 f(z)\mathrm{d}z-\sum_{n\in Z,n\neq 0}\sum_{k=0}^{\infty}f^{(k)}(z)\Big|_0^1\left(\frac{1}{2i\pi n}\right)^{k+1}\mathrm{e}^{i2\pi nz}$$

$$f(z)=\int_0^1 f(z)\mathrm{d}z+\sum\nolimits_{k=0}^{\infty}\left[f^{(k)}(1)-f^{(k)}(0)\right]\frac{1}{(k+1)!}B_{k+1}(z) \tag{72}$$

for examples

$$z^m=\int_0^1 z^m\mathrm{d}z+\sum\nolimits_{k=1}^{m}\binom{m}{k-1}\frac{1}{k}B_k(z)\quad 0\le z<1 \tag{73}$$

Now from the relations

$$mz^{m-1}=\partial_z z^m=\left(\mathrm{e}^{\partial_z}-1\right)B_m(z)=B_m(z+1)-B_m(z)$$

$$2z^m=\left(\mathrm{e}^{\partial_z}+1\right)E_m(z)=E_m(z+1)+E_m(z)$$

we see that entire functions may also be expanded into series of Bernoulli as so as into series of Eulerpoly nomials.

For examples, because

$$2B_m(z)=2B_m+\cdots+\binom{m}{k}B_{m-\mathrm{k}}2z^k+\cdots+B_0 2z^m$$

we may write

$$\begin{aligned}2B_m(z)&=\left(1+\mathrm{e}^{\partial_z}\right)\sum_{k=0}^{m-1}\binom{m}{k}\frac{1}{m-k}B_{m-k}E_k(z)+B_0 2z^m\\&=\left(1+\mathrm{e}^{\partial_z}\right)\sum_{k=0}^{m-1}\binom{m}{k}\frac{1}{2-2^{m-k+1}}E_{m-k-1}(0)E_k(z)\\&\quad+\left(E_m(z+1)+E_m(z)\right)\end{aligned} \tag{74}$$

$$\frac{z}{1-z}=z^1+z^2+\cdots=\left(e^{\partial_z}-1\right)\left(B_1(z)+B_2(z)+\cdots+B_m(z)+\cdots\right) \tag{75}$$

$$\frac{2z}{1-z}=E_1(z)+E_1(z+1)+\cdots+\left(E_k(z)+E_k(z+1)\right)+\cdots \tag{76}$$

$$\begin{aligned}\ln(1-z)&=\int\frac{1}{1-z}=z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\\&=\frac{1}{2}\left(1+e^{\partial_z}\right)\left(E_1(z)+\frac{1}{2}E_2(z)+\frac{1}{3}E_3(z)+\cdots+\frac{1}{k}E_k(z)+\cdots\right)\end{aligned}$$

The above relations may be resumed in Table 6.

## 10. Remarks and Conclusions

The main particularity of this work is the use of the Lagrange translation or shift Operator $e^{a\partial_z}$ that is curiously let apart by quasi all authors although this is seen here to be very useful and easy to utilize. From it, Euler polynomials $E_m(z)$ may be presented under the form of an Appell type polynomial which gives directly many algebraic properties concerning $E_m(-z)$, $E_m((2n+1)z)$, $E_m(z+a)$, many relations with sums of powers, many known and new relations with the Bernoulli polynomials $B_m(z)$, noticeably $E_m(B(z))=:B_m(E(z))$; the symbolic relation $E_m(z+y)=:(E(z)+y)^m$, a formula simultaneously giving $E_m(0)$ and $E_m(z)$; relations between $E_{2m-1}(0)$ and Euler-Riemann zeta

Table 6. Series of and on Euler polynomials.

| | |
|---|---|
| $E_{m-1}(z)$ | $=-\frac{m!}{(i\pi)^m}2\sum_{k=-\infty}^{\infty}\frac{1}{(2k+1)^m}e^{i\pi(2k+1)z}$ |
| $E_{2n-1}(z)$ | $=(-1)^n\frac{(2n-1)!}{\pi^{2n}}4\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi z}{(2k+1)^{2n}}$ |
| $E_{2n-1}(0)$ | $=(-1)^n\frac{(2n-1)!}{\pi^{2n}}4\left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\cdots+\frac{1}{(2k+1)^{2n}}+\cdots\right)$ |
| $E_{2n}(z)$ | $=(-1)^n\frac{(2n)!}{\pi^{2n+1}}4\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi z}{(2k+1)^{2n+1}}$ |
| $\zeta(2m)$ | $=(2\pi)^{2m}\frac{m}{1-2^{2m}}\frac{(-1)^{m-1}}{2(2m)!}E_{2m-1}(0)$ |
| $2B_m(z)$ | $=\sum_{k=0}^{m-1}\binom{m}{k}\frac{1}{2-2^{m-k+1}}E_{m-k-1}(0)\left(E_m(z+1)+E_k(z)\right)+\left(E_m(z+1)+E_m(z)\right)$ |
| $2z^m$ | $=E_m(z)+E_m(z+1)$ |
| $\frac{2z}{1-z}$ | $=E_1(z)+E_1(z+1)+E_2(z)+E_2(z+1)+\cdots$ |
| $\ln(1-z)$ | $=\frac{1}{2}\left(1+e^{\partial_z}\right)\left(E_1(z)+\frac{1}{2}E_2(z)+\cdots+\frac{1}{k}E_k(z)+\cdots\right)$ |

functions $\zeta(2m)$ as so as between $\pi^m E_{2m-1}\left(\frac{1}{n}\right)$ and series on $\frac{1}{k^{2m+1}}\sin\frac{k\pi}{n}$, $\frac{1}{k^{2m}}\cos\frac{2k\pi}{n}$ generalizing $\zeta(2m)$ are given.

Last but not least, Fourier series of Euler polynomials and Euler series of functions are discussed and shown.

We think that this work has some significative value for the comprehension of the Euler polynomials. Nevertheless it may be completed with many works on them, one may find in literature for example by Vergara-Hermosilla [16] concerning the properties of Hurwitz polynomials and by Ghisa, D. [17] concerning Euler Product Dirichlet Functions.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Notations

$A\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)z^m$ $\equiv$ Appel-type polynomials;

$B_m(z)$ = Bernoulli polynomials;

$B_m \equiv B_m(0)$ = Bernoulli numbers;

$E_m(z)$ = Euler polynomials;

$E_m = 2^m E_m\left(\frac{1}{2}\right)$ = Euler numbers;

Symbolic relation $E_m(z) =: (E(0)+z)^m$ where $E^m(0) \equiv E_m(0)$;

$S_m(n) = \sum_{k=0}^{n-1} k^m$ = Powers sums;

$\sum_{k=0}^{n-1} (-1)^k z^m$ = Alternating sums of powers;

$\zeta(2m)$ = Euler-Riemann zeta function.