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A Relation between Resolvents of Subdifferentials and Metric Projections to Level Sets

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Abstract

An equation concerning with the subdifferential of convex functionals defined in real Banach spaces and the metric projections to level sets is shown. The equation is compared with the resolvents of general monotone operators, and makes the geometric properties of differential equations expressed by subdifferentials clear. Hence, it can be expected to be useful in obtaining the steepest descents defined by the convex functionals in Banach spaces. Also, it gives a similar result to the Lagrange multiplier method under certain conditions.

Keywords

Subdifferential, Convex Functional, Monotone Operator, Resolvent, Lagrange Multiplier, Banach Space, Metric Projection

1. Introduction

The subdifferentials of lower semi-continuous convex functionals defined on real Banach spaces play important roles in many researches of nonlinear differential equations. In fact, for example, −Laplacian or $-p$-Laplacian operator with a usual boundary condition is the subdifferential of lower-semicontinuous convex functional $\varphi(v) := p^{-1}\int_\Omega |\nabla v|^p \,\mathrm{d}x$ defined in $L^2(\Omega)$.

Throughout this paper, let X be a real Banach space, X^* be the dual space, and $F : X \to X^*$ be the duality mapping of X. Let $\varphi : X \to (-\infty, +\infty]$ be a proper lower-semicontinuous convex functional. The effective domain of φ, which is denoted by $D(\varphi)$, is the following;

$$D(\varphi) := \{x \in X : \varphi(x) < \infty\}.$$

The level set of φ for $\lambda > \inf_X \varphi$ is denoted by $C(\varphi,\lambda)$, *i.e.*,

$$C(\varphi,\lambda) := \{x \in D(\varphi) : \varphi(x) \le \lambda\}.$$

Since φ is lower-semicontinuous and convex, the level sets are closed and convex in *X*.

The subgradients of φ at $x \in D(\varphi)$ are the elements $f \in X^*$ satisfying

$$\varphi(x) \le \varphi(\xi) + (f, x-\xi), \quad \forall \xi \in X.$$

The subdifferential of φ at x is the set of all subgradients of φ at x, and denoted by $\partial\varphi(x)$, *i.e.*,

$$\partial\varphi(x) := \{f \in X^* : f \text{ is a subgradient of } \varphi \text{ at } x\},$$

$$D(\partial\varphi) := \{x \in D(\varphi) : \partial\varphi(x) \ne \emptyset\}.$$

It is known that $\partial\varphi : D(\partial\varphi) \subset X \to X^*$ is a maximal monotone operator ([1] [2] [3]).

For every closed convex subset $C \subset X$, the metric projection from *X* onto *C*, which is denoted by Proj_C, is defined as below.

$$\mathrm{Proj}_C x := \left\{z \in C : \|x - z\| = \min_{\zeta \in C} \|x - \zeta\|\right\}, \quad x \in X.$$

As is seen in **Figure 1**, in general, $\mathrm{Proj}_C x$ is not unique. In this paper, we denote arbitral $z \in \mathrm{Proj}_C x$ by $\mathrm{Proj}_C x$ for simplicity.

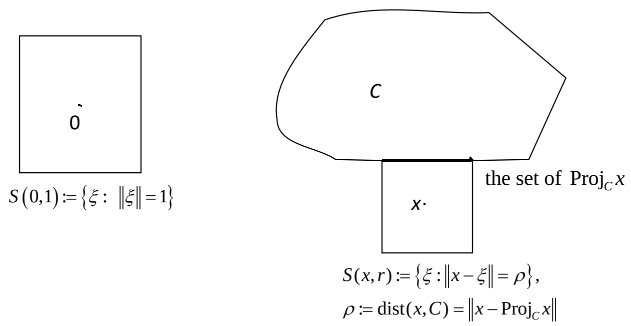

Figure 1. An example where $\mathrm{Proj}_C x$ is not unique.

If *X* is reflexive, then $\mathrm{Proj}_C x \ne \emptyset$ for $\forall x \in X$. (In fact, there is a sequence $\{z_n\} \subset C$ such that $\|x - z_n\| \to \min_{\zeta \in C} \|x - \zeta\|$. Since *X* is reflexive, the bounded subset $\{z_n\}$ is weakly compact. Thus, some subsequence of it converges to $\exists z \in X$. By the closed convexity of *C*, $z \in C$.) If *X* is strictly convex, then $\mathrm{Proj}_C x$ is either single or empty.

In general, if both *X* and X^* are strictly convex and reflexive, then every maximal monotone operator $A : D(A) \subset X \to X^*$ satisfies the following ([1]).

(i) For $\forall \lambda > 0$ (equivalently, for $\exists \lambda > 0$), $R(A + \lambda F) = X^*$.

(ii) For $\forall \lambda > 0$ and $\forall x \in X$, there is a unique solution $x_\lambda \in D(A)$ to the relation

$$F(x_\lambda - x) + \lambda A x_\lambda \ni 0. \tag{1.1}$$

(iii) For $\lambda > 0$ and $x \in X$, let $x_\lambda \in D(A)$ be the unique solution of (1.1) and put

$$J_\lambda x := x_\lambda, \quad A_\lambda x := -\frac{1}{\lambda} F(x_\lambda - x). \tag{1.2}$$

Then $J_\lambda : X \to X$ and $A_\lambda : X \to X^*$ satisfy the following;
A_λ is single valued and monotone, J_λ and A_λ are bounded,
both $\|A_\lambda x\| \le |Ax|$ and $\lim_{\lambda \to 0} J_\lambda x = x$ hold for $\forall x \in D(A)$, and so on.

The subdifferential operators satisfy more properties other than (i) to (iii) above. For instance, the following (A) (B) are known.

(A) If *X* is a real Hilbert space *H*, then for $\forall x \in D(\varphi) \setminus C(\varphi, \lambda)$, relations

$$\mathrm{Proj}_{C(\varphi,\lambda)} x \in D(\partial\varphi), \quad \mathrm{Proj}_{C(\varphi,\lambda)} x - x + \mu \partial\varphi\left(\mathrm{Proj}_{C(\varphi,\lambda)} x\right) \ni 0 \tag{1.3}$$

hold with $\exists \mu \equiv \mu(x) > 0$.

Although μ in (1.3) is depending on x, while λ in (1.1) is common to all $x \in X$, (1.3) seems sufficiently useful to obtain solutions of $(\mathrm{d}/\mathrm{d}t)u \in -\partial\varphi(u)$ in *H*. (1.3) is proved without using the above properties (i) to (iii), but geometric properties of convex functionals' graphs in Hilbert spaces (see [3]).

(B) Let *g* be a given smooth functional satisfying $g(x) \ge 0$ on *X*. Put

$$K := \{x \in X : g(x) = 0\},$$

and suppose that *K* is convex and closed in *X*. Let I_K be the indicator functional of *K*, *i.e.*,

$$I_K(x) := 0, \text{ if } x \in K, \ := +\infty, \text{ otherwise.}$$

Then, I_K is a lower-semicontinuous convex functional and its subdifferential is below.

$$\partial I_K(x_0) = \{y \in X^* : (y, x_0 - \xi) \ge 0 \text{ for } \forall \xi \in K\}, \quad D(\partial I_K) = K.$$

Let φ_0 be a proper lower-continuous convex functional defined on *X*. Then, the convex functional

$$\varphi := \varphi_0 + I_K$$

is useful for conditional extremum problem on *K*.

For example (cf. [4], obstacle problems), let $X := L^2(\Omega)$,
$\varphi_0(x) := 2^{-1} \int_\Omega |\nabla x(\omega)|^2 \,\mathrm{d}\omega$, and $g(x) := 2^{-1} \int_\Omega \left[\{x(\omega) - k(\omega)\}_+\right]^2 \mathrm{d}\omega$, where $\alpha_+ := \max\{\alpha, 0\}$ and $k : \Omega \to \mathbf{R}$ is smooth. Then,
$K = \{x \in L^2(\Omega) : x(\omega) \le k(\omega), \text{a.e. } \omega \in \Omega\}$. Since *K* is closed and convex, I_K is a lower semi-continuous convex functional, and
$\partial I_K(x_0) = \left\{y \in L^2(\Omega) : \int_\Omega y(\omega)(x_0(\omega) - k(\omega))\mathrm{d}\omega \ge 0\right\}$ for $x_0 \in D(\partial I_K) \equiv K$.
Thus, $\varphi := \varphi_0 + I_K$ is useful in the obstacle problem $x(\omega) \le k(\omega)$.

Concerning the above (A) (B), our theorem and remarks show the following.

(A)' Same result of (A) holds under more general assumptions; (i) *X* is an arbitral real Banach space, (ii) $x \in \overline{D(\varphi)} \setminus C(\varphi, \lambda)$, and (iii) $\mathrm{Proj}_{C(\varphi,\lambda)} x$ exists.

Here, as is mentioned above, if X is reflexive, then assumption (iii) always holds.

It seems that $F(\mathrm{d}u/\mathrm{d}t) \in -\partial\varphi(u)$ in X is not solved even if X is reflexive. The author hopes that our theorem will contribute to solving this problem.

(B)' Let $\varphi := \varphi_0 + I_K$ and $x \notin K$. Then, in general, $\mathrm{Proj}_{C(\varphi,\lambda)}x$ may fail to satisfy (1.3) in H, or, in the case of Banach space X,

$$\mathrm{Proj}_{C(\varphi,\lambda)}x \in D(\partial\varphi), \quad F\left(\mathrm{Proj}_{C(\varphi,\lambda)}x - x\right) + \mu\partial\varphi\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right) \ni 0 \tag{1.4}$$

(see Remark 2.2).

Suppose that $\mathrm{Proj}_{C(\varphi,\lambda)}x$ satisfies (1.4), and that codimension of K is finite. In general, as is seen in Figure 2 and Figure 3, if one takes arbitral $h \in F\left(\mathrm{Proj}_{C(\varphi,\lambda)}x - x\right)$, then h may falt to satisfy (1.4). If h satisfies (1.4), then $\mathrm{Proj}_{C(\varphi,\lambda)}x + h^{-1}(0)$ is a kind of hyperplane tangent of $C(\varphi,\lambda)$ at $\mathrm{Proj}_{C(\varphi,\lambda)}x$, because every $y \in \partial\varphi\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right)$ is the same.

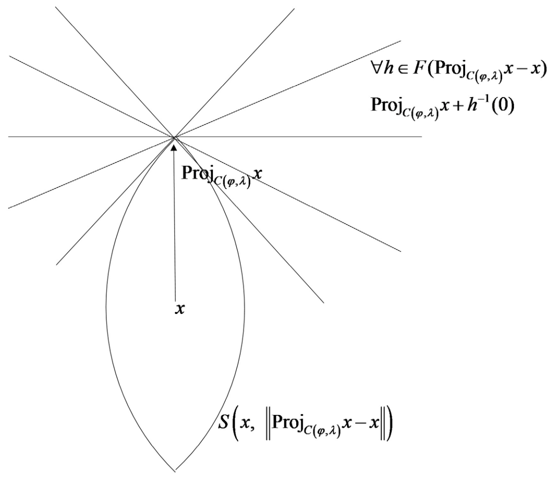

Figure 2. In spaces where the unit sphere has corners, $F(\mathrm{Proj}_C x - x)$ may not be unique.

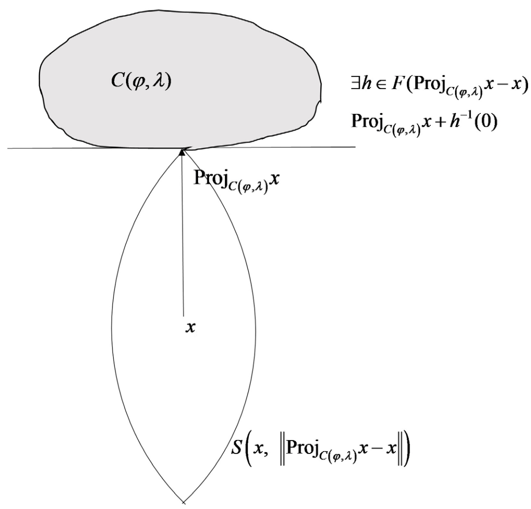

Figure 3. For $h \in F(\mathrm{Proj}_C x - x)$ that satisfies (1.4), $\mathrm{Proj}_C x + h^{-1}(0)$ is a kind of hyperplane tangent of $C(\varphi,\lambda)$, and also of $B(x, \|\mathrm{Proj}_C x - x\|)$.

In this paper, for simplicity, $F\left(\mathrm{Proj}_{C(\varphi,\lambda)}x-x\right)$ denotes $\exists h\in F\left(\mathrm{Proj}_{C(\varphi,\lambda)}x-x\right)$ such that (1.4) holds. Then, since

$$\partial\left(\varphi_0+I_K\right)\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right)\subset\partial\varphi_0\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right)+\partial I_K\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right)$$

holds by $\mathrm{codim}(K)<+\infty$, (1.4) implies that $\exists y_1\in\partial\varphi_0\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right)$, $\exists y_2\in\partial I_K\left(\mathrm{Proj}_{C(\varphi,\lambda)}x\right)$, $\exists r_1,r_2\in\mathbf{R}$ such that

$$r_1y_1+r_2y_2=F\left(x-\mathrm{Proj}_{C(\varphi,\lambda)}x\right). \tag{1.5}$$

On the other hand, let $x_0\in D(\partial\varphi_0)$ be obtained by Lagrange multiplier method in the problem of minimizing $\varphi_0(x)$ under the condition $\psi(x)=0$ with smooth function ψ. Then, Lagrange multiplier method implies that $\exists y_3\in\partial\varphi_0(x_0)$, $\exists r_3\in\mathbf{R}$ such that

$$y_3+r_3d\psi(x_0)=0. \tag{1.6}$$

Put $g(x):=\{\psi(x)\}^2$ and $K:=g^{-1}(0)$. Suppose that K is convex. Then, by (1.6), for $\exists y_4\in\partial I_K(x_0)$, $\exists r_4\in\mathbf{R}$,

$$y_3+r_4y_4=0. \tag{1.7}$$

Hence, if we put $F\left(x-\mathrm{Proj}_{C(\varphi,\lambda)}x\right)=0$ in (1.5), then the form of (1.5) is the same as (1.7).

2. Results

As is mentioned in Section 1, let X be a real Banach space with duality mapping F, and φ be a proper lower-continuous convex functional defined in X. Fix $\forall\lambda>\inf_X\varphi$. In the following, we denote the metric projection $\mathrm{Proj}_{C(\varphi,\lambda)}$ by P, and Px means arbitral element of Px, for simplicity.

Let $x\in X\setminus C(\varphi,\lambda)$ be such that Px exists. Then, $B(x,\|x-Px\|)$ has inner points, and any inner point of $B(x,\|x-Px\|)$ does not included in $C(\varphi,\lambda)$. Thus, Hahn-Banach theorem implies that $\exists h\in F(x-Px)$ satisfying

$$\begin{aligned}&C(\varphi,\lambda)\subset\{\xi\in X:(h,\xi-Px)\le 0\}\\ &\text{and } B(x,\|x-Px\|)\subset\{\xi\in X:(h,\xi-Px)\ge 0\}\end{aligned} \tag{2.1}$$

(cf. [5]).

Fix an arbitral $h\in F(x-Px)$ such that (2.1) holds.

Theorem 2.1. *Suppose that*

$$D(\varphi)\cap\{\xi\in X:(h,\xi-Px)>0\}\neq\varnothing. \tag{2.2}$$

Then,

(i) $Px\in D(\partial\varphi)$.

(ii) *The inclusion*

$$\left\{\alpha\frac{h}{\|h\|}:\alpha\in[\alpha_-,\alpha_+]\right\}\subset\partial\varphi(Px) \tag{2.3}$$

holds with α_-,α_+ *defined by as blow;*

$$\kappa(\xi):=\inf_{\xi+h^{-1}(0)}\varphi.$$

$$\alpha_- := \lim_{t\uparrow 0} \frac{\kappa\left(Px + t\dfrac{x-Px}{\|x-Px\|}\right) - \kappa(Px)}{t},$$

$$\alpha_+ := \lim_{t\downarrow 0} \frac{\kappa\left(Px + t\dfrac{x-Px}{\|x-Px\|}\right) - \kappa(Px)}{t}.$$

Remark 2.1. *Assumption* (2.2) *holds if either*

$$x \in D(\varphi) \quad \textit{or} \quad D(\varphi) \quad \textit{is dense in } X.$$

Hence, assertion (A)' *mentioned in Section* 1 *follows from Theorem* 2.1.

Remark 2.2. *In Theorem* 2.1, *the assumption* (2.2) *is needed. In fact, if* (2.2) *does not hold, then there are two types of examples as below.*

(i)' $Px \notin D(\partial\varphi)$

(ii)' $Px \in D(\partial\varphi)$ *holds, but* (2.3) *does not hold.*

The examples of (i)' (ii)' *are given in Section* 3, *and assertion* (B)' *in Section* 1 *concerns with these examples.*

3. Proofs of Results

3.1. Proof of Theorem 2.1

We verify convexity of κ. Fix $\forall w_1, w_2 \in X$ and $\forall t \in (0,1)$. For $\forall \varepsilon > 0$, $\exists y_i \in w_i + h^{-1}(0)$ $(i = 1,2)$ such that

$$\varphi(y_i) - \varepsilon < \inf\left\{\varphi(y) : y \in w_i + h^{-1}(0)\right\} \equiv \kappa(w_i).$$

Then, the relation $ty_1 + (1-t)y_2 \in tw_1 + (1-t)w_2 + h^{-1}(0)$ implies

$$\begin{aligned} t\kappa(w_1) + (1-t)\kappa(w_2) &> t\varphi(y_1) + (1-t)\varphi(y_2) - \varepsilon \\ &\geq \varphi(ty_1 + (1-t)y_2) - \varepsilon \geq \kappa(tw_1 + (1-t)w_2) - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitral, κ is convex.

Now we know that both κ and φ are convex, and $\kappa(w) \leq \varphi(w)$ for $\forall w \in X$. Suppose that

$$\kappa(Px) = \varphi(Px) \ (=: \lambda). \tag{3.1}$$

Then, by definition of subdifferential,

$$\partial\kappa(Px) \subset \partial\varphi(Px).$$

On the other hand, by definition of κ,

$$\partial\kappa(Px) = \left\{\alpha\frac{h}{\|h\|} : \alpha_- \leq \alpha \leq \alpha_+\right\}.$$

Thus, the proof of Theorem 2.1 is completed if (3.1) is shown.

To verify (3.1) by contradiction, suppose that (3.1) does not hold. Then, by definition of κ, $\exists w_0 \in Px + h^{-1}(0)$ with $\varphi(w_0) < \lambda$. By (2.2), $\exists y_0 \in D(\varphi) \cap \{w : (h, w - Px) > 0\}$. Take $t \in (0,1)$ sufficiently small such that

$$t\varphi(y_0) + (1-t)\varphi(w_0) < \lambda.$$

Then, since φ is convex,

$$ty_0+(1-t)w_0 \in C(\varphi,\lambda)\cap\{w:(h,w-Px)>0\}. \tag{3.2}$$

On the other hand, we have verified (2.1) which contains the inclusion $C(\varphi,\lambda)\subset\{w\in X:(h,w-Pv)\le 0\}$. This is a contradiction to (3.2). Therefore, Theorem 2.1 is proved.

3.2. Example of (i)' in Remark 2.2

Suppose $\dim X=\infty$. Take $\varphi_0: X\to[0,\infty]$ which satisfies

$$D(\varphi_0) \text{ is dense in } X \text{ and } C(\varphi_0,r) \text{ are compact.} \tag{3.3}$$

Since $\dim X=\infty$, (3.3) yields $D(\varphi_0)\setminus D(\partial\varphi_0)\ne\varnothing$. For example, $X:=L^2(\Omega)$ with bounded $\Omega\subset\mathbf{R}^n$, $\varphi_0(x):=2^{-1}\int_\Omega|\nabla x(\omega)|^2\,\mathrm{d}\omega$, $D(\varphi_0):=H_0^1$.

Fix any $x_1\in X\setminus\{0\}$ and $y\in F(x_1)$, where F is the duality mapping of X. Define the nonnegative convex functional g by $g(\xi):=\{(y,\xi)\}^2$, $\xi\in X$. Put

$$\varphi:=\varphi_0+I_K \text{ with } K:=g^{-1}(0)\equiv y^{-1}(0)$$

Then, since K is still an infinite dimensional linear subspace, same properties of (3.3) hold if we take $\varphi_0|_K$ and K instead of φ_0 and X, respectively. Take x_0 such that

$$x_0\in D(\varphi_0|_K)\setminus D(\partial(\varphi_0|_K)),\ (\varphi_0|_K)(x_0)<\lambda.$$

Then, for $x:=x_0+x_1$,

$$Px=x_0\notin D(\partial\varphi)$$

holds as is seen in Figure 4.

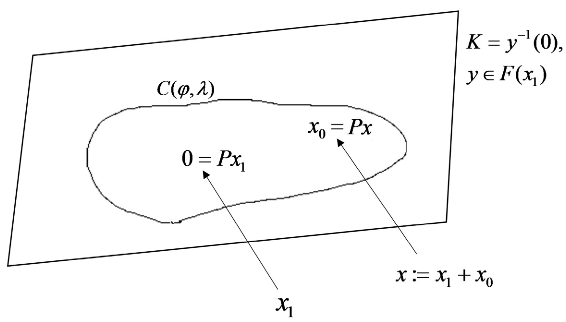

Figure 4. Since K is a hyperplane of X and $\mathrm{Proj}_C x_1=0$, for every $x_0\in K$ and $r\in\mathbf{R}$, one has $\mathrm{Proj}_K(rx_1+x_0)=x_0$.

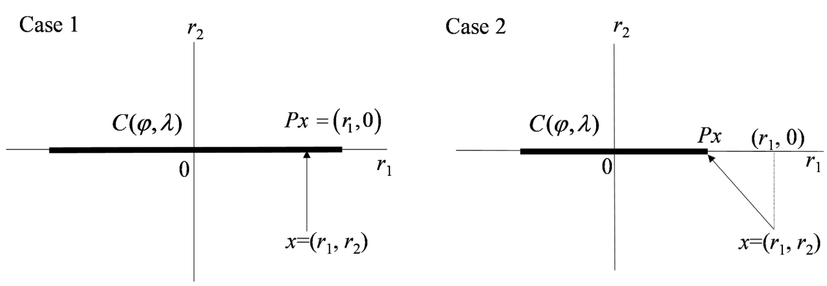

Figure 5. How $\mathrm{Proj}_{C(\varphi,\lambda)}x$ is determined differs depending on λ.

3.3. Example of (ii)' in Remark 2.2

Let $X := \mathbf{R}^2$ with $\|(r_1, r_2)\| := \sqrt{r_1^2 + r_2^2}$. Put

$$\varphi_0((r_1, r_2)) := \|(r_1, r_2)\|^2,\ g((r_1, r_2)) := r_2^2,\ K := g^{-1}(0) = \{(r_1, r_2) : x_2 = 0\}.$$

By definition, $\varphi := \varphi_0 + I_K$ is the following.

$$\varphi((r_1, r_2)) = r_1^2,\ \text{if } r_2 = 0; \quad = +\infty, \text{otherwise},$$

and

$$C(\varphi, \lambda) = \left\{(r_1, 0) : r_1 \in \left[-\sqrt{\lambda}, \sqrt{\lambda}\right]\right\},$$

$$\partial\varphi((r_1, 0)) = \{(2r_1, \rho) : \rho \in \mathbf{R}\},\ D(\partial\varphi) = K.$$

Let $x := (r_1, r_2)$ with $r_1 \neq 0$. Then, as is seen in **Figure 5**, the following cases hold. Case 2 satisfies (2.2).

Case 1. If $r_1^2 < \lambda$, then (1.3) does not hold. In fact, for $\forall \mu \in \mathbf{R}$,

$$x - Px = (0, r_2) \notin \mu\partial\varphi(Px) \equiv \{\mu(2r_1, \rho) : \rho \in \mathbf{R}\}.$$

Case 2. If $r_1 > \sqrt{\lambda}$, then (1.3) with $\mu = \left(r_1 - \sqrt{\lambda}\right) / \left(2\sqrt{\lambda}\right)$ holds, since

$$Px = \left(\sqrt{\lambda}, 0\right),\ x - Px = \left(r_1 - \sqrt{\lambda}, r_2\right),\ \partial\varphi(Px) = \left\{\left(2\sqrt{\lambda}, \rho\right) : \rho \in \mathbf{R}\right\}.$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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