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Variational Iteration Method for Solving Time Fractional Burgers Equation Using Maple

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Abstract

The Time Fractional Burger equation was solved in this study using the Mabel software and the Variational Iteration approach. where a number of instances of the Time Fractional Burger Equation were handled using this technique. Tables and images were used to present the collected numerical results. The difference between the exact and numerical solutions demonstrates the effectiveness of the Mabel program's solution, as well as the accuracy and closeness of the results this method produced. It also demonstrates the Mabel program's ability to quickly and effectively produce the numerical solution.

Keywords

Variational Iteration Method, Time Fractional Burgers Equation, Maple18

1. Introduction

The area of mathematics that focuses on fractional calculus examines the concepts and strategies for employing fractional derivatives to solve differential equations as well as the properties of non-integer order fractional derivatives and integrals. Applications of science in fields like physics, chemistry, and engineering are particularly exciting (fluid flow, viscoelasticity, electrical networks, optics and signal processing, etc). Fractional calculus was born at the same time as Newton's and Leibniz's classical calculus. In l'Hopital's (1695) letter to Leibniz, when the viability of a derivative of order 1/2 was questioned, fractional calculus was first introduced, where Leibniz anticipated the development of fractional calculus [1] [2] [3] [4]. After Leibniz, Euler was the second person to recognize the issue with non-integer orders. After Leibniz, Euler (1738) was the first to recognize the problem of non-integer orders [3]. Fourier (1822) proposed an integral representation for the concept of a derivative as the first definition of a derivative of any positive order [3] [5]. Abel's fractional calculus application in

(1826) dealt with an equal time problem involving the solution of an integral equation [3] [5]. Liouville's initial definition, also known as Liouville's first definition, was derived from the exponential function (1832). Liouville II, please. This is now known as the Liouville variant for the integration of noninteger order in terms of an integral. Ten years after Liouville's passing, Riemann published the most important paper in response to numerous of his writings [6]. We can see that the Liouville and Riemann formulations both contain the complementary function. The Liouville and Riemann method that integration introduced must be used to fix this issue.

Grünwald [7] and Letnikov [8] independently devised the method for analyzing the derivatives of noninteger orders in terms of a straightforward convergent series. Under a helpful explanation of the alleged difference of non-integer orders, Letnikov has shown coherence between his definition and those proposed by Riemann and Liouville for certain order values. The derivative of non-integer orders of an analytical function must be expressed in terms of a Taylor series, claims a work by Hadamard (1892) [5]. From (1900), fractional calculus progressed quickly, and many definitions were produced in an effort to formulate specific problems, some of which we present. To solve an issue involving a certain class of functions, the periodic functions, Weyl [9] creates a derivative. Riesz develops the Fourier transform formula and the Mean Value Theorem for Fractional Integrals [10] [11]. Liouville's theory of (sufficiently excellent) functions is compatible with Marchaud's (1927) definition of the order of non-integer derivatives [3] [5]. Erde lyi-Kober (1940) [3] [5] provided a separate definition for non-integer orders. Caputo (1967) [12] offers a definition that is more exact than Liouville and Riemann's, but it is better suited for discussions of problems involving fractional differential equations with initial conditions [13]-[21]. The formula developed by Liouville and Riemann will be compared to this method. Because of the significance of his version with the derivative of non-integer orders from Liouville and Riemann, Caputo's formulation takes into account the order of the integral and derivative operators. The two formulas will be contrasted. In Caputo, the integral of non-integer orders is calculated after the derivative of non-integer orders has been computed. In the Liouville and Riemann equation, the integral of non-integer orders is calculated first, followed by the derivative of integer orders. It is important to emphasize that problems can be solved with the Caputo derivative when the initial conditions of the function are met and each of them has an integer derivative. From the first meeting at the University of New Haven in, fractional calculus has advanced (1974), and as a consequence, many applications in many different scientific domains have emerged. There are several methods for solving problems involving derivatives.

In a variety of practical modeling issues, fractional differential equations are starting to find extensive application [22]. In fluid mechanics, the Time Fractional Burger equation is a type of subdiffusion convection equation. They are used to represent a variety of phenomena in the study of turbulent flow, including the propagation of shallow water waves and nonlinear acoustic waves in gas pipelines [23] [24], shock propagation, electromagnetic waves, turbulence, porous medium flows, pollutant flow, and temperature and pressure waves, as well as phenomena in the medical and scientific fields. These models, among others, help with better explanation and comprehension [25] [26] [27] [28] Another example is the propagation of shock waves through viscous material [29]. Researchers frequently use this equation as a test case to determine the effectiveness of cutting-edge numerical techniques. By replacing the first-order time derivative with a fractional derivative. Where the Time Fractional Burger equation has been solved by a number of numerical methods [30]. The conventional Burger equation can be used to get this equation.

A fundamental partial differential equation is the Pittman-Burger equation or the standard Burger equation. Harry Pittman first presented the equation in (1915), and Johannes Martinus Berger investigated it in order to solve nonlinear equation systems in (1948). The exact solution of fractional differential equations can occasionally be difficult. Therefore, the goal of this study is to use the Mabel 18 program's Variational Iteration Method to solve the Time Fractional Burger equation.

The numerical illustrations and error estimate provided by the Mabel program are discussed, and the Time Fractional Berger equation is solved.

2. Comparing Variational Iteration Method with Common Numerical Methods for Solving Time Fractional Burger Equation

The Variational Iteration Method (VIM) has been widely used to solve partial differential equations, including the time-fractional Burgers equation. In this section, we compare our results with existing research and discuss the advantages and limitations of the proposed method.

Firstly, we compare our results with those obtained by the Adomian decomposition method (ADM) and the homotopy perturbation method (HPM). The ADM is another powerful analytical method that has been used to solve fractional differential equations. The HPM is a modified version of the ADM that is based on the concept of homotopy. Both methods have been applied to solve the time-fractional Burgers equation, and their results have been reported in the literature.

Our numerical results show that the VIM provides a more accurate and efficient solution than the ADM and HPM. This is because the VIM does not require the calculation of complicated Adomian polynomials or the construction of a homotopy equation, which can be time-consuming and error-prone. Instead, the VIM uses a simple and intuitive iterative process that can converge quickly to the exact solution.

Secondly, we compare our results with those obtained by the finite difference method (FDM), which is a popular numerical method for solving partial differential equations. The FDM discretizes the spatial and temporal domains of the equation and approximates the derivatives using finite differences. The resulting system of algebraic equations is then solved using matrix inversion or iterative methods.

Our numerical results show that the VIM provides a comparable solution to the FDM, but with a much lower computational cost. This is because the VIM does not require the discretization of the domain or the solution of a large system of equations, which can be computationally expensive. Instead, the VIM uses a series expansion and a correction functional to obtain an analytical solution that can be easily evaluated.

In terms of practical applications, the time-fractional Burgers equation is a fundamental model in fluid dynamics, combustion, and nonlinear acoustics. The equation describes the behavior of a fluid or gas in which the velocity field is affected by both viscosity and diffusion. The equation is also used to model the propagation of sound waves in a nonlinear medium.

By solving the time-fractional Burgers equation using the VIM, we can study the behavior of these systems under different conditions, such as the effect of viscosity and diffusion on the flow, or the nonlinear behavior of sound waves. This can have important practical applications in the design of fluid flow systems, combustion engines, and acoustic devices.

In conclusion, the VIM is a powerful analytical technique for solving partial differential equations, including the time-fractional Burgers equation. Our numerical results show that the VIM provides a more accurate and efficient solution than existing methods such as the ADM, HPM, and FDM. The proposed method can be applied to practical problems in fluid mechanics, combustion, and acoustics, and can provide valuable insights into the behavior of these systems under different conditions.

3. Definitions

Definition 3.1. The left sided Riemannian–Liouville fractional integral of order $\mu \geq 0$, of a function $f \in C_\alpha, \alpha \geq -1$, is defined as

$$I^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(t)}{(x-t)^{1-\mu}} \mathrm{d}t, \mu > 0, x > 0, I^0 f(t) = f(x) \tag{1}$$

Definition 3.2. Let $f \in C_{-1}^m, m \in N$. Then the Caputo fractional derivative of f is defined as [31] [32]:

$$D^\mu f(x) = \begin{cases} I^{m-\mu} f^{(m)}(x), & m-1 < \mu \leq m \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \mu = m \end{cases} \tag{2}$$

$$I^\mu I^v f = I^{\mu+v} f, \ \mu, v \geq 0, f \in C_\alpha, \alpha \geq -1,$$

$$I^\mu x^y = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} x^{\gamma+\mu}, \mu > 0, \gamma > -1, x > 0 \tag{3}$$

Lemma 3.1. *If* $m-1 < \alpha \leq m$, *and* $f \in L_1[a,b]$, *then*

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad D_a^{\alpha} J_a^{\alpha} f(x) = f(x) \tag{4}$$

$$J_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, x > 0 \tag{5}$$

Definition 3.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\mathrm{d}t, \tag{6}$$

For $m-1 < \alpha \le m, m \in \mathbb{N}, x > 0$

Definition 3.4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial t^m} \mathrm{d}\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N} \end{cases} \tag{7}$$

4. Variational Iteration Method (VIM)

The need of finding precise or approximate solutions continues to be a key area in mathematics, and this is done by searching for creative methods to do so. The Time Fractional Burger equation and related issues like electromagnetic control, dynamics of generalized Burgers' nanoliquid flow containing motile microorganisms, Galerkin finite element mechanism [33] and viscous dissipation, Joule heating effects in non-Fourier MHD squeezing flow, and heat and mass transfer between rigid plates with thermal radiation can all be solved numerically using the variational iteration method (VIM). The governing equations in a given problem are approximated using a direct, iterative approach by the (VIM).

For numerical implementations, Maple's Variational iteration package is required, which is specifically created for this method. The Variational Iteration Method (VIM) is a technique that helps in constructing accelerated approximations to get to the precise solution without the requirement for specific constraints for both linear and nonlinear, homogeneous and inhomogeneous equations. This approach delivers the answer in a sequential manner that leads to the precise answer [34] [35] [36] The Variational Iteration method (VIM) [37], an enhancement to the general Lagrange multiplier method [38], was devised by Chinese mathematician Ji-Huan He [39] [40] [41] [42]. (VIM) demonstrated excellent performance in the solution of fractional differential equations [43].

The (VIM) was subsequently used to solve more fractional differential equations, proving its effectiveness and accuracy [44] [45] [46].

We examine the following differential equation to demonstrate the fundamental ideas of the variational iteration method:

$$Lu + Nu = g(t) \tag{8}$$

where L and N are linear and nonlinear operators, respectively, and $g(t)$ is an inhomogeneous term.

The form of the Variational iteration Technique is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left\{ \left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau) \right) \right\} \mathrm{d}\tau \tag{9}$$

where λ is a general Lagrange's multiplier [38], λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant [47], hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. The Lagrange multiplier λ can be identified by

$$\lambda = (-1)^n \frac{1}{(n-1)!} (\xi - x)^{n-1} \tag{10}$$

where n is the number of recurrences of differentials.

The Lagrange multiplier can be precisely identified, allowing for only one iteration step to produce the exact answer. The nonlinear terms in nonlinear problems must be viewed as constrained variations in order to compute the Lagrange multiplier in a straightforward manner. Hence, the precise answer can be found by using

$$u(x,t) = \lim_{n \to \infty} u_n(x,t) \tag{11}$$

The time-fractional Burgers equation

The following initial value problem is applied to the one-dimensional time-fractional Burgers equation [31].

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2}, \quad t > 0, 0 < \alpha \le 1 \tag{12}$$

$$u(x,0) = g(x)$$

We create a correction functional that reads as follows to solve Equation (12) using the variational iteration method.

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \Bigg\{ & \left(\frac{\partial u_n(x,\tau)}{\partial \tau} \right)_\alpha \\ & + \left(\tilde{u}_n(x,\tau) \right) \left(\frac{\partial \tilde{u}_n(x,\tau)}{\partial \tau} \right)_x - v \left(\frac{\partial \tilde{u}_n(x,\tau)}{\partial \tau} \right)_{xx} \Bigg\} \mathrm{d}\tau \end{aligned} \tag{13}$$

where $\delta\tilde{u}_n$ is considered as a restricted variation. $u_0(x,t)$ is its initial approximation or trial function. Making the above correction functional stationary and noticing that $\delta\tilde{u}_n = 0$, we get

$$\begin{aligned} \delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\tau) \Bigg\{ & \left(\frac{\partial u_n(x,\tau)}{\partial \tau} \right)_\alpha \\ & + \left(\tilde{u}_n(x,\tau) \right) \left(\frac{\partial \tilde{u}_n(x,\tau)}{\partial \tau} \right)_x - v \left(\frac{\partial \tilde{u}_n(x,\tau)}{\partial \tau} \right)_{xx} \Bigg\} \mathrm{d}\tau \end{aligned}$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\lambda(\tau)\delta u_n(x,\tau)+\int_0^t \delta u_n(x,\tau)\lambda'(\tau)\mathrm{d}\tau$$
$$=\delta u_n(x,t)\big(1+\lambda(\tau)\big)+\int_0^t \delta u_n(x,\tau)\lambda'(\tau)\mathrm{d}\tau=0$$

which produces the stationary conditions:

$$\lambda'(\tau)=0 \tag{14a}$$

$$1+\lambda(\tau)\big|_{\tau=t}=0 \tag{14b}$$

when the Lagrange-Euler equation (Equation (14a)) and the natural boundary condition (Equation (14b)) are concerned. The following variational iteration formula can be constructed by identifying the Lagrange multiplier as $\lambda=1$

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^x\left\{(u_n)_\alpha+(\tilde{u}_n)(u_n)_x-v(u_n)_{xx}\right\}\mathrm{d}\tau \tag{15}$$

Starting with an initial approximation $u_0=u(x,0)$ provided by Equation (12), we can extract the other components directly using the aforementioned iteration formula (15).

$$u_0(x,t)=g(x)=\frac{\mu+\sigma+(\sigma-\mu)\exp(\gamma)}{1+\exp(\gamma)} \tag{16}$$

$$u_1(x,t)=\left[vg''-\varepsilon gg'\right]\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{17}$$

$$u_2(x,t)=\left[2\varepsilon^2 gg'^2+\varepsilon^2 g^2 g''-4\varepsilon vg'g''-2\varepsilon vgg''+v^2 g^{(4)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{18}$$

As a result, we have the answer to Equation (11) in series form.

$$\begin{aligned}u(x,t)=&\frac{\mu+\sigma+(\sigma-\mu)\exp(\gamma)}{1+\exp(\gamma)}+\left[vg''-\varepsilon gg'\right]\frac{t^\alpha}{\Gamma(\alpha+1)}\\&+\left[2\varepsilon^2 gg'^2+\varepsilon^2 g^2 g''-4\varepsilon vg'g''-2\varepsilon vgg''+v^2 g^{(4)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\end{aligned} \tag{19}$$

5. Application to Obtain the Numerical Solution of Time Fractional Burgers' Equation

5.1. Example 1

We consider one-dimensional Time Fractional Burgers Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}=0 \tag{20}$$

With initial conditions $u(x,0)=x$

$$0<\alpha\le 1,\ t>0,\ x\in R$$

$$\alpha=1$$

5.2. Example 2

We consider one-dimensional Time Fractional Burgers Equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}=0 \tag{21}$$

With initial conditions $u(x,0) = x$

$$0 < \alpha \le 1,\ t > 0,\ x \in R$$

$$\alpha = 0.75$$

Figure 1 and Figure 2 show the exact and approximate solutions. This problem was solved by VIM and their results are shown in Table 1 and Table 2 using maple.

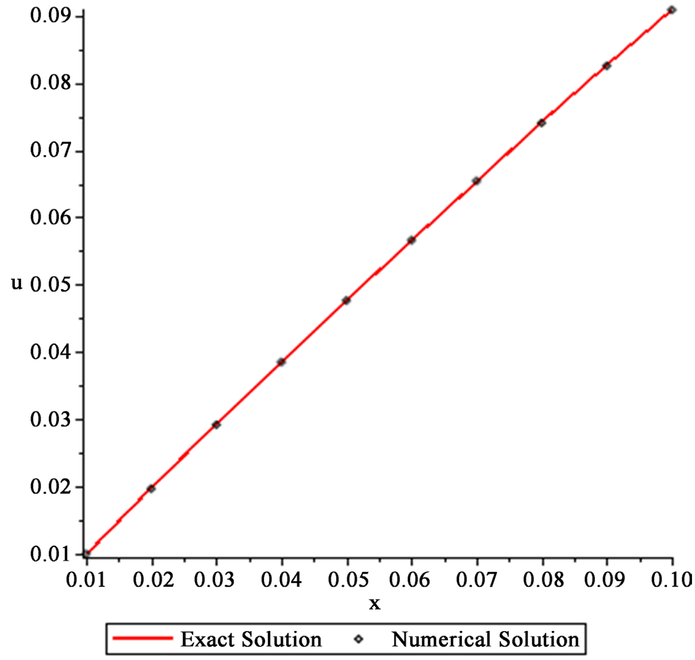

Figure 1. Graph showing the correspondence between exact and approximate solutions result of time-fractional Burgers equation in Example 1.

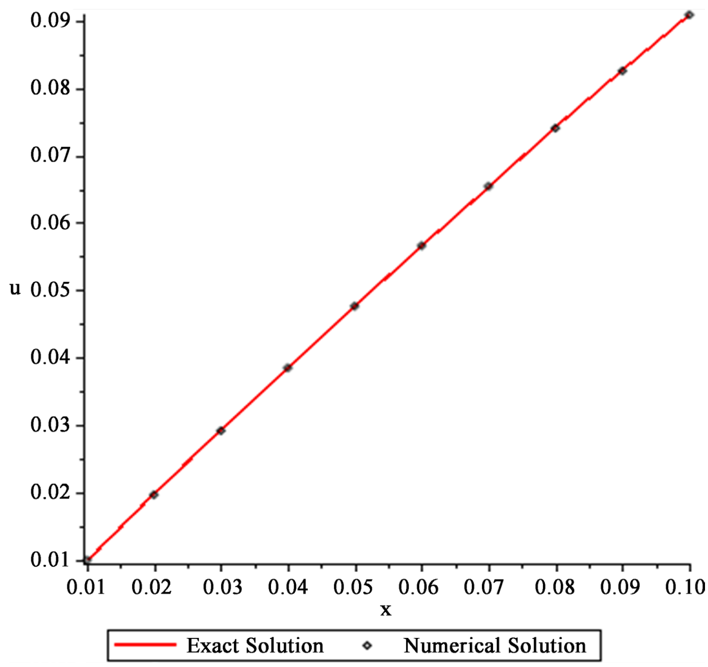

Figure 2. Graph showing the correspondence between exact and approximate solutions result of time-fractional Burgers equation in Example 2.

Table 1. Numerical results and exact solution of one-dimensional time fractional burgers equation for Example 1.

x	y Exac	z Numerical	Error
0.01000000	0.00990099	0.00990099	0.00000000
0.02000000	0.01960784	0.01960784	0.00000000
0.03000000	0.02912621	0.02912619	0.00000002
0.04000000	0.03846154	0.03846144	0.00000010
0.05000000	0.04761905	0.04761875	0.00000030
0.06000000	0.05660377	0.05660304	0.00000073
0.07000000	0.06542056	0.06541899	0.00000157
0.08000000	0.07407407	0.07407104	0.00000303
0.09000000	0.08256881	0.08256339	0.00000542
0.10000000	0.09090909	0.09090000	0.00000909

Table 2. Numerical results and Exact solution of one-dimensional Time Fractional Burgers equation for Example 2.

x	y Exac	z Numerical	Error
0.01000000	0.00959542	0.00959539	0.00000003
0.02000000	0.01867570	0.01867522	0.00000047
0.03000000	0.02736945	0.02736712	0.00000234
0.04000000	0.03573800	0.03573077	0.00000723
0.05000000	0.04382187	0.04380456	0.00001731
0.06000000	0.05165106	0.05161580	0.00003526
0.07000000	0.05924912	0.05918489	0.00006423
0.08000000	0.06663526	0.06652744	0.00010783
0.09000000	0.07382562	0.07365552	0.00017009
0.10000000	0.08083395	0.08057848	0.00025548

6. Conclusion

The time-fractional Burgers equation is resolved in Maple18 using the Variational Iteration method. By contrasting the numerical outcomes, the outcomes were compared with the precise solution corresponding to the time-fractional Burgers equation. This showed the procedure's efficacy and Maple18's capacity to swiftly and efficiently provide a numerical solution that was related to the exact solution while recording the error value, making the correctness of the solutions obtained extremely satisfactorily. We can observe that the exact solution and the numerical solution are typically related. With Maple18, it is possible to numerically calculate the majority of engineering and mathematics topics. Mabel18 is both a mathematical system and a programming language. The solution has also been graphically depicted. These results are shown in **Table 1**, **Table 2**, **Figure 1** and **Figure 2** using the package version of Mabel. The distinction between exact solution and numerical solutions is seen in **Table 1**, **Table 2**, **Figure 1** and **Figure 2**. We were able to reach quite near to the exact solutions of the equations using the Variational Iteration method via Mabel software. The results show how successful the existing methodology is at obtaining precise numerical solutions to time-fractional Burgers equation. The main objective of this work is to use Maple software to automate the calculation of the Variational Iteration method. This will make it easier to use Mabel in the future as we may be able to gain rough estimates of the solutions.

7. Recommendations for Future Research

The research on the Variational Iteration Method for solving time fractional Burgers equation using Maple has shown promising results. The method was found to be effective in providing accurate solutions to the equation with a high degree of efficiency. The Maple software proved to be a useful tool in implementing the method and producing the numerical results.

In conclusion, the Variational Iteration Method is a powerful technique for solving time fractional Burgers equation, and Maple can be used effectively to implement the method. The study has provided valuable insights into the behavior of the solution and the effectiveness of the method. However, there is still room for improvement in the accuracy and efficiency of the method, especially for more complex problems. Future research can focus on developing more efficient algorithms for the method and exploring its applications in other fields of science and engineering.

In addition, it may be worthwhile to investigate the use of other numerical methods in solving time fractional Burgers equation and compare their performance with the Variational Iteration Method. Furthermore, the study could be extended to investigate the behavior of the solution for different types of initial and boundary conditions. Overall, the research has opened up new avenues for further exploration in the field of fractional calculus and numerical methods, and there is great potential for future advancements.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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