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Like a Sum Is Generalized into an Integral, a Product May Be Generalized into an Inteduct

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Abstract

It is well known that an integral is nothing but a continuous form of a sum. Is it possible to do the same thing with a product? The answer is yes and done for the first time in this publication. The new operator is called inteduct. As an integral is a proper tool to calculate the arithmetic mean of a function, the inteduct gives the geometric mean of a function. This defines a new branch of mathematics. Most applications may lay way ahead. Only some are discussed here. One is applying the inteduct to probability theory. There it is possible e.g., to determine a *function* for a life expectation rather than just a numerical value. Another application is to distinguish chaos from randomness within numerically given values. At least for the logistic map there exists a direct connection between Lyapunov exponent and inteduct. To distinguish between chaos and randomness is particularly important in finance. While randomness implies ergodicity, chaos is non-ergodic. And many fundamental financial theories from portfolio theory to market efficiency require ergodicity.

Keywords

Geometric Mean, Chaos, Finance, Ergodicity

1. Introduction

Going back to Newton or Leibniz, an integral is a continuous generalization of a sum. Like the arithmetic average is a sum, the average value of a function can be expressed by an integral.

It is a very different question whether there exists some analogy for a product. To scrutinize it is the main point of this publication. As the answer is yes, it is justified in its own right to publish about this new operator which we will call INTEDUCT (as a merger of *integral* and *product*).

In Chapter 2 we will define the inteduct and give some analytic calculations for an inteduct over some functions. It is comparable to a table of integrals which can be found in e.g. [1]. Like the integral gives the (arithmetic) average of an infinite number of functional values, the inteduct gives the geometric mean of an infinite number of functional values. As the geometric mean is defined for positive numbers only, so is the inteduct. In that sense the inteduct is the continuous generalization of the geometric mean.

Inteduct and integral both give some average and both take an infinite amount of numbers as an input. However, in the case of an inteduct the number is countable infinite ($\aleph_0$) while the integral needs non-countable infinite numbers ($\aleph_1$). An integral over a function showing functional values for rational numbers only (e.g. Dirichlet function) does not exist as a Riemann integral. With a Lebesgue integral the result will always be zero. In that sense an integral over a probability distribution is always zero as probabilities are by definition rational number (number of possible states divided by number of all states). In reality one continuously adds the irrational numbers in a probability distribution which looks arbitrary. Taking the integral over that distribution gives results which are in perfect accordance with very many experiments. One has to ask, whether this is pure luck. Though the inteduct is no panacea to solve this contradiction, it is at least a measure which can deal with rational numbers only. Only recently [2] [3] some questions were raised. Comparing random numbers between 0 and 1 to chaotically varying numbers as given by e.g. a logistic map (29) shows no differences at first glance. Statistical properties like average and variance are identical. However, some limits are completely different. Furthermore, random numbers are ergodic, which is a prerequisite for many statistical considerations in finance from portfolio theory [4] to Fama's market efficiency [5]. As one knows from [6] or [7], fluctuations in e.g. stock prices vary chaotically rather than randomly. So there is no ergodicity. If the values are created by a mathematical formula like the logistic map (37), there are mathematical tools (e.g. Lyapunov exponent or Hausdorff dimension) to determine the degree of chaos. However, stock prices (and also radio signals from outer space) are given by observations rather than formulas. To distinguish between randomness or chaotic behavior is at least extremely difficult. This brings us to the application discussed in Chapter 3.2. There we show that at least within the logistic map there is a direct connection between Lyapunov exponent and the inteduct over the values. How it is for other chaotic systems is so far unclear and left for future work. From this it is clear, that most applications of an inteduct are to solve problems to be discovered in the future. To the best of the knowledge of the authors this is the first time such *product operator* is discussed in a publication.

In Chapter 3.1., we have a more back-to-earth application. As probabilities are positive numbers, the inteduct should have applications in probability theory especially when probabilities are multiplied. In Chapter 3.1., we consider a mortality table. It has normal annual values (e.g. **Table 1**). From these given values e.g., the probability of reaching a certain age can be calculated. As the probabilities in a mortality table are discrete measured values, the life expectation is just a number determined numerically. However, the values of a mortality table can be fitted. Via inteduct one can give a functional form of the average life expectation.

We close with short conclusions in Chapter 4. There we also discuss future work.

2. Definition and Theorems

An integral is nothing but a sum with an infinite number of summands and especially infinitely small summands. Its geometric interpretation is an area *A* under the graph of a function $f(x)$ between *a* and *b*:

$$A = \int_a^b \mathrm{d}x\, f(x) \tag{1}$$

Maybe more interesting for many applications is the average value of $f(x)$ between *a* and *b*:

$$\frac{1}{b-a}\int_a^b \mathrm{d}x\, f(x) \tag{2}$$

An area under a curve (1) and average (2) are well known even in high school mathematics. There are standard books [1] on it. Especially when scrutinizing the average, there are also more recent considerations [2].

The integral is a linear measure. There are also non-linear *averages*. There is e. g. a geometric mean. Having *N* variables x_k (*k* runs from 1 to *N*) the geometric mean is defined as:

$$\min\{x_k\} \le \left(\prod_{k=1}^{N} x_k\right)^{1/N} \le \max\{x_k\} \tag{3}$$

Of course, all $x_k > 0$. The inequalities in (3) are trivial. The geometric interpretation of (3) is as follows. Having an *N* dimensional cuboid, the geometric mean of (3) is nothing but the edge length of an *N* dimensional cube having the same volume as the original cuboid. So far this is also high school mathematics and it can be found in textbooks such as [1]. Please note that the average in (2) is continuous while the (geometric) mean in (3) is discrete. Making (3) continuous is the main purpose of this paper.

Considering a function $f(x)$ within an interval $x \in [a,b]$ and $f(x) > 0$ $\forall x \in [a,b]$, one may divide the interval $[a,b]$ in smaller and smaller pieces leading to the definition of *inteduct*:

$$\prod_{a\ b}^{x} f(x) \equiv \lim_{n\to\infty}\left(\prod_{k=0}^{n} f\left(a+\frac{k}{n}\cdot(b-a)\right)\right)^{\frac{1}{n+1}} \tag{4}$$

The sign Π for the inteduct is chosen for convenience as it exists in Word and LaTeX as an operator for product. There is an infinite number of continuous transformations of

$$\sum \to \int \tag{5}$$

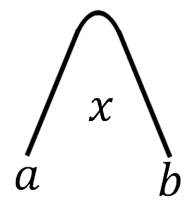

Figure 1. Suggestion for an inteduct operator of a function of *x* between *a* and *b*.

So there is no strict construction for a symbol for inteduct. However, with some handwaving argumentation one may conclude from (5) that is a reasonable symbol. However, more important than the symbol of **Figure 1** is the meaning of (4). It is the continuous form of a geometric mean just like (2) is the continuous form of the discrete summation average. Like in the case of the *ordinary* geometric mean, (3) is the edge length of a cube of *infinite* dimension having the same volume as a cuboid of infinite dimension having the edge lengths $f(x)$. Though (2) and (4) give some form of average, there is a fundamental difference between (2) and (4). It is of minor importance that (2) is a linear operation while (4) is not. Both, (2) and (4) consider an infinite number of functional values of $f(x)$. However, in (4) it is countable infinite and in (2) it is not. This limitation of (2) is mostly ignored. Some remarks can be found in [2] or [3], respectively. An integral like (1) is often used in probability theory. A probability is by definition a rational number. Taking the integral of (1) only over rational values of $f(x)$ will yield always to a result identical to zero. Strictly speaking, applying an integral in probability theory has serious flaws. However, as reality shows that using the integral appears to be mostly correct. But especially when chaos is involved (cf. [2] or [3]) some limitations become obvious. The same would be true for quantum mechanics when integrating over an infinite number of states. Again, the number of states may be infinite but it is always countable infinite. This indicates that the main applications of the inteduct lay way ahead. Maybe it has even applications where ordinary numbers fail such as describing games like go (a game played with black and white pieces on a board of 361 crosses) or chess where surreal numbers appear promising ([8] [9]) or while treating things like envy quantitatively where at least real numbers fail [10].

But now back to the inteduct of (4). It exits when the limit in (4) exits. In what follows we give sum formulas on how to calculate an inteduct analytically. For $f(x) = \text{constant}$ the inteduct is identical to this constant. For $f(x) = x$ (and $0 < a < b$) we have

$$\prod_{a\ b}^{x} x = \lim_{n\to\infty}\left(\prod_{k=0}^{n} a + \frac{k}{n}\cdot(b-a)\right)^{\frac{1}{n+1}} \tag{6}$$

One can write

$$\left(\prod_{k=0}^{n} a + \frac{k}{n}\cdot(b-a)\right)^{\frac{1}{n+1}} = a^{\frac{1}{n+1}}\cdot\left(\frac{b-a}{n}\right)^{\frac{n}{n+1}}\left(P_h\left(1+\frac{a\cdot n}{b-a}, n\right)\right)^{\frac{1}{n+1}} \tag{7}$$

where the Pochhammer symbol is defined as:

$$P_h(x,n) \equiv x\cdot(x+1)\cdot\dots\cdot(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{8}$$

Expressing the Pochhammer symbol by gamma functions and using a generalized Stirling formula

$$\Gamma(x) = \sqrt{2\pi}x^{x-1/2}\mathrm{e}^{-x}\mathrm{e}^{-\mu(x)} \quad \text{with } 0<\mu(x)<\frac{1}{12x} \quad \text{if } x>0 \tag{9}$$

one ends up with

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n}\left(a+\frac{k}{n}\cdot(b-a)\right)\right)^{\frac{1}{n}} = \frac{1}{\mathrm{e}}\cdot\frac{b^{\frac{b}{a-b}}}{a^{\frac{b}{b-a}}} \tag{10}$$

Because

$$\lim_{n\to\infty}\left(f(a,b,n)\right)^{\frac{1}{1+n}} = \lim_{n\to\infty}\left(f(a,b,n)\right)^{\frac{1}{n}\cdot\left(1+O\left(\frac{1}{n}\right)\right)} = \lim_{n\to\infty}\left(f(a,b,n)\right)^{\frac{1}{n}} \tag{11}$$

One may change the exponent $1/n$ in (10) to $1/n+1$ and get eventually

$$\prod_{a\ b}^{x} x = \frac{1}{\mathrm{e}}\cdot\frac{b^{\frac{b}{a-b}}}{a^{\frac{b}{b-a}}} \tag{12}$$

Trivially we have

$$\prod_{a\ b}^{x} f(x)\cdot g(x) = \prod_{a\ b}^{x} f(x)\cdot\prod_{a\ b}^{x} g(x) \tag{13}$$

And with $a<b<c$

$$\prod_{a\ c}^{x} f(x) = \left(\prod_{a\ b}^{x} f(x)\right)^{\frac{b-a}{c-a}}\cdot\left(\prod_{b\ c}^{x} f(x)\right)^{\frac{c-b}{c-a}} \tag{14}$$

From (12) with (13) one easily gets

$$\prod_{a\ b}^{x} x^m = \frac{1}{\mathrm{e}^m}\cdot\frac{b^{\frac{mb}{b-a}}}{a^{\frac{ma}{b-a}}} \tag{15}$$

Furthermore as $\mathrm{e}^a\cdot\mathrm{e}^b = \mathrm{e}^{a+b}$ we have

$$\prod_{a\ b}^{x} \mathrm{e}^{m\cdot x} = \exp\left(m\cdot\frac{a+b}{2}\right) \tag{16}$$

In the same way one gets

$$\prod_{a\ b}^{x} \mathrm{e}^{-m\cdot x^2} = \exp\left(-m\cdot\frac{a^2+a\cdot b+b^2}{3}\right) \tag{17}$$

With (13) to (17) it is now possible to take the inteduct over a variety of *product like* functions. It comes as no surprise that it is not simple to calculate an inteduct over a sum of functions. With the Chebichev inequality [1] $(a_1+a_2)\cdot(b_1+b_2)\le 2\cdot(a_1\cdot b_1+a_2\cdot b_2)$ we can just show

$$\prod_{a\ b}^{x} f(x)+g(x) \le 2\cdot\left(\prod_{a\ b}^{x} f(x)+\prod_{a\ b}^{x} g(x)\right) \tag{18}$$

It is clear that the inteduct does exist over a uniformly continuous function. That does not mean that the inteduct does not exist for not uniformly continuous functions. Consider the slightly changed Dirichlet function

$$D(x)=\begin{cases}1 & x\in\mathbb{Q}^+\\ & \text{für}\\ 2 & x\in\mathbb{R}^+\setminus\mathbb{Q}\end{cases} \tag{19}$$

This function is nowhere continuous and of course not uniformly continuous. The Riemann integral does not exist. The inteduct

$$\prod_{0}^{y}{}_{x} D(y)=\begin{cases}1 & x\in\mathbb{Q}^+\\ & \text{für}\\ 2 & x\in\mathbb{R}^+\setminus\mathbb{Q}\end{cases} \tag{20}$$

is again a Dirichlet function. Though it is formally possible to take an inteduct of a Dirichlet function, the result should not be taken too seriously. The Dirichlet function of (19) has an average of 2 as can be shown by taking the Lebesgue integral. It is also plausible as *most* functional values are 2. The inteduct essentially shows that a geometric mean does not exist here (at least if calculated by an inteduct).

In order to see how the averaging with (2) or (4), respectively works consider the function

$$f(x)=a+\sin x \quad \text{with } a>1 \tag{21}$$

The averages via integral or inteduct, respectively are given by:

$$F(x)=\frac{1}{x-0}\int_0^x \mathrm{d}y\, f(y) \tag{22}$$

$$\mathfrak{F}(x)=\prod_{0}^{y}{}_{x} f(y) \tag{23}$$

(22) can be easily calculated to

$$F(x)=a+\frac{1-\cos x}{x} \tag{24}$$

(23) is more cumbersome to calculate. It can be expressed by a complicated series

$$\mathfrak{F}(x)=a\lim_{n\to\infty}\left(1+\frac{1}{a}\sum_{k=0}^{n}\sin\left(\frac{k}{n}x\right)+\frac{1}{a^2}\sum\text{"double"}+\frac{1}{a^3}\sum\text{"triple"}+\cdots\right)^{\frac{1}{n+1}} \tag{25}$$

The exact form of the dummies "double", "triple", and so forth can be found elsewhere. They are of no particular interest as the limit in (25) must be taken numerically anyway. E.g. $a=2$ one can plot the result in **Figure 2**.

Without surprise both averages go to a (here 2) if $x\to\infty$.

So far we have shown how an inteduct is defined and how to calculate it. In chapter 3 we will discuss *some* applications.

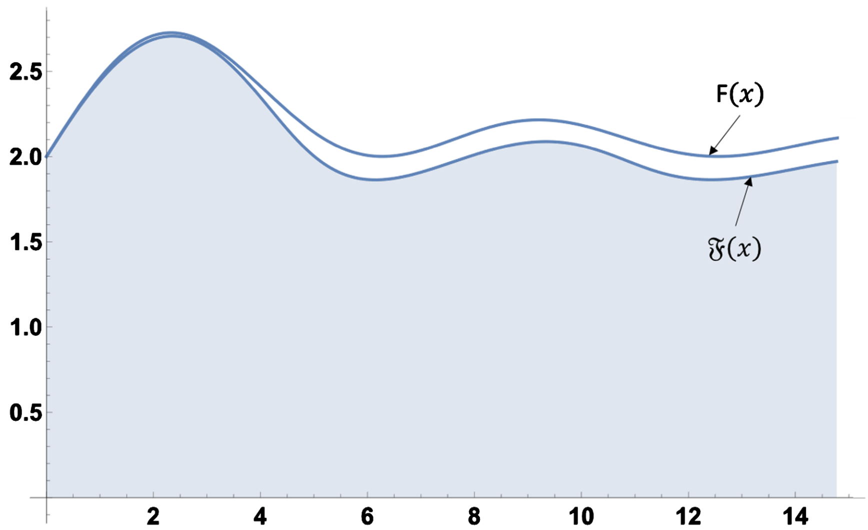

Figure 2. Average via inteduct ($\mathfrak{F}(x)$) and integral ($F(x)$), respectively.

3. Applications

In this chapter we will discuss applications of the newly defined inteduct. Please note that there may be many more. In 3.1 we will show how the inteduct is useful in ordinary probability theory. As probabilities are by definition positive, it is clear that an inteduct might be suitable here. In 3.2 we will discuss possible applications to scrutinize chaos.

3.1. Application to Mortality Table

A mortality table gives the probability (likelihood) to die in your first year on earth, second year, and so forth. For sure it depends on gender, time and country as an example please see **Table 1**. It is normally even taught in high school how to calculate an average age or life expectation from it. Given a birth rate, it is slightly more advanced to calculate an age distribution. The probability $p(\tau)$ to reach a certain age τ is e.g. given by:

$$p(\tau) = \prod_{t=1}^{\tau} 1 - p_d(t) \tag{26}$$

Of course, it is not possible to construct a function $p(\tau)$ via (26). It gives just the numerical values. As the $p_d(\tau)$ are given numerically anyway, this is sufficient for most practical purposes.

Though it may be sufficient to express the mortality of **Table 1** within steps of one year, it is of course possible to express it finer and finer leading to a continuous function eventually. An almost perfect fit of the p_d of **Table 1** is given by[1]

$$1 - p_d(t) = \frac{\alpha}{\alpha + \exp\frac{t}{\tau}} \quad \text{with } \alpha \approx 184.225 \text{ and } \tau \approx 8.48029 \text{ years} \tag{27}$$

[1]It follows from a non-linear least square fit. Except for the first couple of years it fits almost perfectly here. These early years are of little quantitative importance in most cases. They can be included easily be a slightly more complicated fit.

Table 1. Mortality table, men 2014/16 Germany, source:Statistisches Bundesamt.

Year up to age of τ	Probability $p_d(\tau)$ to die within that year
1	3.5×10^{-3}
2	2.7×10^{-4}
3	1.6×10^{-4}
…	…
11	7.8×10^{-5}
…	…
98	0.34
99	0.36
100	0.38

The main point of (27) is that we have a function now. Instead of (26) we can write the probability $p(T)$ to live up to the age of T as an inteduct:

$$p(T) = \left(\prod_{0}^{t} {}_{T} \frac{\alpha}{\alpha + \exp\frac{t}{\tau}} \right)^{T} \tag{28}$$

We have now a real function $p(T)$ which can be calculated numerically with arbitrary precision.

3.2. The Inteduct to Scrutinize Chaos

As a second example one may look to chaos as defined in textbooks like [11]. Our focus here lies in applied situations in business and economics as can be found [6] [7] and [12] [13]. In such situations there is no given formula and one cannot calculate e.g. the Lyapunov exponent or a fractal dimension to prove chaos. There are just data of e.g. stock prices. They are often assumed to vary randomly. However, in reality they are chaotic. This makes a huge difference, as randomness implies ergodicity while chaos is not ergodic. The consequences are various. Just see [6].

More generally speaking, it is next to impossible to decide for experimental data (be it stock prices or radio signals from outer space) whether they are random or chaotic. To scrutinize it further consider the archetype of chaotic numbers (logistic map):

$$x_{n+1} = 4 \cdot x_n \cdot (1 - x_n) \tag{29}$$

The x_n are chaotic as long as the starting value x_0 fulfills the following conditions:

$$0 < x_0 < 1 \text{ and } x_0 \neq \frac{1}{2}\left(1 - \cos\left(\frac{\pi}{2^k}\right)\right) \text{ with } k \in \mathbb{N}_0 \tag{30}$$

It is easy to prove that (29) leads to a Lyapunov exponent of $\ln 2$ and a

Hausdorff dimension $D = 4/3$.[2] (see e.g. [2]) Consider in contrast random numbers z_i within an interval $(0,1)$. Their Hausdorff dimension is 2 which proves them as non-chaotic.

As the x_n and z_i are mathematically defined objects, it is easy to prove or disprove chaos. Given a set of numbers x_n and z_i numerically, they are very hard to distinguish. One will e.g. end up with

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} z_i = \frac{1}{2} \tag{31}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} x_i = \frac{1}{2} \tag{32}$$

in a numerical calculation. Statistically the sets of z_i and x_i are indistinguishable. However, building the geometric mean one gets

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} z_i \right)^{\frac{1}{n+1}} \to \text{does not exist} \tag{33}$$

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} x_i \right)^{\frac{1}{n+1}} = \frac{1}{2} \tag{34}$$

Please note that (34) converges quite slowly. For $n = 10.000$ one just obtains ≈ 0.46. (33) looks numerically as if it would converge. For $50.000 \le n \le 100.000$ one typically gets 0.366 ± 1.02 × 10^{-3}.

(33) and (34) show that the continuous geometric mean or better inteduct does exist for chaotic numbers of the logistic map but not for random numbers. It is left to future work whether this applies to all sets of chaotic numbers or at least what kind of chaotic maps it applies to.

For more details for the detection of chaos via inteduct please consider the following. A useful approach would be to describe an iterative map like e.g. (29) by a function $f_n(x)$. In the case of the logistic map $x = x_0$ and $f_n(x) = x_n$. The Lyapunov exponent $\lambda(x)$ as defined in e.g. [11] is then given by

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{\mathrm{d}f_n(x)}{\mathrm{d}x} \right| \tag{35}$$

As $f_n(x) = f_1(f_{n-1}(x)) = f_1(f_1(f_{n-2}(x))) = \cdots$ and by applying the chain rule one gets

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=1}^{n} \frac{\mathrm{d}f_1(f_{n-i})}{\mathrm{d}f_{n-i}} \right| = \lim_{n\to\infty} \ln \left| \prod_{i=1}^{n} \frac{\mathrm{d}f_1(f_{n-i})}{\mathrm{d}f_{n-i}} \right|^{\frac{1}{n}} \tag{36}$$

For a generalized logistic map

$$x_{n+1} = a \cdot x_n \cdot (1 - x_n) \tag{37}$$

We have $f_1(x) = a \cdot x \cdot (1 - x)$ and therefore

[2]Of course, one has to map the numbers x_n of (29) to an interval $[0,1]$ where 0 corresponds to x_0 and 1 to $x_{n\to\infty}$. In this sense (29) is a function mapping $(0,1)$ on $(0,1)$. The random numbers z_i can be considered in the same way.

$$\lambda(x) = \ln a + \lim_{n\to\infty} \ln\left(\prod_{i=1}^{n}\left|1-2\cdot f_{n-i}\right|\right)^{\frac{1}{n}} \tag{38}$$

As the logistic map is symmetric around 1/2, the limit in (34) is identical to the limit in (38). Therefore there exists at least for the logistic map a direct connection between Lyapunov exponent and inteduct. In other words, at least for the logistic map the Lyapunov exponent can be calculated by an inteduct, cf. (34). The main importance of it is that the Lyapunov exponent defined in (35) needs a mathematical function (e.g. logistic map) to be calculated. The inteduct can be calculated from measured numbers. It is left to future work to show a connection between the Lyapunov exponent and inteduct via (36). But this will be far from straight forward.

4. Conclusions and Future Work

We have shown that it is possible to define (see (4)) an inteduct which is a continuous generalization of the geometric mean just like the integral is the continuous generalization of the arithmetic average. In (12) and (15) to (17) we give analytic calculations of an inteduct for some functions. (13) and (14) show how to handle a product of functions or how to split the path the inteduct is running, respectively.

The most obvious application lies in treating statistics. Probabilities are positive numbers by definition. So an inteduct is always well defined. We show just one example in Chapter 3.1 (mortality table).

In 3.2 we have shown a direct connection between Lyapunov exponent and inteduct for a logistic map (37). To extend this to other maps would be a great step forward. Probably it will not be possible to show it for any map. Maybe certain properties and especially symmetries are required. But this is all left to future work.

Maybe most important is the fact that an inteduct is a new measure completely independent from integral. It may have applications where standard analysis fails such as particular situations in game theory [10] or even describing games like go or chess, cf. [8] or [9]. However, such and further speculations are also left for future work.

As an inteduct is a generalization of a geometric mean, one should ask what a geometric mean is good for. There are some situations where the arithmetic mean is even ludicrous, cf. [2]. The geometric mean gives an average where it is more important that the individual values are similar. With that in mind it is possible to explain the Cob Douglas (production) function which will be scrutinized in[3].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

[3]Grabinski, M. and Klinkova, G. (2023) Explaning Cob Douglas with the New Mathematics of Inteduct. Work in Progress.

References

[1] Bronshtein, I.N., Semendyayev, K.A., Musiol, G. and Muehlig, H. (2007) Handbook of Mathematics. 5th English Edition, Springer, Berlin Heidelberg.

[2] Grabinski, M. and Klinkova, G. (2019) Wrong Use of Averages Implies Wrong Results from Many Heuristic Models. *Applied Mathematics,* **10***,* 605-618. https://doi.org/10.4236/am.2019.107043

[3] Grabinski, M. and Klinkova, G. (2020) Scrutinizing Distributions Proves That IQ Is Inherited and Explains the Fat Tail. *Applied Mathematics,* **11***,* 957-984. https://doi.org/10.4236/am.2020.1110063

[4] Markovitz, H.M. (1952) Portfolio Selection. *Journal of Finance,* **7**, 77-91. https://doi.org/10.1111/j.1540-6261.1952.tb01525.x

[5] Fama, E.F. (1965) The Behavior of Stock-Market Prices. *Journal of Business*, **5**, 34-105. https://doi.org/10.1086/294743

[6] Klinkova, G. (2018) The Effects of Chaos on Business Operations. Ph.D. Thesis, Hochschule Neu-Ulm University, Neu-Ulm. http://uni-neu-ulm.de/Klinkova_PhD

[7] Klinkova, G. and Grabinski, M. (2017) Due to Instability Gambling Is the Best Model for Most Financial Products. *Archives of Business Research*, **5**, 255-261. https://doi.org/10.14738/abr.53.3029

[8] Conway, J.H. (1978) A Gamut of Game Theories. *Mathematics Magazine*, **51**, 5-12. https://doi.org/10.1080/0025570X.1978.11976668

[9] Henle, J. (2021) J. H. Conway and the End of the Story of Numbers. *The Mathematical Intelligencer*, **43**, 10-14. https://doi.org/10.1007/s00283-021-10053-1

[10] Lunkenheimer, M., Kracklauer, A., Klinkova, G. and Grabinski, M. (2022) Homo Economicus to Model Human Behavior Is Ethically Doubtful and Mathematically Inconsistent. Preprint.

[11] Schuster, H.G. (1984) Deterministic Chaos. WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim.

[12] Grabinski, M. (2004) Is There Chaos in Management or Just Chaotic Management? Reprint from: Complex Systems, Intelligence and Modern Technology Applications, Paris. http://www.h-n-u.de/Veroeffentlichungen/CSIMTA%202004.pdf

[13] Filipe, J.A., Ferreira, M.A.M., Coelho, M. and Pedro, M.I. (2010) Chaos, Anti-Chaos and Resources: Dealing with Complexity. *Aplimat—Journal of Applied Mathematics*, **3**, 83-90.