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The Natural Transform Decomposition Method for Solving Fractional Klein-Gordon Equation

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Abstract

In this paper, a coupling of the natural transform method and the Admoian decomposition method called the natural transform decomposition method (NTDM), is utilized to solve the linear and nonlinear time-fractional Klein-Gordan equation. The (NTDM), is introduced to derive the approximate solutions in series form for this equation. Solutions have been drawn for several values of the time power. To identify the strength of the method, three examples are presented.

Keywords

Natural Transform, Adomian Decomposition Method, Fractional Klein-Gordon Equation

1. Introduction

The fractional calculus has acquired importance in applied mathematics, and it has been used to model several fields in physical science such as viscoelasticity control, diffusion, etc. Fractional differential equations appear in different regions of engineering and physics science, such as continuum traffic flow [1], the oscillation of earthquake [2], fluid-dynamic traffic [3] and so on. The fractional order and integer order differential equations have been studied and solved by utilizing several methods [4]-[12]. Many problems in science and engineering have been modeled by the Klein-Gordon equation, such as classical and quantum mechanics, solitons and condensed physics. The Klein-Gordon equation has been extensively solved by many researchers using several analytic methods [13] [14] [15] [16]. The product of this equation has been properly described by a fractional version of them. The integral transform methods are the most important methods for solving many problems, some of them have been coupled with the other analytical methods to obtain analytic solutions [17] [18] [19] [20] and it has proven effective in solving problems. The natural transform method [21] [22] [23] [24] is an example of the integral transform methods. It is a moderate and effective new method for solving differential equations. The natural transform decomposition method (NTDM) is a method combined of the natural transform and a domain decomposition method. It has been used to solve the fraction model [25]-[31]. The main aim of this work is to employ the natural transform decomposition method to solve the linear and nonlinear Klein-Gordon equations of fractional order.

2. Natural Transform

Definition 1 [21] [22] [23] [24] *The natural transform of a function* $g(t)$ *defined by the integral*:

$$\mathbb{N}^{+}\left[g(t)\right]=\varphi(s,u)=\int_{0}^{\infty}g(ut)\mathrm{e}^{-st}\mathrm{d}t, u>0, s>0, \tag{1}$$

where u and s are the natural variables.

Property 1

$$\mathbb{N}^{+}\left[t^{\gamma}\right]=\frac{u^{\gamma}\Gamma[\gamma+1]}{s^{\gamma+1}}, \gamma>-1. \tag{2}$$

Theorem 1 *If* $n\in N$ *where* $n-1\le\gamma<n$ *and* $\varphi(s,u)$ *is natural transform of the function* $g(t)$, *then the natural transform of Caputo fractional derivative of* $\frac{\partial^{\gamma}g(x,t)}{\partial t^{\gamma}}$ *is given by*

$$\mathbb{N}^{+}\left[\frac{\partial^{\gamma}g(x,t)}{\partial t^{\gamma}}\right]=\frac{s^{\gamma}}{u^{\gamma}}\varphi(s,u)-\sum_{k=0}^{n-1}\frac{s^{\gamma-(k+1)}}{u^{\gamma}-k}\left[\frac{\partial^{\gamma}g(x,t)}{\partial t^{\gamma}}\right]_{t=0}. \tag{3}$$

Definition 2 *The inverse natural transform of* $\varphi(s,u)$ *is defined by*

$$\mathbb{N}^{-}\left[\varphi(s,u)\right]=g(t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\varphi(s,u)\mathrm{e}^{\frac{st}{u}}\mathrm{d}t, u>0, s>0. \tag{4}$$

3. Analysis of Method

In this part, we apply (NTMD) to the following general fractional Klein Gordon equation of the form:

$$\frac{\partial^{\gamma}V(x,t)}{\partial t^{\gamma}}=a\frac{\partial^{2}V(x,t)}{\partial x^{2}}+bV(x,t)+\mathcal{F}(V(x,t))+\mu(x,t),$$
$$1<\gamma\le 2 \text{ and } x,t>0, \tag{5}$$

subject to:

$$V(x,0)=f_{1}(x), V_{t}(x,0)=f_{2}(x), \tag{6}$$

where a,b are constants, $\mu(x,t)$ is a source term and $\mathcal{F}(V(x,t))$ is a non-linear function. Taking natural transform to Equation (5), we get

$$\frac{s^{\gamma}}{u^{\gamma}}\varphi(x,s,u)-\frac{s^{\gamma-1}}{u^{\gamma}}V(x,0)-\frac{s^{\gamma-2}}{u^{\gamma-1}}V_t(x,0)$$
$$=\mathbb{N}^{+}\left[a\frac{\partial^2 V(x,t)}{\partial x^2}+bV(x,t)+\mathcal{F}(V(x,t))+\mu(x,t)\right]. \tag{7}$$

Substitution the initial condition (6) into Equation (7), we get

$$\varphi(x,s,u)=\frac{f_1(x)}{s}+\frac{uf_2(x)}{s^2}$$
$$+\frac{s^{\gamma}}{u^{\gamma}}\mathbb{N}^{+}\left[a\frac{\partial^2 V(x,t)}{\partial x^2}+bV(x,t)+\mathcal{F}(V(x,t))+\mu(x,t)\right]. \tag{8}$$

Operating with inverse natural transform of (8) gives

$$V(x,t)$$
$$=\psi(x,t)+\mathbb{N}^{-}\left[\frac{s^{\gamma}}{u^{\gamma}}\mathbb{N}^{+}\left[a\frac{\partial^2 x^2 V(x,t)}{\partial x^2}+bV(x,t)+\mathcal{F}(V(x,t))+h(x,t)\right]\right], \tag{9}$$

where $\psi(x,t)$ represents the term deriving from the source term and the directed initial conditions. The natural decomposition method represents solution as infinite series

$$V(x,t)=\sum_{n=0}^{\infty}V_n(x,t), \tag{10}$$

and the term $\mathcal{F}(V(x,t))$ is as follows

$$\mathcal{F}(V(x,t))=\sum_{n=0}^{\infty}A_n, \tag{11}$$

where A_n can be calculated by

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\beta^n}\left[N\sum_{i=0}^{n}\beta^i V_i\right]_{\beta=0}. \tag{12}$$

Substitution Equation (10) and Equation (11) into Equation (9), yields

$$\sum_{n=0}^{\infty}V_n(x,t)=\psi(x,t)+\mathbb{N}^{-}\left[\frac{s^{\gamma}}{u^{\gamma}}\mathbb{N}^{+}\left[a\sum_{n=0}^{\infty}\frac{\partial^2 V_n(x,t)}{\partial x^2}+b\sum_{n=0}^{\infty}V_n(x,t)+\sum_{n=0}^{\infty}A_n\right]\right]. \tag{13}$$

We obtain the recursive relation

$$V_0(x,t)=\psi(x,t)$$
$$V_{n+1}(x,t)=\mathbb{N}^{-}\left[\frac{s^{\gamma}}{u^{\gamma}}\mathbb{N}^{+}\left[a\frac{\partial^2 V_n(x,t)}{\partial x^2}+bV_n(x,t)+A_n\right]\right]. \tag{14}$$

The approximate solution can be written as a series form

$$V(x,t)=\sum_{n=0}^{\infty}V_n(x,t) \tag{15}$$

4. Applications

Now, we explain the efficacy of the method by the three examples:

Example 1 *Consider the linear inhomogeneous fractional Klein-Gordon equation*:

$$\frac{\partial^{\gamma}V(x,t)}{\partial t^{\gamma}}=\frac{\partial^{2}V(x,t)}{\partial x^{2}}+V(x,t)+6x^{3}t+\left(x^{3}-6x\right)t^{3},\ t>0,\ x\in\mathbb{R}\text{ and }1<\gamma\leq 2,\quad(16)$$

subject to:

$$V(x,0)=0,\ V_t(x,0)=0$$

Solution:

Subsequent the discussion presented above, the system of Equation (13) *becomes*:

$$V_0(x,t)=\frac{6x^3t^{\gamma+1}}{\Gamma[\gamma+2]}+\frac{6\left(x^3-6x\right)t^{\gamma+3}}{\Gamma[\gamma+4]},$$

$$V_{n+1}(x,t)=\mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty}\frac{\partial^2V_n(x,t)}{\partial x^2}-\sum_{n=0}^{\infty}V_n(x,t)\right]\right],\ n\geq 0.\quad(17)$$

This gives

$$V_1(x,t)=\frac{36x^3t^{2\gamma+1}}{\Gamma[2\gamma+2]}-\frac{6x^3t^{2\gamma+1}}{\Gamma[2\gamma+2]}+\frac{72xt^{2\gamma+3}}{\Gamma[2\gamma+4]}-\frac{6x^3t^{2\gamma+3}}{\Gamma[2\gamma+4]}$$

$$V_2(x,t)=-\frac{72xt^{3\gamma+1}}{\Gamma[3\gamma+2]}+\frac{6x^3t^{3\gamma+1}}{\Gamma[3\gamma+2]}+\frac{6x^3t^{3\gamma+3}}{\Gamma[3\gamma+4]}-\frac{108xt^{3\gamma+3}}{\Gamma[3\gamma+4]}$$

$$V_3(x,t)=\frac{108xt^{4\gamma+1}}{\Gamma[4\gamma+2]}-\frac{6x^3t^{4\gamma+2}}{\Gamma[4\gamma+2]}+\frac{144xt^{4\gamma+3}}{\Gamma[4\gamma+4]}-\frac{6x^3t^{4\gamma+3}}{\Gamma[4\gamma+4]}$$

$$\vdots$$

Therefore, the approximate solution is given by (**Figure 1** *and* **Figure 2**):

$$\begin{aligned}V(x,t)=&\frac{6x^3t^{\gamma+1}}{\Gamma[\gamma+2]}+\frac{6(x^3-6x)t^{\gamma+3}}{\Gamma[\gamma+4]}+\frac{36x^3t^{2\gamma+1}}{\Gamma[2\gamma+2]}-\frac{6x^3t^{2\gamma+1}}{\Gamma[2\gamma+2]}+\frac{72xt^{2\gamma+3}}{\Gamma[2\gamma+4]}\\&-\frac{6x^3t^{2\gamma+3}}{\Gamma[2\gamma+4]}-\frac{72xt^{3\gamma+1}}{\Gamma[3\gamma+2]}+\frac{6x^3t^{3\gamma+1}}{\Gamma[3\gamma+2]}+\frac{6x^3t^{3\gamma+3}}{\Gamma[3\gamma+4]}-\frac{108xt^{3\gamma+3}}{\Gamma[3\gamma+4]}\\&+\frac{108xt^{4\gamma+1}}{\Gamma[4\gamma+2]}-\frac{6x^3t^{4\gamma+2}}{\Gamma[4\gamma+2]}+\frac{144xt^{4\gamma+3}}{\Gamma[1\gamma+4]}-\frac{6x^3t^{4\gamma+3}}{\Gamma[1\gamma+4]}+\cdots\end{aligned}$$

Then the result at $\gamma=2$ *is*

$$V(x,t)=x^3t^3$$

Example 2 *Consider the non-linear inhomogeneous fractional Klein-Gordon equation*:

$$\frac{\partial^{\gamma}V(x,t)}{\partial x^{\gamma}}=\frac{\partial^{2}V(x,t)}{\partial x^{2}}-V^2(x,t)+2x^2-2t^2+x^4t^4,\ t>0,\ x\in\mathbb{R}\text{ and }1<\gamma\leq 2,\quad(18)$$

subject to:

$$V(x,0)=0,\ V_t(x,0)=0.\quad(19)$$

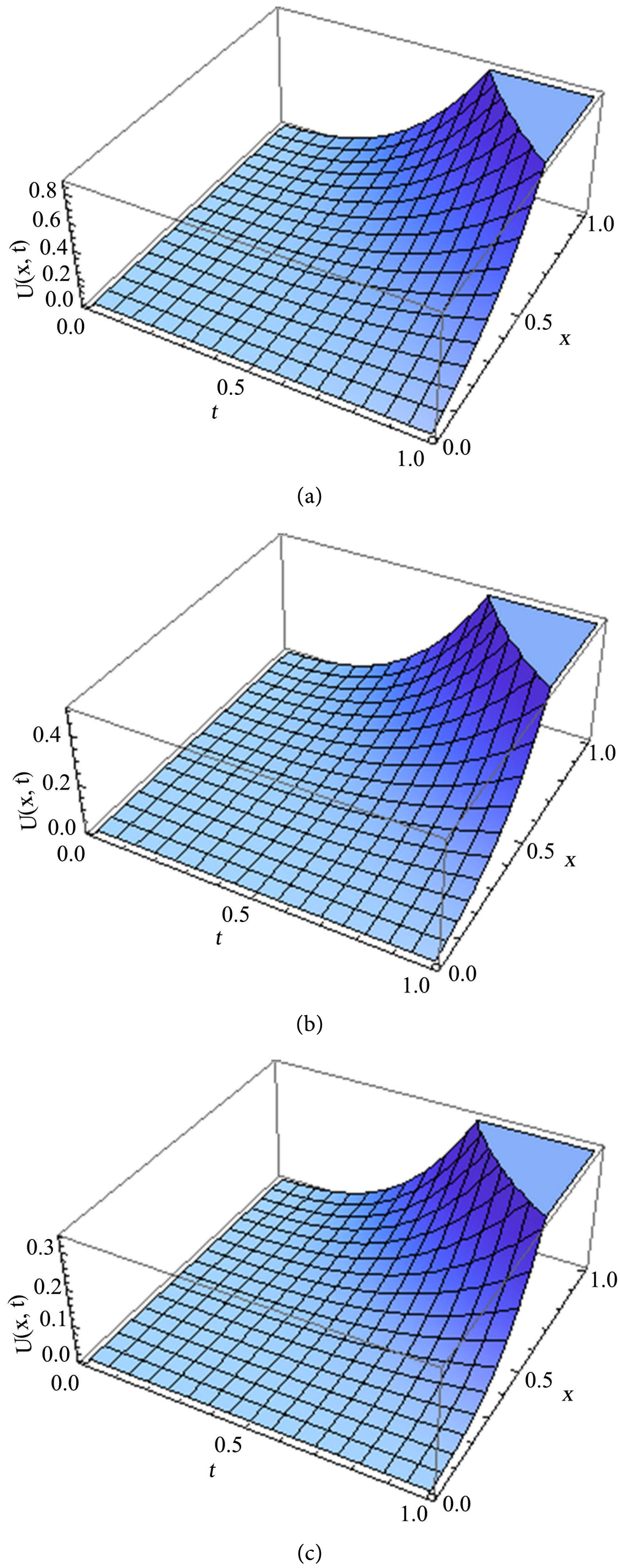


Figure 1. The approximate solutions for Example 1 when (a) $\gamma = 1.5$; (b) $\gamma = 1.75$; (c) $\gamma = 1.90$.

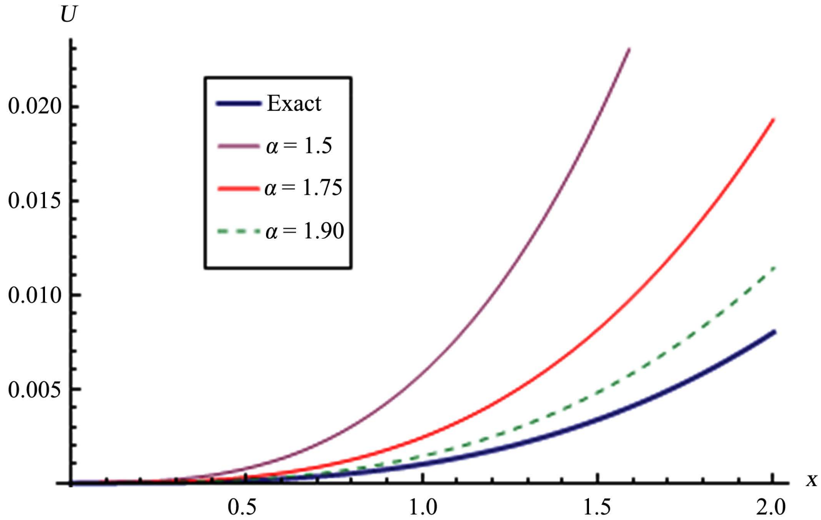


Figure 2. The approximate and exact solutions for Example 1 when $\gamma = 1.5, \gamma = 1.75, \gamma = 1.90$.

Solution: *Proceeding as in Example 1, Equation (13) becomes*:

$$V_0(x,t) = \frac{2x^2 t^{\gamma}}{\Gamma[\gamma+1]} - \frac{4t^{\gamma+2}}{\Gamma[\gamma+3]} + \frac{24x^4 t^{\gamma+4}}{\Gamma[\gamma+5]}$$

$$V_{n+1} = \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[\partial^2 V_n(x,t)\partial x^2 - A_n\right]\right], n \geq 0. \quad (20)$$

This gives

$$V_1(x,t) = \frac{4t^{2\gamma}}{\Gamma[2\gamma+1]} + \frac{288x^2 t^{2\gamma+4}}{\Gamma[2\gamma+5]} - \frac{4x^4 t^{3\gamma+4}\Gamma[2\gamma+1]}{\Gamma^2[\gamma+1]\Gamma[3\gamma+1]} + \frac{16x^2 t^{3\gamma+2}\Gamma[2\gamma+3]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[3\gamma+3]} - \frac{16t^{3\gamma+4}\Gamma[2\gamma+5]}{\Gamma^2[\gamma+3]\Gamma[3\gamma+5]} - \frac{96x^6 t^{3\gamma+4}\Gamma[2\gamma+5]}{\Gamma[\gamma+1]\Gamma[\gamma+5]\Gamma[3\gamma+5]} + \frac{192x^4 t^{3\gamma+6}\Gamma[2\gamma+7]}{\Gamma[\gamma+3]\Gamma[\gamma+5]\Gamma[3\gamma+7]} - \frac{576x^8 t^{3\gamma+3}\Gamma[2\gamma+9]}{\Gamma^2[\gamma+5]\Gamma[3\gamma+9]}.$$

$$V_2(x,t) = \frac{576t^{3\gamma+4}}{\Gamma[3\gamma+5]} - \frac{48x^2 t^{4\gamma}\Gamma[2\gamma+1]}{\Gamma^2[\gamma+1]\Gamma[4\gamma+1]} - \frac{16x^3 t^{4\gamma}\Gamma[3\gamma+1]}{\Gamma[\gamma+1]\Gamma[2\gamma+1]\Gamma[4\gamma+1]} - \frac{2880x^4 t^{4\gamma+4}\Gamma[2\gamma+5]}{\Gamma[\gamma+1]\Gamma[\gamma+5]\Gamma[4\gamma+5]} - \frac{192x^5 t^{4\gamma+4}\Gamma[3\gamma+5]}{\Gamma[\gamma+5]\Gamma[2\gamma+1]\Gamma[4\gamma+5]} - \frac{1152x^5 t^{4\gamma+4}\Gamma[3\gamma+5]}{\Gamma[\gamma+1]\Gamma[2\gamma+5]\Gamma[4\gamma+5]} + \frac{2304x^2 t^{4\gamma+6}\Gamma[2\gamma+7]}{\Gamma[\gamma+3]\Gamma[\gamma+5]\Gamma[4\gamma+7]} + \frac{2304x^2 t^{4\gamma+6}\Gamma[2\gamma+7]}{\Gamma[\gamma+3]\Gamma[2\gamma+5]\Gamma[4\gamma+7]} + \frac{16x^4 t^{5\gamma}\Gamma[2\gamma+1]\Gamma[4\gamma+1]}{\Gamma^3[\gamma+1]\Gamma[3\gamma+1]\Gamma[5\gamma+1]} - \frac{32x^5 t^{5\gamma+2}\Gamma[2\gamma+1]\Gamma[4\gamma+3]}{\Gamma^2[\gamma+1]\Gamma[\gamma+3]\Gamma[3\gamma+1]\Gamma[5\gamma+3]}$$

$$
\begin{aligned}
&-\frac{64x^5t^{5\gamma+2}\Gamma[2\gamma+3]\Gamma[4\gamma+3]}{\Gamma^2[\gamma+1]\Gamma[\gamma+3]\Gamma[3\gamma+3]\Gamma[5\gamma+3]}\\
&+\frac{192x^9t^{5\gamma+4}\Gamma[2\gamma+1]\Gamma[4\gamma+5]}{\Gamma^2[\gamma+1]\Gamma[\gamma+5]\Gamma[3\gamma+1]\Gamma[5\gamma+5]}\\
&+\frac{128x^3t^{5\gamma+4}\Gamma[2\gamma+3]\Gamma[4\gamma+5]}{\Gamma[\gamma+1]\Gamma^2[\gamma+3]\Gamma[3\gamma+3]\Gamma[5\gamma+5]}\\
&+\frac{64x^3t^{5\gamma+4}\Gamma[2\gamma+5]\Gamma[4\gamma+5]}{\Gamma[\gamma+1]\Gamma^2[\gamma+3]\Gamma[3\gamma+5]\Gamma[5\gamma+5]}\\
&+\frac{348x^9t^{5\gamma+4}\Gamma[2\gamma+5]\Gamma[4\gamma+5]}{\Gamma^2[\gamma+1]\Gamma[\gamma+5]\Gamma[3\gamma+5]\Gamma[5\gamma+5]}\\
&-\frac{768x^7t^{5\gamma+6}\Gamma[2\gamma+3]\Gamma[4\gamma+7]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[\gamma+5]\Gamma[3\gamma+3]\Gamma[5\gamma+7]}\\
&-\frac{128x^5t^{5\gamma+6}\Gamma[2\gamma+7]\Gamma[4\gamma+7]}{\Gamma^3[\gamma+3]\Gamma[3\gamma+5]\Gamma[3\gamma+1]\Gamma[5\gamma+7]}\\
&-\frac{768x^7t^{5\gamma+6}\Gamma[2\gamma+5]\Gamma[4\gamma+7]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[\gamma+5]\Gamma[3\gamma+5]\Gamma[5\gamma+7]}\\
&-\frac{768x^7t^{5\gamma+6}\Gamma[2\gamma+7]\Gamma[4\gamma+7]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[\gamma+5]\Gamma[3\gamma+3]\Gamma[5\gamma+7]}\\
&+\frac{4608x^11t^{5\gamma+8}\Gamma[2\gamma+5]\Gamma[4\gamma+9]}{\Gamma[\gamma+1]\Gamma^2[\gamma+5]\Gamma[3\gamma+5]\Gamma[5\gamma+9]}\\
&+\frac{768x^5t^{5\gamma+8}\Gamma[2\gamma+5]\Gamma[4\gamma+9]}{\Gamma[\gamma+3]\Gamma[\gamma+5]\Gamma[3\gamma+5]\Gamma[5\gamma+9]}\\
&+\frac{1536x^5t^{5\gamma+8}\Gamma[2\gamma+7]\Gamma[4\gamma+9]}{\Gamma^2[\gamma+3]\Gamma[\gamma+5]\Gamma[3\gamma+7]\Gamma[5\gamma+9]}\\
&+\frac{230x^{11}t^{5\gamma+8}\Gamma[2\gamma+9]\Gamma[4\gamma+9]}{\Gamma[\gamma+1]\Gamma^2[\gamma+5]\Gamma[3\gamma+9]\Gamma[5\gamma+9]}\\
&-\frac{9216x^{11}t^{5\gamma+10}\Gamma[2\gamma+7]\Gamma[4\gamma+11]}{\Gamma[\gamma+3]\Gamma^2[\gamma+5]\Gamma[3\gamma+7]\Gamma[5\gamma+11]}\\
&-\frac{4608x^9t^{5\gamma+10}\Gamma[2\gamma+9]\Gamma[4\gamma+11]}{\Gamma[\gamma+3]\Gamma^2[\gamma+5]\Gamma[3\gamma+9]\Gamma[5\gamma+11]}\\
&+\frac{27648x^{13}t^{5\gamma+12}\Gamma[2\gamma+9]\Gamma[4\gamma+13]}{\Gamma^2[\gamma+5]\Gamma[3\gamma+9]\Gamma[5\gamma+13]}\\
&+\frac{32t^{4\gamma+12}\Gamma[2\gamma+3]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[5\gamma+3]}+\frac{32256x^6t^{4\gamma+8}\Gamma[2\gamma+9]}{\Gamma^2[\gamma+5]\Gamma[4\gamma+9]}\\
&-\frac{32t^{4\gamma+12}\Gamma[2\gamma+3]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[5\gamma+3]}-\frac{13824x^7t^{4\gamma+8}\Gamma[3\gamma+9]}{\Gamma[\gamma+5]\Gamma[2\gamma+5]\Gamma[4\gamma+9]}\\
&\vdots
\end{aligned}
$$

Therefore, the series solution is given by (**Figure 3** *and* **Figure 4**):

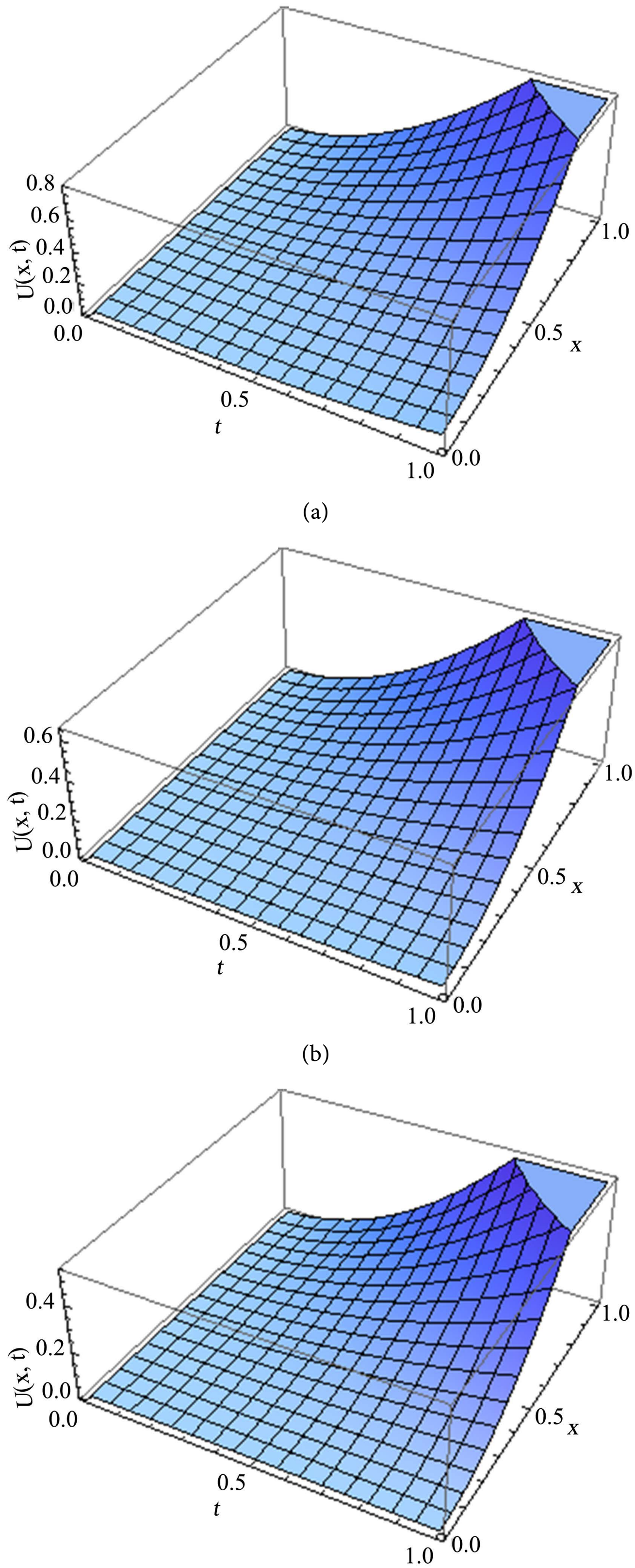


Figure 3. The approximate solutions for Example 2 when (a) $\gamma = 1.8$; (b) $\gamma = 1.9$; (c) $\gamma = 2$.

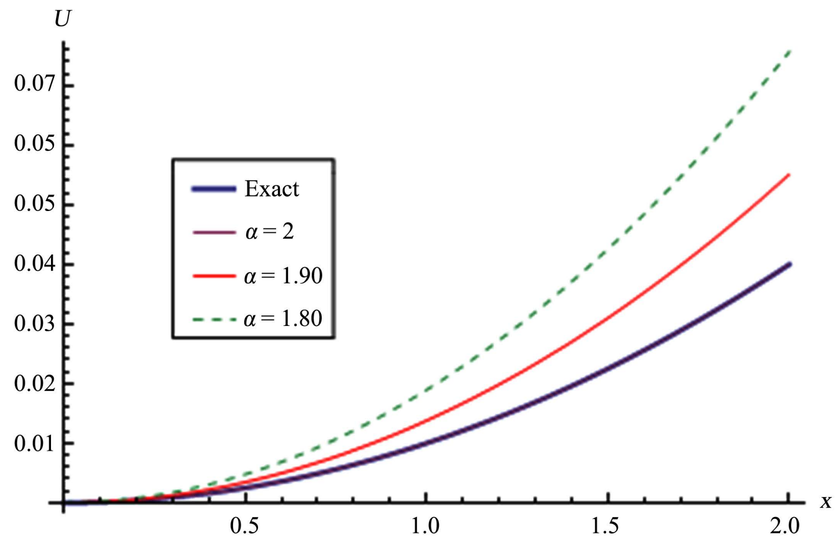


Figure 4. The approximate and exact solutions for Example 1 when $\gamma = 1.8, \gamma = 1.9, \gamma = 2$.

$$\begin{aligned} V(x,t) = {} & \frac{2x^2t^\gamma}{\Gamma[\gamma+1]} - \frac{4t^{\gamma+2}}{\Gamma[\gamma+3]} + \frac{24x^4t^{\gamma+4}}{\Gamma[\gamma+5]} + \frac{4t^{2\gamma}}{\Gamma[2\gamma+1]} + \frac{288x^2t^{2\gamma+4}}{\Gamma[2\gamma+5]} \\ & - \frac{4x^4t^{3\gamma+4}\Gamma[2\gamma+1]}{\Gamma^2[\gamma+1]\Gamma[3\gamma+1]} + \frac{16x^2t^{3\gamma+2}}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[3\gamma+3]} \\ & - \frac{16t^{3\gamma+4}\Gamma[2\gamma+5]}{\Gamma^2[\gamma+5]\Gamma[3\gamma+9]} - \frac{96x^6t^{3\gamma+4}\Gamma[2\gamma+5]}{\Gamma[\gamma+1]\Gamma[\gamma+5]\Gamma[3\gamma+5]} \\ & + \frac{192x^4t^{3\gamma+6}\Gamma[2\gamma+7]}{\Gamma[\gamma+1]\Gamma[\gamma+5]\Gamma[3\gamma+7]} - \frac{576x^8t^{3\gamma+3}\Gamma[2\gamma+9]}{\Gamma^2[\gamma+5]\Gamma[3\gamma+9]} + \frac{576t^{3\gamma+4}}{\Gamma[3\gamma+5]} \\ & - \frac{48x^2t^{4\gamma}\Gamma[2\gamma+1]}{\Gamma^2[\gamma+1]\Gamma[4\gamma+1]} - \frac{16x^3t^{4\gamma}\Gamma[3\gamma+1]}{\Gamma[\gamma+1]\Gamma[2\gamma+1]\Gamma[4\gamma+1]} \\ & - \frac{2880x^4t^{4\gamma+4}\Gamma[2\gamma+5]}{\Gamma[\gamma+1]\Gamma[\gamma+5]\Gamma[4\gamma+5]} + \cdots \end{aligned}$$

For $\gamma = 2$, *the exact solution is*

$$V(x,t) = x^2t^2$$

Example 3 *Consider the non-linear inhomogeneous fractional Klein-Gordon equation with cubic nonlinearity*:

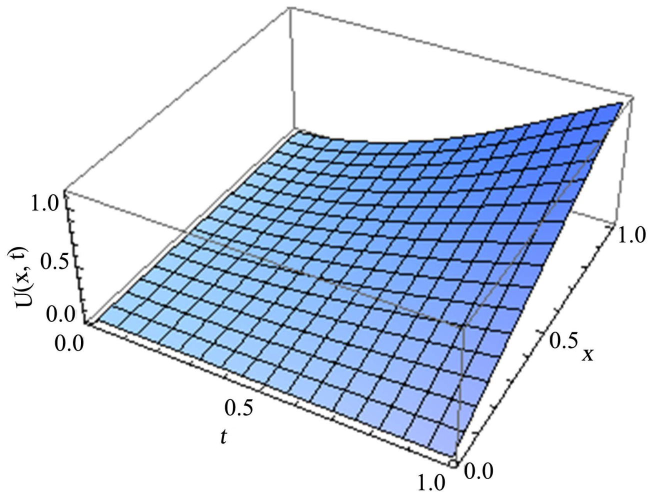
$$\frac{\partial^\gamma V(x,t)}{\partial x^\gamma} + \frac{\partial^2 V(x,t)}{\partial x^2} + V(x,t) + V^3(x,t) = 2x + xt^2 + x^3t^6, \quad t > 0,\ x \in \mathbb{R} \text{ and } 1 < \gamma \le 2 \tag{21}$$

Subject to

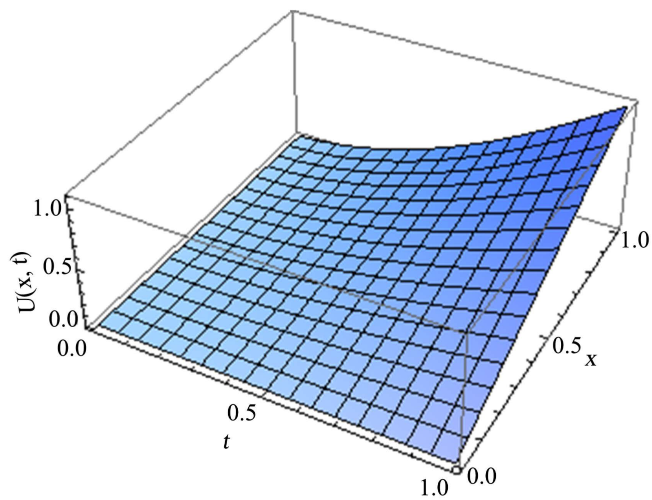
$$V(x,0) = 0, V_t(x,0) = 0$$

Using the previous aforesaid, we get

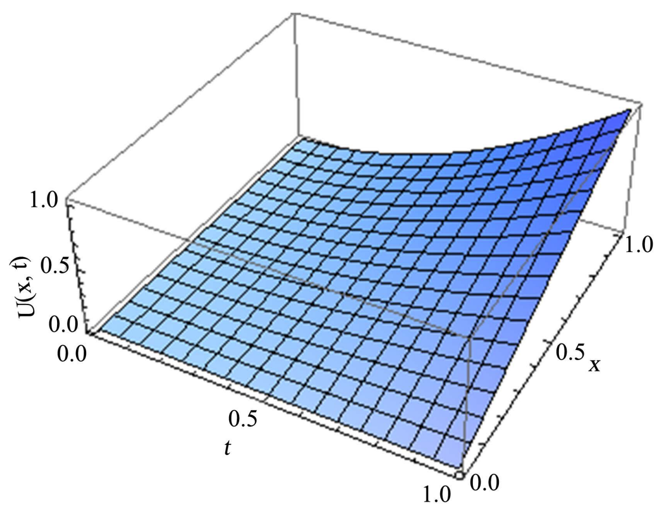
$$V_0(x,t) = \frac{2xt^\gamma}{\Gamma[\gamma+1]} + \frac{2x^4t^{\gamma+2}}{\Gamma[\gamma+3]} + \frac{720x^4t^{\gamma+6}}{\Gamma[\gamma+7]}$$



(a)



(b)



(c)

Figure 5. The approximate solutions for Example 3 when (a) $\gamma = 1.75$; (b) $\gamma = 1.85$; (c) $\gamma = 1.95$.

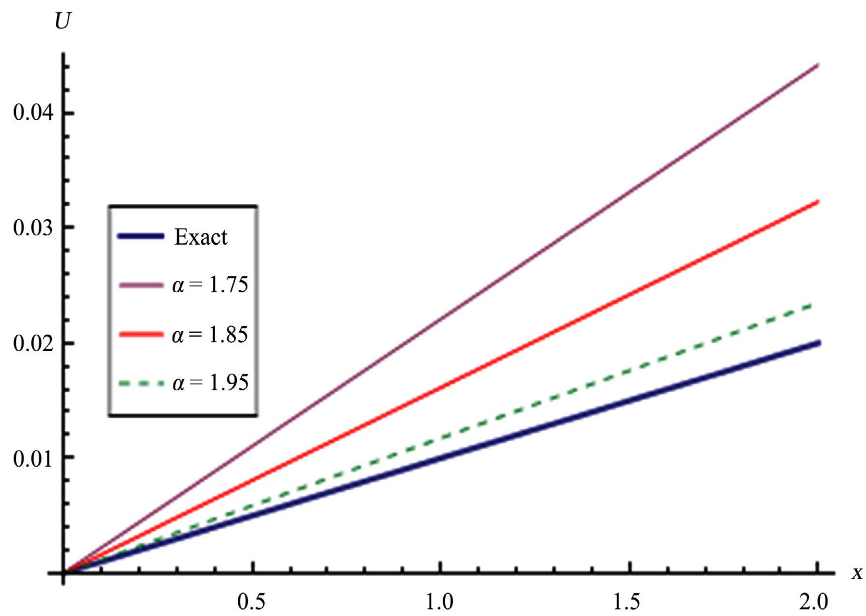


Figure 6. The approximate and exact solutions for Example 3 when $\gamma = 1.75$, $\gamma = 1.85$, $\gamma = 1.95$.

$$V_1(x,t) = -\frac{2xt^{2\gamma}}{\Gamma[2\gamma+1]} - \frac{2xt^{2\gamma+2}}{\Gamma[2\gamma+3]} - \frac{4320xt^{2\gamma+6}}{\Gamma[2\gamma+7]} - \frac{720x^3t^{2\gamma+6}}{\Gamma[2\gamma+7]}$$
$$-\frac{8x^3t^{4\gamma}\Gamma[3\gamma+1]}{\Gamma^3[\gamma+1]\Gamma[4\gamma+1]} - \frac{24x^3t^{4\gamma+2}\Gamma[3\gamma+3]}{\Gamma^2[\gamma+1]\Gamma[\gamma+3]\Gamma[4\gamma+3]}$$
$$-\frac{24x^3t^{4\gamma+4}\Gamma[3\gamma+5]}{\Gamma[\gamma+1]\Gamma^2[\gamma+3]\Gamma[4\gamma+5]} - \frac{8x^3t^{4\gamma+6}\Gamma[3\gamma+7]}{\Gamma^3[\gamma+3]\Gamma[4\gamma+7]}$$
$$-\frac{8640x^5t^{4\gamma+6}\Gamma[3\gamma+7]}{\Gamma^2[\gamma+1]\Gamma[\gamma+7]\Gamma[4\gamma+7]} - \frac{17280x^5t^{4\gamma+8}\Gamma[3\gamma+9]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[\gamma+7]\Gamma[4\gamma+9]}$$
$$-\frac{8640x^5t^{4\gamma+10}\Gamma[3\gamma+11]}{\Gamma^2[\gamma+3]\Gamma[\gamma+7]\Gamma[4\gamma+11]} - \frac{3110400x^7t^{4\gamma+12}\Gamma[4\gamma+13]}{\Gamma[\gamma+1]\Gamma^2[\gamma+7]\Gamma[4\gamma+13]}$$
$$-\frac{3110400x^5t^{4\gamma+14}\Gamma[3\gamma+15]}{\Gamma[\gamma+3]\Gamma^2[\gamma+7]\Gamma[4\gamma+15]} - \frac{373248x^9t^{4\gamma+18}\Gamma[3\gamma+19]}{\Gamma^2[\gamma+7]\Gamma[4\gamma+19]}.$$

Therefore, the series solution is given by (**Figure 5** *and* **Figure 6**):

$$V(x,t) = \frac{2xt^{\gamma}}{\Gamma[\gamma+1]} + \frac{2x4t^{\gamma+2}}{\Gamma[\gamma+3]} + \frac{720x^4t^{\gamma+6}}{\Gamma[\gamma+7]} - \frac{2xt^{2\gamma}}{\Gamma[2\gamma+1]} - \frac{2xt^{2\gamma+2}}{\Gamma[2\gamma+3]}$$
$$-\frac{4320xt^{2\gamma+6}}{\Gamma[2\gamma+7]} - \frac{720x^3t^{2\gamma+6}}{\Gamma[2\gamma+7]} - \frac{8x^3t^{4\gamma}\Gamma[3\gamma+1]}{\Gamma^3[\gamma+1]\Gamma[4\gamma+1]}$$
$$-\frac{24x^3t^{4\gamma+2}\Gamma[3\gamma+3]}{\Gamma^2[\gamma+1]\Gamma[\gamma+3]\Gamma[4\gamma+3]} - \frac{24x^3t^{4\gamma+4}\Gamma[3\gamma+5]}{\Gamma[\gamma+1]\Gamma^2[\gamma+3]\Gamma[4\gamma+5]}$$
$$-\frac{8x^3t^{4\gamma+6}\Gamma[3\gamma+7]}{\Gamma^3[\gamma+3]\Gamma[4\gamma+7]} - \frac{8640x^5t^{4\gamma+6}\Gamma[3\gamma+7]}{\Gamma^2[\gamma+1]\Gamma[\gamma+7]\Gamma[4\gamma+7]}$$

$$-\frac{17280x^5t^{4\gamma+8}\Gamma[3\gamma+9]}{\Gamma[\gamma+1]\Gamma[\gamma+3]\Gamma[\gamma+7]\Gamma[4\gamma+9]}-\frac{8640x^5t^{4\gamma+10}\Gamma[3\gamma+11]}{\Gamma^2[\gamma+3]\Gamma[\gamma+7]\Gamma[4\gamma+11]}$$
$$-\frac{3110400x^7t^{4\gamma+12}\Gamma[4\gamma+13]}{\Gamma[\gamma+1]\Gamma^2[\gamma+7]\Gamma[4\gamma+13]}-\frac{3110400x^5t^{4\gamma+14}\Gamma[3\gamma+15]}{\Gamma[\gamma+3]\Gamma^2[\gamma+7]\Gamma[4\gamma+15]}$$
$$-\frac{373248x^9t^{4\gamma+18}\Gamma[3\gamma+19]}{\Gamma^2[\gamma+7]\Gamma[4\gamma+19]}+\cdots$$

For $\gamma = 2$, *the exact solution is*

$$V(x,t) = xt^2$$

5. Conclusion

In this research, we successfully applied (NTDM) to obtain an approximate solution for the linear and nonlinear fractional Klein Gordon equation. We have tested the method on three examples, which revealed that the method is highly effective. Indeed, in **Figure 4**, we see that when $\gamma = 2$, the approximate solution is almost identical to the exact solution.

Conflicts of Interest

The author declares that they have no competing interests.

References

[1] He, J.-H. (1998) Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media. *Computer Methods in Applied Mechanics and Engineering*, **167**, 57-68. https://doi.org/10.1016/S0045-7825(98)00108-X

[2] He, J.-H. (1998) Nonlinear Oscillation with Fractional Derivative and Its Applications. *International Conference on Vibrating Engineering*, Dalian, 288-291.

[3] He, J.-H. (1999) Some Applications of Nonlinear Fractional Differential Equations and Their Approximations. *Bulletin of Science, Technology & Society*, **15**, 86-90.

[4] Tamsir, M. and Srivastava, V.K. (2016) Analytical Study of Time-Fractional Order Klein-Gordon Equation. *Alexandria Engineering Journal*, **55**, 561-567. https://doi.org/10.1016/j.aej.2016.01.025

[5] Srivastava, V.K., Awasthi, M.K. and Tamsir, M. (2013) RDTM Solution of Caputo Time Fractional-Order Hyperbolic Telegraph Equation. *AIP Advances*, **3**, Article ID: 032142. https://doi.org/10.1063/1.4799548

[6] Zhang, Y.-N., Sun, Z.-Z. and Liao, H.-L. (2014) Finite Difference Methods for the Time Fractional Diffusion Equation on Non-Uniform Meshes. *Journal of Computational Physics*, **265**, 195-210. https://doi.org/10.1016/j.jcp.2014.02.008

[7] Keskin, Y. and Oturanc, G. (2010) Reduced Differential Transform Method: A New Approach to Factional Partial Differential Equations. *Nonlinear Science Letters A*, **1**, 61-72.

[8] Momani, S. and Odibat, Z. (2007) Comparison between the Homotopy Perturbation Method and the Variational Iteration Method for Linear Fractional Partial Differential Equations. *Computers & Mathematics with Applications*, **54**, 910-919. https://doi.org/10.1016/j.camwa.2006.12.037

[9] Sudha Priya, G., Prakash, P., Nieto, J.J. and Kayar, Z. (2013) Higher-Order Numer-

ical Scheme for the Fractional Heat Equation with Dirichlet and Neumann Boundary Conditions. *Numerical Heat Transfer*, *Part B*: *Fundamentals*, **63**, 540-559. https://doi.org/10.1080/10407790.2013.778719

[10] Elbadri, M. (2022) Initial Value Problems with Generalized Fractional Derivatives and Their Solutions via Generalized Laplace Decomposition Method. *Advances in Mathematical Physics*, **2022**, Article ID: 3586802. https://doi.org/10.1155/2022/3586802

[11] Elbadri, M. (2013) A New Homotopy Perturbation Method for Solving Laplace Equation. *Advances in Theoretical and Applied Mathematics*, **4**, 37-242.

[12] Elbadri, M. and Elzaki, T.M. (2013) New Modification of Homotopy Perturbation Method and the Fourth-Order Parabolic Equations with Variable Coefficients. *Pure and Applied Mathematics Journal*, **21**, 242-247. https://doi.org/10.11648/j.pamj.20150406.13

[13] Chowdhury, M.S.H. and Hashim, I. (2009) Application of Homotopy-Perturbation Method, to Klein-Gordon and Sine-Gordon Equations. *Chaos*, *Solitons & Fractals*, **39**, 1928-1935. https://doi.org/10.1016/j.chaos.2007.06.091

[14] Lu, J. (2009) An Analytical Approach to the Sine Gordon Equation Using the Modified Homotopy Perturbation Method. *Computers & Mathematics with Applications*, **58**, 2313-2319. https://doi.org/10.1016/j.camwa.2009.03.071

[15] Eslami, M. and Biazar, J. (2014) Analytical Solution of the Klein-Gordon Equation by a New Homotopy Perturbation Method. *Computational Mathematics and Modeling*, **25**, 124-134. https://doi.org/10.1007/s10598-013-9213-y

[16] Odibat, Z. and Momani, S. (2007) A Reliable Treatment of Homotopy Perturbation Method for Klein-Gordon Equations. *Physics Letters A*, **365**, 351-357. https://doi.org/10.1016/j.physleta.2007.01.064

[17] Elbadri, M. (2010) Comparison between the Homotopy Perturbation Method and Homotopy Perturbation Transform Method. *Applied Mathematics*, **9**, 130-137. https://doi.org/10.4236/am.2018.92009

[18] Singh, J., Kumar, D. and Rathore, S. (2012) Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinear Klein-Gordon Equations. *Journal of Information and Computing Science*, **7**, 131-139.

[19] Jafari, H., Khalique, C.M. and Nazari, M. (2011) Application of the Laplace Decomposition Method for Solving Linear and Nonlinear Fractional Diffusion-Wave Equations. *Applied Mathematics Letters*, **24**, 1799-1805. https://doi.org/10.1016/j.aml.2011.04.037

[20] Hosseinzadeh, H., Jafari, H. and Roohani, M. (2010) Application of Laplace Decomposition Method for Solving Klein-Gordon Equation. *World Applied Sciences Journal*, **8**, 809-813.

[21] Khan, Z.H. and Khan, W.A. (2008) N-Transform Properties and Applications. *NUST Journal of Engineering Sciences*, **1**, 127-133.

[22] Rawashdeh, M.S. and Al-Jammal, H. (2018) Theories and Applications of the Inverse Fractional Natural Transform Method. *Advances in Difference Equations*, **2018**, Article No. 222. https://doi.org/10.1186/s13662-018-1673-0

[23] Belgacem, F.B.M. and Silambarasan, R. (2012) Theory of Natural Transform. *Mathematics in Engineering*, *Science and Aerospace* (*MESA*) *Journal*, **3**, 99-124.

[24] Belgacem, F.B.M. and Silambarasan, R. (2012) Advances in the Natural Transform. *AIP Conference Proceedings*, **1493**, Article No. 106. https://doi.org/10.1063/1.4765477

[25] Abdel-Rady, A.S., Rida, S.Z., Arafa, A.A.M. and Abedl-Rahim, H.R. (2015) Natural Transform for Solving Fractional Models. *Journal of Applied Mathematics and Physics*, **3**, 1633-1644. https://doi.org/10.4236/jamp.2015.312188

[26] Elbadri, M., Ahmed, S.A., Abdalla, Y.T. and Hdidi, W. (2020) A New Solution of Time-Fractional Coupled KdV Equation by Using Natural Decomposition Method. *Abstract and Applied Analysis*, **2020**, Article ID: 3950816. https://doi.org/10.1155/2020/3950816

[27] Rawashdeh, M.S. and Maitama, S. (2015) Solving Nonlinear Ordinary Differential Equations Using the NDM. *Journal of Applied Analysis and Computation*, **5**, 77-88. https://doi.org/10.11948/2015007

[28] Rawashdeh, M. and Maitama, S. (2017) Finding Exact Solutions of Nonlinear PDEs Using the Natural Decomposition Method. *Mathematical Methods in the Applied Sciences*, **40**, 223-236. https://doi.org/10.1002/mma.3984

[29] Khan, H., Shah, R., Kumam, P. and Arif, M. (2019) Analytical Solutions of Fractional-Order Heat and Wave Equations by the Natural Transform Decomposition Method. *Entropy*, **21**, Article No. 597. https://doi.org/10.3390/e21060597

[30] Rawashdeh, M. and Al-Jammal, H. (2016) Numerical Solutions for Systems of Nonlinear Fractional Ordinary Differential Equations Using the FNDM. *Mediterranean Journal of Mathematics*, **13**, 4661-4677. https://doi.org/10.1007/s00009-016-0768-7

[31] Shah, R., Kha, H., Kumam, P., Arif, M. and Baleanu, D. (2019) Natural Transform Decomposition Method for Solving Fractional-Order Partial Differential Equations with Proportional Delay. *Mathematics*, **7**, Article No. 532. https://doi.org/10.3390/math7060532