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An Example of a Bounded Potential $q(x)$ on the Half-Line, for Estimates of $A(\alpha)$ Amplitude

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Abstract

We show an example of a bounded potential on the half-line obtained as the image of an Inverse Transformation Operator of the Bessel singular potential of the Reduced Radial Schrödinger Equation, and show us the Estimates of the $A(\alpha)$ amplitude.

Keywords

The Amplitude $A(\alpha)$ as a Function of the Phase *a*, Estimates of the $A(\alpha)$ Amplitude, A Bounded Potential on the Half Line

1. Introduction

By way of contextualizing, we begin with a summary of the cited article [1], highlighting the results related to the amplitude $A(\alpha)$ and the bounded potential on the half line. The last *Theorem* (10.2) stands out, where the estimates of the amplitude $A(\alpha)$ appear with respect to the norm of the bounded potential on the half-line.

They start considering Schrödinger operators

$$-\frac{d^2}{dx^2}+q, \quad (1.1)$$

in $L_2(0,b)$ for $0<b<\infty$ or $b=\infty$ and real-valued locally integrable *q*.

They are interested in cases for $b=\infty$, that is

Case 2: *q* is "essentially" bounded from below in the sense that

$$\sup_{a>0}\left(\int_a^{a+1}\max(-q(x),0)\right)dx<\infty. \quad (1.4)$$

Case 3: (1.4) fails but (1.1) is *limit point at* ∞, that is, for each

$z \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$,

$$-u'' + qu = zu, \quad (1.5)$$

has a unique solution, up to a multiplicative constant, which is L_2 at ∞.

Case 4: (1.1) is *limit circle at infinity*; that is, every solution of (1.5) is $L_2(0,\infty)$ at infinity if $z \in \mathbb{C}_+$. We then pick a boundary condition by picking a nonzero solution u_0 of (1.5) for $z = i$. Other functions *u* satisfying the associated boundary condition at infinity then are supposed to satisfy

$$\lim_{x\to\infty}\left[u_0(x)u'(x) - u_0'(x)u(x)\right] = 0. \quad (1.6)$$

1.1. The Function Weyl-Titchmarscht $m(z)$

$m(z)$, is defined for $z \in \mathbb{C}_+$ as follows. Fix $z \in \mathbb{C}_+$. Let $u(x,z)$ be a nonzero solution of (1.5) which satisfies the boundary condition at *b*. That is, in Cases 2 and 3: $\int_R^\infty |u(z,x)|^2 \,\mathrm{d}x < \infty, \forall R > 0$ and

$$m(z) := \frac{u'(0_+,z)}{u(0_+,z)}. \quad (1.7)$$

In Case 4, it satisfies (1.6). And, more generally

$$m(z) := \frac{u'(x,z)}{u(x,z)}. \quad (1.8)$$

$m(z,x)$ satisfies the *Riccati equation* (with $m' = \frac{\partial m}{\partial x}$)

$$m'(z,x) = q(x) - z - m(z,x)^2. \quad (1.9)$$

$m(z,x)$ is an analytic function of *z* for $z \in \mathbb{C}_+$, and

Case 2: For some $\beta \in \mathbb{R}$, m has an analytic continuation to $\mathbb{C}\setminus[\beta,\infty)$ with *m* real on $(-\infty,\beta)$.

Case 3: In general, *m* cannot be continued beyond $\mathbb{C}_+$ (there exist *q's* where *m* has a dense set of polar singularities on $\mathbb{R}$).

Case 4: *m* is meromorphic in $\mathbb{C}$ with a discrete set of poles (and zeros) on $\mathbb{R}$ with limit points at both $+\infty$ and $-\infty$.

Moreover, if $z \in \mathbb{C}_+$ then $m(z,x) \in \mathbb{C}_+$; so *m* satisfies a *Herglotz representation theorem*,

$$m(z) = c + \int_{\mathbb{R}}\left[\frac{1}{\lambda - z} + \frac{1}{1+\lambda^2}\right]\mathrm{d}\rho(\lambda), \quad (1.10)$$

where ρ is a positive measure called the spectral measure, which satisfies

$$\int_{\mathbb{R}} \frac{\mathrm{d}\rho(\lambda)}{1+\lambda^2} < \infty, \quad (1.11)$$

$$\mathrm{d}\rho(\lambda) = w - \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}\left(m(\lambda + i\varepsilon)\right)\mathrm{d}\lambda. \quad (1.12)$$

And it was found

$$c = \mathrm{Re}\, m(i).$$

1.2. Existence of Function Amplitude $A(\alpha)$

Previous results

Theorem 1.2 ([1], *Theorem* 2.1). Let $q \in L_1(0,\infty)$. *Then, there exists a function* $A(\alpha)$ on $(0,\infty)$ so that $A-q$ is continuous and satisfies (1.16)

Theorem 1. *There exists a function* $A(\alpha)$ *for all* $\alpha \in [0,b)$ *so that* $A \in L_1(0,a)$, *for all* $a < b$ *and*

$$m(-\kappa^2) = -\kappa - \int_0^a A(\alpha)e^{-2\alpha\kappa}d\alpha + \tilde{O}(e^{-2\alpha\kappa}), \quad (1.15) \tag{1}$$

as $\kappa \to \infty$ *with* $-\frac{\pi}{2} + \varepsilon < \arg(\kappa) < -\varepsilon < 0$. $f = \tilde{O}(g)$ *if* $g \to 0$ *and for all* $\varepsilon > 0$, $(f/g)|g|^{\varepsilon} \underset{\kappa\to\infty}{\to} 0$. *Moreover,* $A-q$ *is continuous and*

$$|(A-q)\alpha| \le \left[\int_0^\alpha |q(x)|dx\right]^2 \exp\left(\alpha\int_0^\alpha |q(x)|dx\right). \quad (1.16)$$

One of your purposes here is to prove this result if one only assumes (1.3) (*i.e.* in Cases 3 and 4).

Previous results

Theorem 2. ([1], *Theorem* 2.1) *Let* $q \in L_1(0,\infty)$. *Then, there exists a function* $A(\alpha)$ *on* $(0,\infty)$ *so that* $A-q$ *is continuous and satisfies* (1.16) *such that* $\mathrm{Re}(\kappa) > \frac{1}{2}\|q\|_1$,

$$m(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha)e^{-2\kappa\alpha}d\alpha. \ (1.17)$$

Theorem 1.1 in all cases follows from *Theorem* 1.2 and the following result which we will prove in Section 3.

Theorem 3. *Let* q_1, q_2 *be potentials defined on* $(0,b_j)$ *with* $b_j > a$ *for* $j = 1,2$. *Suppose that* $q_1 = q_2$ *on* $[0,a]$. *Then, in the region* $\arg(\kappa) \in (-\pi/2+\varepsilon, -\varepsilon)$, $|\kappa| \ge K_0$, *we have that*

$$|m_1(-\kappa^2) - m_2(-\kappa^2)| \le C_{\varepsilon,\delta} \exp(-2a\,\mathrm{Re}(\kappa)), \ (1.18)$$

where $C_{\varepsilon,\delta}$ *depends only on* ε, δ, *and* $\sup_{0\le x\le a}\left(\int_x^{x+\delta}|q_j(y)|dy\right)$, *where* $\delta > 0$ *is any number so that* $a + \delta \le b_j$, $j = 1,2$.

1.3. The Connection between the Spectral Measure dρ and the A-Amplitude

Your basic formula says that

$$A(\alpha) = -2\int_{-\infty}^{\infty} \lambda^{-1/2} \sin(2\alpha\sqrt{\lambda})d\rho(\lambda). \ (1.21)$$

ρ gives nonzero weight to $(-\infty,0]$, they interpret

$$\lambda^{-1/2}\sin(2\alpha\sqrt{\lambda}) = \begin{cases} 2\alpha, & \text{if } \lambda = \alpha, \\ (-\lambda)^{-1/2}\sinh(2\alpha\sqrt{-\lambda}), & \text{if } \lambda < 0. \end{cases}$$

Consistent with the fact that $\lambda^{-1/2}\sin(2\alpha\sqrt{\lambda})$ defined on $(0,\infty)$ extends to an entire function of λ.

1.4. *A* Satisfies the Simple Differential Equation in the Distributional Sense

$$\frac{\partial A}{\partial x}(\alpha,x)=\frac{\partial A}{\partial \alpha}(\alpha,x)+\int_0^a A(\alpha-\beta,x)A(\beta,x)\,\mathrm{d}\beta. \quad (1.26)$$

This is prove in [1] for $q\in L_1(0,a)$ (and some other *q*s) and so holds in the generality of this paper since *Theorem* 1.3 implies $A(\alpha,x)$ for $\alpha+x\le a$ is only a function of $q(y)$ for $y\in[0,a]$.

Moreover, by (1.16), they have

$$\lim_{\alpha\downarrow 0}\left|A(\alpha,x)-q(\alpha+x)\right|=0, \quad (1.27)$$

uniformly in x on compact subsets of the real line, so by the uniqueness theorem for solutions of (1.26), *A on* $[0,a]$ *determines q* on $[0,a]$.

1.5. The Riccati Equation and the Atkinson Method and the Exponential Bounds for *m*

As explained in the introduction, the Riccati equation and a priori control on m_j allow one to obtain exponentially small estimates on m_1-m_2 (*Theorem* 1.5).

Proposition 4. (*Proposition* 2.1) *Let* $m_1(x),m_2(x)$ *be two absolutely continuous functions on* $[a,b]$ *so that for some* $Q\in L^1(a,b)$,

$$m_j'(x)=Q(x)-m_j(x)^2,\ j=1,2,\ x\in(a,b),$$

then $\left[m_1(a)-m_2(a)\right]=\left[m_1(b)-m_2(b)\right]\exp\left(\int_a^b\left[m_1(y)+m_2(y)\right]\mathrm{d}y\right)$.

As an immediate corollary, they have the following (this implies *Theorem* 1.3).

Theorem 5. (*Theorem* 2.2) *Let* $m_j(x,-\kappa^2)$ *be functions defined for* $x\in[a,b]$ *and* $\kappa\in K$ *some region of* $\mathbb{C}$. *Suppose that for each* κ *in K*, m_j *is absolutely continuous in x and satisfies* (*N.B.*: *q is the same for* m_1 *and* m_2),

$$m_j'(x,-\kappa^2)=q(x)+\kappa^2-m_j(x,-\kappa^2)^2,\ j=1,2.$$

Suppose C is such that for each $x\in[a,b]$ *and* $\kappa\in K$,

$$\left|m_j(x,-\kappa^2)+\kappa\right|\le C,\ j=1,2, \quad (2.2)$$

then

$$\left|m_1(a,-\kappa^2)-m_2(a,-\kappa^2)\right|\le 2C\exp\left[-2(b-a)\left(\mathrm{Re}(\kappa)-C\right)\right]. \quad (2.3)$$

They mention *Theorem* 2.2 places importance on a priori bounds of the form (2.2). Fortunately, by modifying ideas of Atkinson, we can obtain estimates of this form as long as $\mathrm{Im}(\kappa)$ is bounded away from zero.

As a final result

Theorem 6. (*Theorem* 4.1) *If* $q\in L_1(0,\infty)$ *and* $\mathrm{Re}(\kappa)>1$, *then for all a*:

$$\left|m(-\kappa^2)+\kappa+\int_0^a A(\alpha)\mathrm{e}^{-2\kappa\alpha}\mathrm{d}\alpha\right|\le\left[\|q_1\|+\frac{\|q_1\|^2\mathrm{e}^{a\|q_1\|}}{2\,\mathrm{Re}(\kappa)-\|q_1\|}\right]\mathrm{e}^{-2\,\mathrm{Re}(\kappa)}. \quad (4.1)$$

And they get

Corollary 7. (*Corollary* 4.8) *Fix* $b<\infty$, $q\in L_1(0,b)$, *and* $|h|<\infty$ (*let's re-*

member that: $u'(b_-)+hu(b_-)=0$ (1.2), $h\in\mathbb{R}\cup\{\infty\}$, *where* $h=\infty$ *is shorthand for the Dirichlet boundary condition* $u(b_-)=0$) *Fix* $a<b$. *Then, there exist positive constants C and* K_0 *so that for all complex* κ *with* $\mathrm{Re}(\kappa)>K_0$,

$$\left|m(-\kappa^2)+\kappa+\int_0^a A(\alpha)e^{-2\kappa\alpha}d\alpha\right|\le e^{-2\mathrm{Re}(\kappa)}.$$

1.6. The Bounds for the *A* Amplitude for the Potential *q* in the Half-Line

So far, it has been assumed that the potential $q\in L_1(0,\infty)$ or $q\in L_1(0,a)$ for all $a<\infty$ and $h<\infty$.

Now, they assume examples with constant or bounded potentials defined on half-line $(0,b)$ with $b=\infty$, and with $h=\infty$ is shorthand for the Dirichlet boundary condition $\lim_{b\downarrow\infty}u(b_-)=0$.

See [1]: 10. Examples, I: Constant *q*.

Your claim

Theorem 8. (*Theorem* 10.1) *If* $b=\infty$ *and* $q(x)=q_0$, $x\ge 0$, *then if* $q_0>0$,

$$A(\alpha)=\frac{q_0^{1/2}}{\alpha}J_1\left(2\alpha q_0^{1/2}\right),\quad (10.1)$$

where J_1 *is the Bessel function of order one* (*see, e.g.* ([1], *Chapter* 9)); *if* $q_0<0$

$$A(\alpha)=\frac{(-q_0)^{1/2}}{\alpha}I_1\left(2\alpha(-q_0)^{1/2}\right),\quad (10.2)$$

with $I_1(.)$ *the corresponding modified Bessel function of order one* (*see, e.g.* ([1], *Chapter* 9)). *Since* [1], *p.* 375

$$0\le I_1(x)\le e^x,\ x>0.\quad (9.27)$$

This example is especially important because of a monotonicity property.

Theorem 9. (*Theorem* 10.2) *Let* $|q_1(x)|\le -2q_2(x)$ *on* $[0,a]$ *with* $a\le\min(b_1,b_2)$ *then* $|A_1(\alpha)|\le -A_2(\alpha)$, *on* $[0,a]$. *In particular, for any q satisfying* $\sup_{0\le x\le\alpha}|q(x)|<\infty$, *they have that*

$$|A(\alpha)|\le\frac{\gamma(\alpha)}{\alpha}I_1(2\alpha),\quad (10.5)$$

where

$$\gamma(\alpha)=\sup_{0\le x\le\alpha}\left(|q(x)|^{1/2}\right).\quad (10.6)$$

In particular, (9.27) *implies*

$$|A(\alpha)|\le\alpha^{-1}\gamma(\alpha)e^{2\alpha\gamma(\alpha)},\quad (10.7)$$

and if q is bounded,

$$|A(\alpha)|\le\alpha^{-1}\|q\|_\infty\exp\left(2\alpha\|q\|_\infty^{1/2}\right).\quad (10.8)\qquad (2)$$

The article is divided into the following sections.

In Section 2: Background. The results obtained in [2] and [3] are mentioned, which conclude with the existence of the Inverse Transformation Operator $\mathbb{W}$, which transforms the solutions of an initial Sturm-Liouville equation into the solutions of a second Sturm-Liouville equation where the potential transformed by $\mathbb{W}$ is unique see [4]. Later, the Reduced Radial Schröndiger equation is considered where, the singular Bessel potential is the sum of the regular potential and a term with a singularity of quadratic order of the RRSE and the Inverse Transformation Operator $\mathbb{W}$ is applied to RRSE one obtains a Sturm-Liouville equation where, the potential $q_2(x)$ obtained is only the regular potential of the RRSE and the singular term of quadratic order does not appear, instead three additional terms. This is the bounded potential proposed on half-line as our mentioned example.

In Section 3: The bounded potential $q(x)$ on the half-line. This section is the main one of the article, because it is proved that the three additional terms to the potential obtained $q_2(x)$ are bounded on half-line: *Theorem* 25. The Proof is done through several consecutive steps: *Lemmas* 17, 18, 19, 21, 23, 24; in which the bounding of each summand of the proposed potential is proven. Highlighting, the bounding achieved by the *Jost function* through the magnitudes of the Eigen values and the regular potential [5].

Finally, in Section 4: Estimates for the $A(\alpha)$ amplitude for a bounded potential on half-line. We quote the corresponding *Theorem* 10.3 of [3] and the proposed potential is exhibited.

Other articles, where the A amplitude is mentioned as a function of the phase α and estimates are established are: [6] [7] [8].

2. Background

Following what we name the formulation of Marchenko [2] in [3], we obtained the following results.

Consider two problems with symmetrical boundary value problems and defined by for $j = 1,2$ through:

$$-y'' + q_j(x)y = s^2 y, 0 < x < \infty, \tag{3}$$

$$y'(0) - k_j y(0) = 0, \tag{4}$$

where $k_j \in \{h_1, h_2\}$, h_1, h_2 are different real numbers,

$$s \in \{\lambda(h_1), \mu(h_2)\}, \{\lambda(h_1), \mu(h_2)\}, \tag{5}$$

represents the same family of eigenvalues for both problems, $q_j(x)$ are continuous real valued functions. Their uniqueness is determined through their respective spectral distribution function R_j. The aim of the paper is to relate both previous problems in the following way. We will assume the uniqueness of the first problem and determine the uniqueness of the second problem by linking: both spectral distribution functions R_j, both boundary conditions

$$y'(0) - k_j y(0) = 0, \tag{6}$$

and both potential q_j. The stated theorem was: [2], Section 4, pp. 481-483.

Theorem 10. (*Theorem* 1) *Let* R_1 *be a spectral distribution function of the boundary value problem* 2 *with* $j = 1$ *and for*

$$s := \lambda, \mu, \tag{7}$$

the corresponding solution of the equation: $j = 1$

$$\omega_1(\lambda, x) := \omega_1(\lambda, x, h_1),\ \omega_1(\mu, y) := \omega_1(\mu, y, h_1) \tag{8}$$

respectively. We will assume the following hypothesis:

$$1 + c\int_0^x \omega_1^2(\mu, t)\,\mathrm{d}t \neq 0 \quad \forall x \in [0, \infty), \tag{9}$$

and

$$q_2(x) := q_1(x) - 2K_{2,1}(x, x), \tag{10}$$

$$h_2 := h_1 - c,$$

$$R_2 := R_1 - c\delta(\mu - \lambda),$$

where kernel $K_{2,1}(x, x)$ *was defined in Lemma* 1. *Then* R_2 *is a spectral distribution function of the boundary value problem* $j = 2$

$$-y'' + q_2(x)y = \mu^2 y,\ y(0)h_2 - y'(0) = 0. \tag{11}$$

On the other hand, following a series of problems proposed by V. Marchenko 4, that we will name Marchenko's formulation, and relating it to a generalized version of *Theorem* 1 given in 3. The main theorem was proved: [2].

Theorem 11. (*Theorem* 1) *Let's consider two Sturm-Liouville equations*

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j,\ j = 1, 2,\ x \in (a, b),\ b \geq \infty, \tag{12}$$

$q_j(x)$ *continuous only in the interior points of* (a, b). *Consider in particular, the following pair of boundary value problems of Sturm-Liouville on the Half Line*

$$-y'' + q_1(x)y = s^2 y,\ y(0) = 0, \text{ where } s \in \{\lambda, \mu\},\ x > 0, \tag{13}$$

and

$$-y'' + q_2(x)y = s^2 y,\ y(0) = 0 \text{ where } s \in \{\lambda, \mu\},\ x > 0. \tag{14}$$

Let $q_1(x), q_2(x)$ are continuous on $(0, \infty)$. If $y_1(x, \mu) := y_1$ is a fixed solution of the first equation $j = 1$ for $s = \mu$ and let $\varphi_1(\lambda, x) := \varphi_1$ an arbitrary solution of $j = 1$ for $s = \lambda$, then

$$-y_1'' + q_1(x)y_1 = \mu^2 y_1,\ y_1(0, \mu) = 0 \text{ and} \tag{15}$$

$$-\varphi_1'' + q_1(x)\varphi_1 = \lambda^2 \varphi_1,\ \varphi_1(0, \lambda) = 0. \tag{16}$$

Suppose

$$y_1(x, \mu) \neq 0,\ \forall x > 0. \tag{17}$$

If

$$\varphi_2(\lambda, x) := \frac{W(y_1(x, \mu), \varphi_1(\lambda, x))}{y_1(x)(\mu^2 - \lambda^2)},\ \mu \neq \lambda, \tag{18}$$

is solution of $j = 2$ for $s = \lambda$ where the wronskian

$$W(y_1(x, \mu), \varphi_1(\lambda, x)) := y_1(x, \mu)\varphi_1'(\lambda, x) - y_1'(x, \mu)\varphi_1(\lambda, x) \neq 0, \tag{19}$$

$' := \frac{\mathrm{d}}{\mathrm{d}x}$. Then, $\varphi_2(\lambda, x)$ satisfies the equation

$$-\varphi_2'' + q_2(x)\varphi_2 = \lambda^2\varphi_2,\ \varphi_2(0,\lambda) = 0, \tag{20}$$

where

$$q_2(x) = \mu^2 - 2q_1 + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}. \tag{21}$$

According to (16), we define the corresponding *Inverse Transformation Operator* as

$$\mathbb{W} := \frac{W(y_1(x,\mu),\cdot)}{y_1(x)(\mu^2 - \lambda^2)} \Rightarrow \mathbb{W}\varphi_1(x) := \frac{W(y_1(x,\mu),\varphi_1(x))}{y_1(x)(\mu^2 - \lambda^2)} = \varphi_2(x). \tag{22}$$

In [3], Section 3: *The Two Examples*: *Reducced Radial Schrödinger Equation and Schrödinger Equation on the Half-Line*, pp. 492-501, we apply *Inverse Transformation Operator* $\mathbb{W}$ previous to the *Reduced Radial Schrödinger Equation* (*RRSE*)

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi_l(k,r) + \left[V(r) + \frac{l(l+1)}{r^2}\right]\psi_l(k,r) = k^2\psi_l(k,r), \tag{23}$$

where $\psi_l(k,r)$ the partial wave of angular momentum *l* and wave number *k*. (whose main characteristic is the addition a singular term of quadratic order (named *Bessel Singular Potential*) to a regular potential $V(x)$ if $x := r$

$$\int_0^\infty x|V(x)|\mathrm{d}x < \infty. \tag{24}$$

If

$$q_1 := V(r) + \frac{l(l+1)}{r^2}, \tag{25}$$

finally one gets

$$\mathbb{W}q_1 := q_2 = \begin{cases} \mu^2 - 2V(x), & \text{for } 0 < x \le \sqrt[l]{\frac{l+1}{2l}} < 1 \\ \mu^2 - 2V(x) + \frac{\mu}{i^{l-1}}\mathrm{e}^{-i\mu x} + \frac{\varphi_1'}{\varphi_2}, & \text{for } x > 1, l \text{ even} \end{cases},\ (119) \tag{26}$$

we have obtained the uniqueness of the potential which is regular when $x \to 0$ *and, bounded with exponential decrease fast enough when* $x \to \infty$. See [3], Section 3, Formula 119, p. 501.

We will use the following estimation to $\frac{\varphi_1'}{\varphi_2} = O(1)$, that is:

$$\left|\frac{\varphi_1'}{\varphi_2}\right| \le \frac{|\mu^2 - \lambda^2||\lambda|}{2|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}||F_l(\lambda)| + 2|\mu^2 - \lambda^2|}{|F_l(\mu)|} + |(-\mu)^{-l}|\right) := C,\ (170)\text{ - }(171) \tag{27}$$

$$\frac{\varphi_1'}{\varphi_2} = \begin{cases} cx^{2l+1}, & \text{if } 0 < x < 1, \\ O(1), & \text{for } x > 1. \end{cases}\ (172) \tag{28}$$

See Formulas 170 - 172, p. 522, *Appendix of* 3.

3. The Bounded Potential $q(x)$ on the Half-Line

3.1. Preliminaries

We start with the following preliminary results

Theorem 12. *Of* 26

$$q_2 = \mu^2 - 2V(x) + \frac{\mu}{i^{l-1}} e^{-i\mu x} + \frac{\varphi_1'}{\varphi_2} \tag{29}$$

is continuous on $(0,\infty)$, $x > 0$, l *even*

Proof. According to *Theorem* 1 [3], Section 2, pp. 485-493: (11)

$$-y'' + q_1(x) y = s^2 y,\ y(0) = 0,\ \text{where } s \in \{\lambda, \mu\},\ x > 0,$$

where q_1 and q_2 are continuous on $(0,\infty)$. And (15) and (16)

$$-y_1'' + q_1(x) y_1 = \mu^2 y_1,\ y_1(0,\mu) = 0 \ \text{ and}$$

$$-\varphi_1'' + q_1(x)\varphi_1 = \lambda^2 \varphi_1,\ \varphi_1(0,\lambda) = 0.$$

If we define to $\varphi_2(\lambda, x)$ as (18)

$$\varphi_2(\lambda, x) := \frac{W\left(y_1(x,\mu), \varphi_1(\lambda, x)\right)}{y_1(x)\left(\mu^2 - \lambda^2\right)},\ \mu \neq \lambda,$$

then one got 20

$$-\varphi_2'' + q_2(x)\varphi_2 = \lambda^2 \varphi_2,\ \varphi_2(0,\lambda) = 0.$$

Now, according to the proof of *Theorem* 1 after laborious calculus we get the successive equations: (49), (50) and (51), see pp. 491-492 of [3].

$$-\varphi_2''(\lambda, x) = \varphi_2(\lambda, x)\left\{\lambda^2 - \left(-q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}\right)\right\}, \tag{49}$$

that is

$$-\varphi_2''(\lambda, x) + \left(-q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2}\right)\varphi_2(\lambda, x) = \lambda^2 \varphi_2(\lambda, x), \tag{50}$$

then if (21)

$$q_2(x) := -q_1 - \frac{y_1''}{y_1} + \frac{\varphi_1'}{\varphi_2} + \frac{y_1'}{y_1^2} \tag{51}$$

must *be continuous* by hypothesis: see *Theorem* 1 [3], Section 2, pp. 485-493, we got (20)

$$-\varphi_2'' + q_2(x)\varphi_2 = \lambda^2 \varphi_2,\ \varphi_2(0,\lambda) = 0. \tag{30}$$

In the case of *Reduced Radial Schrödinger Equation* (*RRSE*) agree to (25)

$q_1 := V(r) + \dfrac{l(l+1)}{r^2}$ then the *uniqueness and continuity of* q_2 *is* obtained (26)

$$\mathbb{W}q_1 := q_2 = \begin{cases} \mu^2 - 2V(x), & \text{for } 0 < x \le \sqrt[l]{\dfrac{l+1}{2l}} < 1 \\ \mu^2 - 2V(x) + \dfrac{\mu}{i^{l-1}} e^{-i\mu x} + \dfrac{\varphi_1'}{\varphi_2}, & \text{for } x > 1,\ l \text{ even} \end{cases} . \quad (31)$$

The Potential *q* Is "Essentially" Bounded

Definition 13.

$$L^1(a,\infty) := \Big\{ q:(a,\infty) \to \mathbb{C} : q \text{ is Lebesgue measurable and } \int_a^\infty |q(x)| \mathrm{d}x < \infty \Big\}. \quad (32)$$

If $q \in L^1(a,\infty)$ *named q integrated.*

Definition 14

$$L^\infty(a,\infty) := \big\{ q:(a,\infty) \to \mathbb{C} : q \text{ is Lebesgue measurable and } \exists M > 0 \text{ with } |q(x)| \le M \big\}. \quad (33)$$

If $q \in L^\infty(a,\infty)$ *named q essentially bounded.*

Definition 15. *The potential q is "essentially" bounded if*

$$\sup_{a>0} \left(\int_a^{a+1} \max(-q(x), 0) \right) \mathrm{d}x < \infty, \quad (1.4) \quad (34)$$

See [1], Introduction, Case 2, Formula 1.4 and we get the

Lemma 16.

$$\|q\|_{L^1(1,\infty)} \Rightarrow \|q\|_{L^\infty(1,\infty)} \quad (35)$$

Proof. Let $a > 1$, using (34) let $q \in L^1(1,\infty)$, since $\int_a^{a+1} |q(x)| \mathrm{d}x \le \int_a^\infty |q(x)| \mathrm{d}x := \|q\|_{L^1(1,\infty)}$ and $\int_a^{a+1} |q(x)| \mathrm{d}x \underset{a<\xi(a)<a+1}{=} |q(\xi(a))|$ then $\exists \sup_{a>1} |q(\xi(a))| \le \|q\|_{L^1(1,\infty)}$, and $|f(x)| \le \sup_{a>1} |q(\xi(a))|$, $x \in (a,\infty)$ then $\sup_{a>1} |q(\xi(a))| := \|q\|_{L^\infty(1,\infty)}$ and therefore

$$q \in L^\infty(1,\infty).$$

■

3.2. Existence of a Bounded Potential Defined on Half-Line

The existence and uniqueness of the potential $q_2(x)$ (31) was established through the Inverse Transformation Operator $\mathbb{W}$. See [3], Section 3, Formula 119, p. 501.

The main result of the article: *the bounded potential* $q_2(x)$ (31) obtained as the image of the Inverse Transformation Operator

$$\mathbb{W}q_1 = q_2 \in L^\infty(0,\infty),$$

which is composed of three fundamental addends

1) $\mu^2 - 2V(x)$, 2) $\dfrac{\mu}{i^{l-1}} e^{-i\mu x}$ and 3) $\dfrac{\varphi_1'}{\varphi_2}$.

The fundamental result consists of the uniform bounded of three addends separately

$$1)\quad \mu^2 - 2V(x), \tag{36}$$

$$2)\quad \frac{\mu}{i^{l-1}} e^{-i\mu x}, \tag{37}$$

$$3)\quad \frac{\varphi_1'}{\varphi_2}, \tag{38}$$

and conclude that the potential obtained $q_2(x)$ given in (31) is such that $q_2(x) \in L^\infty(0,\infty)$.

Let's observe that initial potential q_1 of named *Problem* 1 *of Theorem* 1: ([3], Section 3, Formula 54, p. 492) *is*

$$q_1 := V(x) + \frac{l(l+1)}{x^2}$$

for the *Reduced Radial Schrödinger Equation* (*RRSE*) see (25) where $V(x)$ satisfy (24): $\int_0^\infty x|V(x)|\,dx < \infty$ implies

$$\int_0^\infty |V(x)|\,dx \le \int_0^\infty x|V(x)|\,dx < \infty, \text{ for } 1 < x, l \text{ even} \tag{39}$$

then

$$V(x) \in L^1(1,\infty), \text{ for } 1 < x, l \text{ even.} \tag{40}$$

We start the first addend of (38).

3.3. The First Addend 1) $\mu^2 - 2V(x) \in L^\infty(0,\infty)$

Lemma 17. *Let* $x > 1$, *for each fixed* $\mu \in \mathbb{C}$, *let* $\frac{l(l+1)}{x^2}, V(x) \in L^1(1,\infty)$ (39), *l even and* (31)

$$q_1 := V(x) + \frac{l(l+1)}{x^2} \in L^1(1,\infty) \tag{41}$$

then

$$1)\quad q_2 = \mu^2 - 2V(x) \in L^\infty(1,\infty). \tag{42}$$

Proof. For $\mu \in \mathbb{C}$ fixed and (26):

$$\mathbb{W} q_1 = q_2 \tag{43}$$

For the *uniqueness* of $q_2 \Leftrightarrow (\nvDash \ntrianglelefteq)$

$$\mathbb{W}^{-1} q_2 = q_1 = V(x) + \frac{l(l+1)}{x^2}, \tag{44}$$

and in this case it is fulfilled: $-y_1'' + q_1(x) y_1 = \mu^2 y_1$ (15) then $\mu^2 - 2q_1(x) = -\frac{y_1''}{y_1} - q_1(x)$, by (31) implies

$$\mu^2 - 2V(x) = -\frac{y_1''}{y_1} - V(x) - \frac{l(l+1)}{x^2}, \tag{45}$$

that is

$$\left|\mu^2 - 2V(x)\right| \le \left|\frac{y_1''}{y_1}\right| + \left|V(x)\right| + \frac{l(l+1)}{x^2}. \tag{46}$$

We have demonstrated the boundedness of each of the above addends

$$\left|\frac{y_1''}{y_1}\right|, \left|V(x)\right|, \frac{l(l+1)}{x^2}, \tag{47}$$

of each of the previous summands of the hypothesis

$$V(x) \in L^1(1,\infty).$$

The summands: $\left|V(x)\right|$ and $\frac{l(l+1)}{x^2}$ are actually integrable, namely

$$\int_1^\infty \left|V(x)\right| \mathrm{d}x < \infty$$

since $\int_1^\infty \left|V(x)\right| \le \int_1^\infty x\left|V(x)\right| < \infty$ by (39). And

$$\int_1^\infty \frac{l(l+1)}{x^2} \mathrm{d}x = l(l+1) < \infty. \tag{48}$$

Then

$$V(x), \frac{l(l+1)}{x^2} \in L^1(1,\infty) \tag{49}$$

Therefore, from (34)

$$V(x), \frac{l(l+1)}{x^2} \in L^\infty(1,\infty). \tag{50}$$

Now, let's estimate term: $\left|\frac{y_1''}{y_1}\right|$, since $\frac{y_1''}{y_1} = q_1(x) - \mu^2 = V(x) - \mu^2 + \frac{l(l+1)}{x^2}$: (15), (25) then

$$\left|\frac{y_1''}{y_1}\right| \le \left|V(x)\right| + \frac{l(l+1)}{x^2} + \left|\mu\right|^2, \tag{51}$$

then

$$\left|V(x)\right| + \frac{l(l+1)}{x^2} \in L^\infty(1,\infty), \tag{52}$$

and (50) then

$$\left|V(x)\right| + \frac{l(l+1)}{x^2} + \left|\mu\right|^2 \in L^\infty(1,\infty). \tag{53}$$

Then by (51)

$$\frac{y_1''}{y_1} \in L^\infty(1,\infty). \tag{54}$$

Therefore, from (45), (46), (50) and (54), we conclude

1) $q_2 = \mu^2 - 2V(x) \in L^\infty(1,\infty)$. ■

Lemma 18. *Let x such that* $0 < x \le \sqrt[l]{\frac{l+1}{2l}} < 1$, *l even, fixed* $\mu \in \mathbb{C}$, *of* (24)

$$\int_0^1 x\left|V(x)\right| \mathrm{d}x < \infty,$$

and of (44)

$$1) \quad q_2(x) = \mu^2 - 2V(x),$$

then

$$1) \quad q_2(x) = \mu^2 - 2V(x) \in L^\infty(0,1). \tag{55}$$

Proof. According to (26), the term: $\mu^2 - 2V(x)$, for $0 < x \le \sqrt[l]{\frac{l+1}{2l}} < 1$, and by (44) implies

$$\mathbb{W}q_1 = q_2 = \mu^2 - 2V(x) = -\frac{y_1''}{y_1} - V(x) - \frac{l(l+1)}{x^2}, \tag{56}$$

then $\left|-\frac{y_1''}{y_1} - V(x) - \frac{l(l+1)}{x^2}\right| \le \left|\frac{y_1''}{y_1}\right| + \left|V(x)\right| + \frac{l(l+1)}{x^2}$. Since $\frac{y_1''}{y_1} = q_1 - \mu^2$ is continuous on $(0,\infty)$, (15), let $\varepsilon > 0$, in particular $\left|\frac{y_1''}{y_1}\right|$ is continuous on $(0,1]$, since $(0,1] \subset \bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right]$, for $\varepsilon/2$, there exists $n \in \mathbf{N}$ such that $\varepsilon/2 > \frac{1}{n}$. In particular, $\left|\frac{y_1''}{y_1}\right|\Big|_{[1/n,1]}$ is continuous, since $\left[\frac{1}{n},1\right]$ is compact then $\left|\frac{y_1''}{y_1}\right|$ is bounded on $\left[\frac{1}{n},1\right]$, that is, there exists $M(n) > 0$ such that $\left|\frac{y_1''}{y_1}(x)\right| \le M(n)$ for all $x \in \left[\frac{1}{n},1\right]$. Then

$$\frac{y_1''}{y_1}(x) \in L^\infty[1/n,1]. \tag{57}$$

So, then it is possible that the function $\frac{y_1''}{y_1}(x)$ is discontinuous only on the interval $(0,\varepsilon/2)$, but its Lebesgue measure: $\lambda(0,\varepsilon/2) = \varepsilon/2 < \varepsilon$, We conclude that $\frac{y_1''}{y_1}(x)$ is bounded except, in the interval $(0,\varepsilon/2)$ that has *zero Lebesgue measure. Therefore*

$$\frac{y_1''}{y_1}(x) \in L^\infty(0,1]. \tag{58}$$

Now, let's estimate the term: $V(x) + \frac{l(l+1)}{x^2}$ and consider the multiplication: $x\left[V(x) + \frac{l(l+1)}{x^2}\right] = xV(x) + \frac{l(l+1)}{x}$, for $x \in (0,1]$, since (24) $\int_0^1 x\left|V(x)\right| \mathrm{d}x < \infty$ then

$$xV(x) \in L^{\infty}(0,1]. \tag{59}$$

And let $\varepsilon > 0$ the term $\frac{l(l+1)}{x}$ is continuous on $\left[\frac{1}{n},1\right]$ and therefore bounded on $\left[\frac{1}{n},1\right]$ and, similarly to $\frac{y_1''}{y_1}(x)$ then $\frac{l(l+1)}{x} \in L^{\infty}\left[\frac{1}{n},1\right]$ and $\frac{l(l+1)}{x}$ is bounded except in the interval $(0,\varepsilon/2)$ which has *zero Lebesgue measure*. *Then*

$$\frac{l(l+1)}{x} \in L^{\infty}(0,1]. \tag{60}$$

From (58), (59) and (60), we conclude that

$$q_2 \in L^{\infty}(0,1].$$

Therefore, of (42) and (55), we conclude

$$q_2 \in L^{\infty}(0,\infty). \tag{61}$$

■

3.4. Second Addend 2) $\frac{\mu}{i^{l-1}}e^{-i\mu x} \in L^{\infty}(0,\infty)$

Lemma 19. *For each fixed* $\mu \in \mathbb{C}$ *f, the term*

$$\frac{\mu}{i^{l-1}}e^{-i\mu x} \in L^{\infty}(0,\infty). \tag{62}$$

Proof. $\left|\frac{\mu}{i^{l-1}}e^{-i\mu x}\right| = \left|\frac{\mu}{i^{l-1}}\right|\left|e^{-i\mu x}\right| = \frac{|\mu|}{|i|^{l-1}} = |\mu|$ for all $x > 0$.

Now, one obtains the following estimates: the term $\frac{\varphi_1'}{\varphi_2}$, be independent of the x-coordinate.

3.5. Third Addend 3) $\frac{\varphi_1'}{\varphi_2} \in L^{\infty}(0,\infty)$

Remark 20. *Two cases: First case:* $x > 1$ *and Second case:* $0 < x \le \sqrt[l]{\frac{l+1}{2l}} < 1$.

Lemma 21. *First case: let* $\lambda,\mu \in \mathbb{C}$, $k \in \{\lambda,\mu\}$ *and*

$$|\operatorname{Im} k| > C_0^{\infty}|V(x)|\,dx, \tag{63}$$

then

$$\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)} \in L^{\infty}(1,\infty). \tag{64}$$

Proof. According to the Estimate (27)

$$\left|\frac{\varphi_1'}{\varphi_2}\right| \le \frac{|\mu^2-\lambda^2||\lambda|}{2|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}||F_l(\lambda)| + 2|\mu^2-\lambda^2|}{|F_l(\mu)|} + |(-\mu)^{-l}|\right)$$

for l even and $x>1$. Which is given in terms of the *Jost functions* $F_l(\lambda)$, $F_l(\mu)$ and the pair of eigenvalues μ,λ. Since [5], Formula 1.4.5, p. 12.

$$F_l(k)=1+\int_0^\infty \mathrm{e}^{ikr}V(r)\varphi(k,r)\,\mathrm{d}r, \tag{65}$$

then

$$|F_l(k)-1|\le C_0^\infty \frac{x|V(k)|}{1+|k|x}\mathrm{d}x, \tag{66}$$

$$\Leftrightarrow$$

$$|F_l(k)|\le 1+C_0^\infty \frac{x|V(x)|}{1+|k|x}\mathrm{d}x, \tag{67}$$

$$\Leftrightarrow$$

$$|F_l(k)|\ge 1-C_0^\infty \frac{x|V(x)|}{1+|k|x}\mathrm{d}x. \tag{68}$$

Now, for $k\in\mathbb{C}$, we have for $C>0$, $x>0$, $|V(x)|$

$$1+|k|x\ge|k|x\Leftrightarrow \frac{Cx|V(x)|}{1+|k|x}\le\frac{Cx|V(x)|}{|k|x}=\frac{C|V(x)|}{|k|}\le\frac{C|V(x)|}{|\mathrm{Im}\,k|},$$

$$\Leftrightarrow$$

$$-\frac{Cx|V(x)|}{1+|k|x}\ge-\frac{C|V(x)|}{|k|}\ge-\frac{C|V(x)|}{|\mathrm{Im}\,k|}, \tag{69}$$

entonces

$-\int_0^\infty \frac{Cx|V(k)|}{1+|k|x}\mathrm{d}x=-C\int_0^\infty\frac{|V(k)|}{|k|}\mathrm{d}x\ge -C\int_0^\infty\frac{|V(k)|}{|\mathrm{Im}\,k|}\mathrm{d}x=-\frac{C}{|\mathrm{Im}\,k|}\int_0^\infty|V(k)|\mathrm{d}x$, that is

$$-C\int_0^\infty\frac{x|V(k)|}{1+|k|x}\mathrm{d}x\ge-\frac{C}{|\mathrm{Im}\,k|}\int_0^\infty|V(k)|\mathrm{d}x,$$

$$\Leftrightarrow$$

$$1-C\int_0^\infty\frac{x|V(k)|}{1+|k|x}\mathrm{d}x\ge1-\frac{C}{|\mathrm{Im}\,k|}\int_0^\infty|V(k)|\mathrm{d}x, \tag{70}$$

then (68) and (70) implies

$$|F_l(k)|\ge1-\frac{C}{|\mathrm{Im}\,k|}\int_0^\infty|V(x)|\mathrm{d}x. \tag{71}$$

Now, using the hypothesis (63)

$$|\mathrm{Im}\,k|>C\int_0^\infty|V(x)|\mathrm{d}x\Leftrightarrow 1-\frac{C}{|\mathrm{Im}\,k|}\int_0^\infty|V(x)|\mathrm{d}x>0,$$

$$\Leftrightarrow$$

$$\frac{1}{|F_l(k)|}\le\frac{1}{1-\frac{C}{|k|}\int_0^\infty|V(x)|\mathrm{d}x},\quad \text{if } |\mathrm{Im}\,k|\ge C\int_0^\infty|V(x)|\mathrm{d}x. \tag{72}$$

Since (67):

$$\begin{aligned}|k| \geq |\operatorname{Im} k| \Rightarrow |F_l(k)| &\leq 1 + C\int_0^\infty \frac{x|V(x)|}{1+|k|x}\mathrm{d}x \leq 1 + C\int_0^\infty \frac{x|V(x)|}{|k|x}\mathrm{d}x \\ &= 1 + \frac{C}{|k|}\int_0^\infty |V(x)|\mathrm{d}x \leq 1 + \frac{C}{|\operatorname{Im} k|}\int_0^\infty |V(x)|\mathrm{d}x \\ &\leq 1+1 = 2.\end{aligned}$$

That is

$$|F_l(k)| \leq 2, \text{ if } |\operatorname{Im} k| \geq C\int_0^\infty |V(x)|\mathrm{d}x. \tag{73}$$

Now, coming back to 27

$$\begin{aligned}\left|\frac{\varphi_1'}{\varphi_2}\right| &\leq \frac{|\mu^2-\lambda^2||\lambda|}{2|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}||F_l(\lambda)| + 2|\mu^2-\lambda^2|}{|F_l(\mu)|} + |(-\mu)^{-l}|\right) \\ &= \frac{|\mu^2-\lambda^2||\lambda|}{2|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}||F_l(\lambda)| + 2|\mu^2-\lambda^2| + |(-\mu)^{-l}||F_l(\mu)|}{|F_l(\mu)|}\right),\end{aligned}$$

using (72) and (73)

$$\begin{aligned}\left|\frac{\varphi_1'}{\varphi_2}\right| &\leq \frac{|\mu^2-\lambda^2||\lambda|}{2|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}||F_l(\lambda)| + 2|\mu^2-\lambda^2| + |(-\mu)^{-l}||F_l(\mu)|}{1 - \frac{C}{|k|}\int_0^\infty |V(x)|\mathrm{d}x}\right) \\ &\leq \frac{|\mu^2-\lambda^2||\lambda|}{2|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}|2 + 2|\mu^2-\lambda^2| + |(-\mu)^{-l}|2}{1 - \frac{C}{|k|}\int_0^\infty |V(x)|\mathrm{d}x}\right) \\ &= \frac{|\mu^2-\lambda^2||\lambda|}{|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}| + |\mu^2-\lambda^2| + |(-\mu)^{-l}|}{1 - \frac{C}{|k|}\int_0^\infty |V(x)|\mathrm{d}x}\right).\end{aligned}$$

$$\left|\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}\right| \leq \frac{|\mu^2-\lambda^2||\lambda|}{|(-\mu)^{-l}|} \times \left(\frac{|\lambda||(-\lambda)^{-l}| + |\mu^2-\lambda^2| + |(-\mu)^{-l}|}{1 - \frac{C}{|\mu|}\int_0^\infty |V(x)|\mathrm{d}x}\right),$$

moreover, since for $z \in \mathbf{C}$, $|(-z)^{-l}| = \left|\frac{1}{(-z)^l}\right| = \frac{1}{|(-z)^l|} = \frac{1}{|-z|^l} = \frac{1}{|z|^l}$, then

$$\left|\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}\right| \leq |\mu^2-\lambda^2||\lambda||\mu|^l \times \left(\frac{\frac{|\lambda|}{|\lambda|^l} + |\mu^2-\lambda^2| + \frac{1}{|\mu|^l}}{1 - \frac{C}{|\mu|}\int_0^\infty |V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right||\lambda||\mu|^l\times\left(\frac{\frac{|\lambda||\mu|^l+\left|\mu^2-\lambda^2\right||\lambda|^l|\mu|^l+|\lambda|^l}{|\lambda|^l|\mu|^l}}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right||\lambda||\mu|^l\times\frac{|\lambda||\mu|^l+\left|\mu^2-\lambda^2\right||\lambda|^l|\mu|^l+|\lambda|^l}{|\lambda|^l|\mu|^l\left(1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x\right)}$$

$$=\left|\mu^2-\lambda^2\right||\lambda|^2\times\frac{|\mu|^l+\left|\mu^2-\lambda^2\right||\lambda|^l|\mu|^l+|\lambda|^l}{|\lambda|^l\left(1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x\right)}$$

$$\left|\mu^2-\lambda^2\right|\frac{|\mu|^l}{|\lambda|^{l-2}}\times\left(\frac{|\mu|^l+\left|\mu^2-\lambda^2\right||\lambda|^l|\mu|^l+|\lambda|^l}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right|\frac{|\mu|^l}{|\lambda|^{l-2}}\left(|\mu|^l+\left|\mu^2-\lambda^2\right||\lambda|^l|\mu|^l+|\lambda|^l\right)\times\left(\frac{1}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right|\left(\frac{|\mu|^l}{|\lambda|^{l-2}}|\mu|^l+\left|\mu^2-\lambda^2\right||\lambda|^l|\mu|^l\frac{|\mu|^l}{|\lambda|^{l-2}}+|\lambda|^l\frac{|\mu|^l}{|\lambda|^{l-2}}\right)\times\left(\frac{1}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right|\left(\frac{|\mu|^{2l}}{|\lambda|^{l-2}}+\left|\mu^2-\lambda^2\right||\lambda|^2|\mu|^{2l}+|\lambda|^2|\mu|^l\right)\times\left(\frac{1}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right|\left(|\mu|^{2l}|\lambda|^{2-l}+\left|\mu^2-\lambda^2\right||\lambda|^2|\mu|^{2l}+|\lambda|^2|\mu|^l\right)\times\left(\frac{1}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right||\lambda|^2\left(|\mu|^{2l}|\lambda|^{-l}+\left|\mu^2-\lambda^2\right||\mu|^{2l}+|\mu|^l\right)\times\left(\frac{1}{1-\frac{C}{|\mu|}\int_0^\infty|V(x)|\mathrm{d}x}\right)$$

$$=\left|\mu^2-\lambda^2\right||\lambda|^2\left(|\mu|^{2l}|\lambda|^{-l}+\left|\mu^2-\lambda^2\right||\mu|^{2l}+|\mu|^l\right)\times\left(\frac{1}{\frac{|\mu|-C\int_0^\infty|V(x)|\mathrm{d}x}{|\mu|}}\right)$$

$$=\left|\mu^2-\lambda^2\right||\lambda|^2\left(|\mu|^{2l}|\lambda|^{-l}+\left|\mu^2-\lambda^2\right||\mu|^{2l}+|\mu|^l\right)\times\left(\frac{|\mu|}{|\mu|-C\int_0^\infty|V(x)|\mathrm{d}x}\right).$$

That is

$$\left|\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}\right| \le \left|\mu^2-\lambda^2\right|\left|\lambda\right|^2\left(\left|\mu\right|^{2l}\left|\lambda\right|^{-l}+\left|\mu^2-\lambda^2\right|\left|\mu\right|^{2l}+\left|\mu\right|^{l}\right)\times\left(\frac{|\mu|}{|\mu|-C\int_0^\infty|V(x)|\,\mathrm{d}x}\right).$$

So, $\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}$ *is bounded uniformely respect to the x-coordinate on the interval* $(1,\infty)$.

Then

$$\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)} \underset{x>1}{=} O\left(\left|\mu^2-\lambda^2\right|\left|\lambda\right|^2\left(\left(\frac{|\mu|}{|\lambda|}\right)^l+\left|\mu^2-\lambda^2\right|\left|\mu\right|^l+1\right)\times\left(\frac{|\mu|^{l+1}}{|\mu|-C\int_0^\infty|V(x)|\,\mathrm{d}x}\right)\right),\ \text{if } |\operatorname{Im}\mu|,|\operatorname{Im}\lambda|>C\int_0^\infty|V(x)|\,\mathrm{d}x.$$

And we get (64)

$$\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}\in L^\infty(1,\infty)\ \text{ when } |\operatorname{Im}\mu|,|\operatorname{Im}\lambda|>C\int_0^\infty|V(x)|\,\mathrm{d}x.$$ ■

Remark 22. *Since*: $1-\frac{C}{|\operatorname{Im}k|}\int_0^\infty|V(x)|\,\mathrm{d}x>0 \Leftrightarrow 1>\frac{C}{|\operatorname{Im}k|}\int_0^\infty|V(x)|\,\mathrm{d}x,$

$$C<\frac{|\operatorname{Im}k|}{\int_0^\infty|V(x)|\,\mathrm{d}x},\ \text{ for } k\in\{\lambda,\mu\}.$$

Therefore

$$0<C<\frac{|\operatorname{Im}k|}{\int_0^\infty|V(x)|\,\mathrm{d}x},\ \text{ when } |\operatorname{Im}k|>C\int_0^\infty|V(x)|\,\mathrm{d}x. \tag{74}$$

Lemma 23. *Second case*: $0<x\le\sqrt[l]{\frac{l+1}{2l}}<1$, *l even, fixed* $\mu\in\mathbb{C}$ *then*

$$\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}\in L^\infty(0,1). \tag{75}$$

Proof. For *l* even, fixed $\lambda\in\mathbb{C}$, according to (28)

$$\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}=cx^{2l+1},\ \text{ for } 0<x\le\sqrt[l]{\frac{l+1}{2l}}<1,$$

then $\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}=cx^{2l+1}\in L^\infty(0,1)$. ■

And we conclude with the following

Lemma 24. *Let* $\lambda,\mu\in\mathbb{C}$, $k\in\{\lambda,\mu\}$, $x>1$, $|\operatorname{Im}k|>C\int_0^\infty|V(x)|\,\mathrm{d}x$, $0<x\le\sqrt[l]{\frac{l+1}{2l}}<1$ *then*

$$\frac{\varphi_1'(\lambda,x)}{\varphi_2(\lambda,x)}\in L^\infty(0,\infty). \tag{76}$$

Finally, summarizing the conclusions of the previous *lemmas*: 17, 18, 19, 21, 23 and 24, we obtain the main theorem.

Theorem 25. *Let* $\lambda,\mu\in\mathbb{C}$, $k\in\{\lambda,\mu\}$, *l even*, $0<x\le\sqrt[l]{\frac{l+1}{2l}}<1$ *and*

$$|\operatorname{Im}k|>C\int_0^\infty|V(x)|\mathrm{d}x.$$

Then

$$\mathbb{W}q_1:=q_2\in L^\infty(0,\infty), \tag{77}$$

where the potential q_2 *is given by* (31).

We finish by displaying the bounds for the amplitude *A*, in terms of the potential q_2 established in [1].

4. Estimates for Amplitude $A(\alpha)$ of a Bounded Potential on Half-Line

We begin by citing the alluded theorem of [1], see Introduction: the bounds for the *A* amplitude for the potential *q* on half-line.

Theorem 26. (*Theorem* 10.3) *Let* $h=\infty$ *and* $q\in L_\infty(0,\infty)$). *Suppose* $\kappa^2>\|q_2\|_\infty$. *Then*

$$m(-\kappa^2)=-\kappa-\int_0^\infty A(\alpha)\mathrm{e}^{--2\alpha\kappa}\mathrm{d}\alpha,$$

with a converget integral and no error term.

$$|A(\alpha)|\le\|q_2(x)\|_{L^\infty(0,\infty)}+\alpha^2\|q_2(x)\|_{L^\infty(0,\infty)}\mathrm{e}^{\alpha^2\|q_2(x)\|_{L^\infty(0,\infty)}}.$$

The example on the half-line displayed is above (31)

$$\mathbb{W}q_1:=q_2=\begin{cases}\mu^2-2V(x), & \text{for } 0<x\le\sqrt[l]{\frac{l+1}{2l}}<1,\\ \mu^2-2V(x)+\frac{\mu}{i^{l-1}}\mathrm{e}^{-i\mu x}+\frac{\varphi_1'}{\varphi_2}, & \text{for } x>1,\ l \text{ even}.\end{cases}$$

And

$$q_2(x)\in L^\infty(0,\infty),$$

according to (61), (62) and (64).

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Dedication

The dedication of the article to my deceased sisters: Alicia Blancarte and Rosa Blancarte.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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