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Variational Iteration Method for Solving Boussinesq Equations Using Maple

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Abstract

In this study, we applied the variational iteration method to solve the Boussinesq time equation. Bossiness's article from 1872 introduced the equations that are now known as the Boussinesq equations. Numerical methods are commonly utilized to solve nonlinear equation systems. Several research papers have documented the values of the variational iteration method and its applications for various categories of differential equations. A comparison of the exact and numerical solutions was obtained using the variational iteration method. The variational iteration method shows that the proposed method is very effective and convenient. The results are shown for different specific cases of the problem. The variational iteration method is useful in numerical simulations and approximate analytical solutions, and it is used to resolve nonlinear differential equations in various situations using Maple. For example, the linear Boussinesq equation was resolved using the variational iteration method. By comparing the numerical results, we found that the variable repetition method produced accurate results and was close to the exact solution, allowing it to be widely applied to the Boussinesq equation. This proves the effectiveness of the method and the capability to quickly and effectively obtain the numerical number solution related to the exact solution using the Maple 18 program. Additionally, the outcomes are extremely precise.

Keywords

Boussinesq Equations, Maple 18, Variational Iteration Method

1. Introduction

The variational iteration method (VIM) was proposed by Ji-Huan He in 1999 [1] [2] [3]. This method has been used in many studies to solve different types of equations. Abdul-Majid Wazwaz used it to determine rational solutions for the

KdV, the K(2, 2), the Burgers, and the cubic Boussinesq equations [4] [5]. M. Javidi constructed solitary wave solutions for the Boussinesq equation using VIM [6]. In addition, the VIM has been proven by numerous authors to be an effective mathematical tool for resolving various nonlinear problems. Tamer Abassy applied the modified VIM to solve the Boussinesq equation [7]. S. A. El-Wakil also used the improved VIM to solve Boussinesq-like B(m, n) equations [8]. It was successfully applied to solve complex ordinary differential equation problems [9], partial differential equations [10], and fractional nonlinear differential equations [11]. E. Rama presented a study for solving various types of problems [12]. Elcin deals with the blow-up solutions of the generalized Boussinesq equation [13] and uses effective programs (Maple) to construct a numerical simulation of the Boussinesq equations [14]. Many applications have been introduced in the literature [15] [16] [17] [18]. Numerical computer programs are developed for these problems, and corresponding examples are provided to verify the accuracy and efficiency of the simplified VIM in solving the Boussinesq equation.

VIM effectively analyzes close-form analytical or approximate solutions of nonlinear evolution equations. If there is an exact solution, this method provides fast, convergent, and successive approximations of it; otherwise, approximations can be used for numerical purposes. Another significant benefit of VIM is its ability to drastically reduce calculation size while retaining the high accuracy of numerical results.

Scientific fields as diverse as fluid dynamics, solid-state physics, plasma physics, and chemical kinetics use nonlinear phenomena. The examination of these scientific models utilized various analytical and numerical techniques because of the rising interest in the theory of solitary waves. The classical Boussinesq equation,

$$u_{tt} = \left(u + u^2 + u_{xx}\right)_{xx} \tag{1}$$

which was derived in 1872 to describe shallow water waves, has the flaw that the Cauchy problem is improperly posed. Therefore, it cannot be used to analyze numerical wave propagation issues.

The equation also occurs in various physical applications, such as vibrations in a nonlinear string, iron sound waves in plasma, and nonlinear lattice waves. Additionally, it was used to address issues with water percolation in porous subsurface strata. Recently, certain novel approaches to solving nonlinear equations, such as the VIM, have garnered considerable interest.

Boussinesq put forth a well-known model of nonlinear dispersive waves in a generalized form:

$$u_{tt} = \left[f(u)\right]_{xx} + u_{xxxx} + h(x,t),\ -\infty < x < \infty, t > 0, \tag{2}$$

where u, f, and h are sufficiently differentiable functions and $f(0) = 0$. The initial conditions associated with the Boussinesq Equation (1) have the following form:

$$u(x,0)=a(x),\ u_t(x,0)=b(x),\ -\infty<x<\infty \tag{3}$$

With $a(x)$ and $b(x)$ given C^∞.

This paper aims to solve the Boussinesq equation and compare the exact and numerical solutions obtained using the method of variable iteration by the Maple 18 program.

To present a clear overview of method, we have chosen two examples, to illustrate the variational iteration method and the obtained solutions are compared with the exact solutions.

2. He's Variational Iteration Method

The variational iteration method is easy to compute the successive approximations. In a fewer iterations we get exact solutions of problems if solution of those problems exist.

To illustrate the methodology of the proposed method, using the VIM, we consider

$$Lu+Nu=g(x), \tag{4}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is a known analytic function. According to the VIM, we can construct a correction functional as follows:

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda\left[Lu_n(\xi)+\left(Nu_n(\xi)\right)^{-}-g(\xi)\right]\mathrm{d}(\xi) \tag{5}$$

where λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth order approximation, and u_n^- is considered a restricted variation, *i.e.*, $\delta[u_n]^-=0$.

Equation (5) is called a correction functional. He's VIM has been shown to solve a large class of nonlinear problems effectively, easily, and accurately, with approximations converging rapidly to accurate solutions.

For linear problems, its exact solution can be obtained by only one iteration step as the Lagrange multiplier can be exactly identified.

3. Application

Two examples are given in this section to illustrate the effects of the proposed method.

3.1. Example 1

We consider the cubic Boussinesq equation:

$$u_{tt}-u_{xx}+2\left(u^3\right)_{xx}-u_{xxxx}=0,\ u(x,0)=\frac{1}{x},\ u_t(x,0)=-\frac{1}{x^2} \tag{6}$$

The exact solution is:

$$u(x,t)=\frac{1}{x+t}$$

The correction functional for (6) reads

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}-\frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right.$$
$$\left.+2\frac{\partial^2(u_n(x,\xi))^3}{\partial x^2}+\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right)\mathrm{d}\xi. \tag{7}$$

This yields the stationary conditions

$$\begin{aligned}&1-\lambda'=0,\\&\lambda(\xi=t)=0,\\&\lambda''=0.\end{aligned} \tag{8}$$

This in turn gives

$$\lambda=\xi-t. \tag{9}$$

Substituting this value of the Lagrangian multiplier into functional (7) gives the iteration formula

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (\xi-t)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2}-\frac{\partial^2(\tilde{u}_n)(x,\xi)}{\partial x^2}\right.$$
$$\left.+2\frac{\partial^2(u_n(x,\xi))^3}{\partial x^2}+\frac{\partial^4 u_n(x,\xi)}{\partial x^4}\right)\mathrm{d}\xi. \tag{10}$$

The given initial values admit the use of

$$u_0(x,t)=\frac{1}{x}-\frac{t}{x^2}. \tag{11}$$

Using (10) we obtain the following successive approximations:

$$\begin{aligned}
u_0(x,t)&=\frac{1}{x}-\frac{t}{x^2},\\
u_1(x,t)&=\frac{1}{x}-\frac{t}{x^2}+\frac{t^2}{x^3}-\frac{t^3}{x^4}+\text{small terms},\\
u_2(x,t)&=\frac{1}{x}-\frac{t}{x^2}+\frac{t^2}{x^3}-\frac{t^3}{x^4}+\frac{t^4}{x^5}-\frac{t^6}{x^7}+\text{small terms},\\
u_3(x,t)&=\frac{1}{x}-\frac{t}{x^2}+\frac{t^2}{x^3}-\frac{t^3}{x^4}+\frac{t^4}{x^5}-\frac{t^6}{x^7}+\frac{t^7}{x^8}-\frac{t^8}{x^9}+\text{small terms},\\
&\vdots\\
u_n(x,t)&=\frac{1}{x}-\frac{t}{x^2}+\frac{t^2}{x^3}-\frac{t^3}{x^4}+\frac{t^4}{x^5}-\frac{t^6}{x^7}+\frac{t^7}{x^8}-\frac{t^8}{x^9}+\cdots
\end{aligned} \tag{12}$$

And in a closed form by

$$u(x,t)=\frac{1}{x+t}. \tag{13}$$

3.2. Example 2

We consider the cubic Boussinesq equation:

$$u_{tt}=u_{xx}+3(u^2)_{xx}+u_{xxxx}, \tag{14}$$

$$u(x,0)=2\frac{ak^{2}e^{kx}}{\left(1+ae^{kx}\right)^{2}},\quad u_{t}(x,0)=-2\frac{ak^{3}\sqrt{1+k^{2}}e^{kx}\left(ae^{kx}-1\right)}{\left(1+ae^{kx}\right)^{3}}$$

The exact solution is:

$$u(x)=2\frac{e^{kx+\sqrt{1+k^{2}}t}}{\left(1+e^{kx+k\sqrt{1+k^{2}}t}\right)^{2}}$$

By same way in example 1 we solved example 2.

Figure 1 and **Figure 2** show the exact and approximate solutions. This problem was solved by VIM and their results are shown in **Table 1** and **Table 2** using maple.

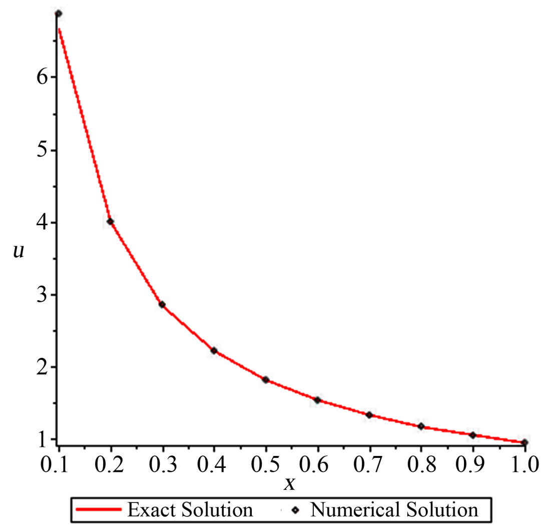

Figure 1. Graph showing the correspondence between exact solution and numerical solution in Example 1.

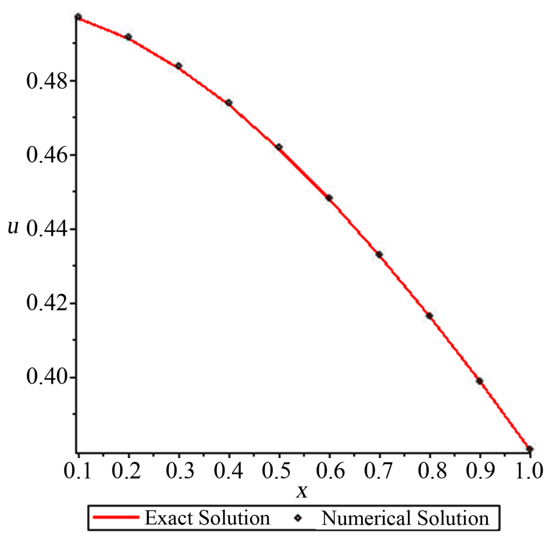

Figure 2. Graph showing the correspondence between exact solution and numerical solution in Example 2.

Table 1. Comparison of VIM with exact solution for Example 1.

x	Exact = $1/(x+t)$	$u(x) = g$	Error
0.10000	6.6666667	6.8750000	0.2083333
0.20000	4.0000000	4.0039062	0.0039062
0.30000	2.8571429	2.8575103	0.0003674
0.40000	2.2222222	2.2222900	0.0000678
0.50000	1.8181818	1.8182000	0.0000182
0.60000	1.5384615	1.5384677	0.0000062
0.70000	1.3333333	1.3333358	0.0000025
0.80000	1.1764706	1.1764717	0.0000011
0.90000	1.0526316	1.0526321	0.0000006
1.00000	0.9523810	0.9523812	0.0000003

Table 2. Comparison of VIM with exact solution for Example 2.

x	Exact = $2\frac{e^{kx+\sqrt{1+k^2}t}}{\left(1+e^{kx+k\sqrt{1+k^2t}}\right)^2}$	$u(x) = g$	Error
0.10000	0.4963749	0.4969902	0.0006153
0.20000	0.4909502	0.4915444	0.0005942
0.30000	0.4832076	0.4837694	0.0005618
0.40000	0.4732955	0.4738151	0.0005196
0.50000	0.4613985	0.4618677	0.0004692
0.60000	0.4477306	0.4481431	0.0004124
0.70000	0.4325273	0.4328787	0.0003514
0.80000	0.4160373	0.4163254	0.0002881
0.90000	0.3985148	0.3987393	0.0002245
1.00000	0.3802125	0.3803746	0.0001622

4. Conclusion

The VIM was used to solve the Boussinesq equations using Maple 18 software. The results were created using tables and figures. Table 1 shows the numerical solution and the right solution. We can see that the numerical solution is generally relevant to the precise answer by comparing the numerical results, proving the method's efficacy and the ability to swiftly and easily obtain the numerical solution relating to the exact solution using Maple 18 software. Furthermore, the results obtained are quite precise.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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