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# On the Well-Posedness of a Class of Hybrid Weakly Singular Integro-Differential Equations

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## Abstract

In this study, a revised version of some numerical methods for a class of hybrid integro-differential equations with weakly singular kernels (Abel types) is presented. These equations were developed from a class of integro-differential equations of first kind originating from an aeroelasticity problem. By manipulating the bounds of initial conditions with random variations, this study numerically demonstrated the well-posedness properties of the equations. Finally, an assumption of separating variables, allowed for linear splines to be chosen as a basis and for the differentiation and integration of the integro-differential part to be interchanged; hence, a numerical scheme was constructed.

## Keywords

Well-Posedness, Hybrid, Weakly Singular, Integro-Differential Equations

## 1. Introduction

The aeroelastic dynamical model is governed by a class of integro-differential equations with weakly singular kernels [1]. In [2], Burns and Ito examined the well-posedness of the first kind equations in weighted product $L_2$-spaces with singular kernels as weights. In this study, we numerically investigated the well-posedness property of hybrid equations. For hybrid type equations, especially for the integro-differential parts, we followed the works in previous studies [3] [4]. For the derivative parts, we revised the results outlined in [5] with second order accuracy difference methods for different boundary conditions. Thereafter, we introduced randomly perturbed noises with different bounds in the initial conditions, and compared the corresponding solutions to solutions without initial perturbations. By setting reasonable tolerance of deviation, we successfully demonstrated the well-posedness property. This paper is organized as follows: Section 1 introduces dynamical systems, Section 2 explains the development of the numerical methods, Section 3 presents examples with numerical results and Section 4 summarizes the study.

## 2. Dynamical Systems

Let us consider weakly singular integro-differential equations of hybrid types with the general normalized form

$$G(x,t,\dot{x})+\frac{\mathrm{d}}{\mathrm{d}t}D_t(x)+L_t(x)=f(t),\ \ t>0, \tag{1}$$

and initial condition $x(s)=\varphi(s)$, for $s\le 0$.

Here, $G(x,t,\dot{x})$ is a function of state, time and time derivative of state. The other terms are such that

$$D_t(x)=\int_{-1}^{0}g(s)x(t+s)\mathrm{d}s, \tag{2}$$

and

$$L_t(x)=\int_{-1}^{0}b(s)x(t+s)\mathrm{d}s. \tag{3}$$

The kernel $g(s)$ belongs to a weakly singular type. In particular, the Abel type $g(s)=|s|^{-p}$ is considered, where $0<p<1$. The kernel $b(s)$ is assumed to be a smooth function for $-1\le s\le 0$.

## 3. Numerical Methods

To develop the numerical algorithms, we separately discretize two variables. For the first variable, $s\in[-1,0]$ is discretized as $-1=\tau_n<\tau_{n-1}<\cdots<\tau_1<\tau_0=0$. For the second variable $t\in[0,1]$, the nodes are $T^0,T^1,\cdots,T^m$, with $0=T^0<T^1<\cdots<T^m=1$. The typical equations we study are

$$G(x(t),t,\dot{x}(t))+\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{0}|s|^{-p}x(t+s)\mathrm{d}s+\int_{-1}^{0}b(s)x(t+s)\mathrm{d}s=f(t). \tag{4}$$

Because the derivative is respect to *t*, we interchange the differentiation and integration of the second term and then apply the property $\frac{\mathrm{d}}{\mathrm{d}t}x(t+s)=\frac{\mathrm{d}}{\mathrm{d}s}x(t+s)$. If we assume that $\kappa(t,s)=x(t+s)$, then

$$\frac{\partial\kappa(t,s)}{\partial t}=\frac{\partial\kappa(t,s)}{\partial s}. \tag{5}$$

Next, suppose that $\kappa(t,s)=\sum_{i=1}^{n}a_i(t)\beta_i(s)$, with the basis $\beta_i(s)$, $i=1,2,\cdots,n-1$, defined as:

$$\beta_i(s)=\begin{cases}\frac{1}{\delta_{i+1}}(s-\tau_{i+1}), & s\in[\tau_{i+1},\tau_i],\\ \frac{1}{\delta_i}(\tau_{i-1}-s), & s\in[\tau_i,\tau_{i-1}],\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$

where $\delta_i = \tau_{i-1} - \tau_i > 0$, $i = 1, \cdots, n$. In particular, $\beta_0(s) = \frac{1}{\tau_0 - \tau_1}(\tau_0 - s)$, $s \in [\tau_1, \tau_0]$ and $\beta_n(s) = \frac{1}{\tau_{n-1} - \tau_n}(\tau_{n-1} - s)$, $s \in [\tau_n, \tau_{n-1}]$.

The semi-discretized form of Equation (4) becomes

$$\begin{aligned} &G\left(a_0(t), t, \dot{a}_0(t)\right) + \int_{-1}^{0} |s|^{-p} \left( \sum_{i=0}^{n} a_i(t) \frac{\mathrm{d}}{\mathrm{d}s} \beta_i(s) \right) \mathrm{d}s \\ &+ \int_{-1}^{0} b(s) \left( \sum_{i=0}^{n} a_i(t) \beta_i(s) \right) \mathrm{d}s = f(t). \end{aligned} \tag{7}$$

With the piecewise linear property of $\beta_i(s)$, the second term of Equation (7) can be further partitioned into $c_0 a_0(t) + c_1 a_1(t) + \cdots + c_n a_n(t)$, where $c_k$, $k = 0, 1, \cdots, n$, depending on $\tau_l$, $l = 0, 1, \cdots, n$. Analogously, the third term of Equation (7) can be discretized as $d_0 a_0(t) + d_1 a_1(t) + \cdots + d_n a_n(t)$, and $d_k$, $k = 0, 1, \cdots, n$, also depending on $\tau_l$, $l = 0, 1, \cdots, n$.

For a fully-discretized form, with a second-order finite difference approximating the derivative term, Equation (7) becomes the following

For $k = 1, \cdots, m-2$,

$$G\left( a_0^k, T^k, \frac{-a_0^{k+2} + 4a_0^{k+1} - 3a_0^k}{2\Delta_k} \right) + \sum_{i=0}^{n} a_i^k c_i + \sum_{i=0}^{n} a_i^k d_i = f\left(T^k\right), \tag{8}$$

for $k = m-1$, $G\left( a_0^{m-1}, T^{m-1}, \frac{a_0^m - a_0^{m-2}}{2\Delta_{m-1}} \right) + \sum_{i=0}^{n} a_i^{m-1} c_i + \sum_{i=0}^{n} a_i^{m-1} d_i = f\left(T^{m-1}\right)$,

and for $k = m$, $G\left( a_0^m, T^m, \frac{3a_0^m - 4a_0^{m-1} + a_0^{m-2}}{2\Delta_m} \right) + \sum_{i=0}^{n} a_i^m c_i + \sum_{i=0}^{n} a_i^m d_i = f\left(T^m\right)$,

with $\Delta_k = T^k - T^{k-1} > 0$, $k = 1, \cdots, m$.

To identify $a_0^k$, $k = 1, \cdots, m$, and apply uniform discretization in both *t* and *s*, we use the transit property $a_i^j = a_{i-1}^{j-1}$ and assume $m = n$. Therefore, we have $a_i^j = a_0^{j-i}$, for $j > i$. For $j \le i$, $a_l^0$, $l = 0, 1, \cdots, n-1$, can be determined by the initial condition.

Without loss of generality, we use the special form $G\left(x(t), t, \dot{x}(t)\right) = \dot{x}(t) - x(t)$ and construct an $n \times n$ linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for the system of algebraic equations, where

$$A = \begin{bmatrix} c_0 - d_0 - 1 - \frac{3}{2\Delta} & \frac{2}{\Delta} & \frac{-1}{2\Delta} & \cdots & \cdots & 0 \\ c_1 - d_1 & c_0 - d_0 - 1 - \frac{3}{2\Delta} & \frac{2}{\Delta} & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ c_{n-3} - d_{n-3} & \cdots & \cdots & c_0 - d_0 - 1 - \frac{3}{2\Delta} & \frac{2}{\Delta} & \frac{-1}{2\Delta} \\ c_{n-2} - d_{n-2} & \cdots & \vdots & c_2 - d_2 & c_0 - d_0 - 1 & \frac{1}{2\Delta} \\ c_{n-1} - d_{n-1} & \cdots & c_3 - d_3 & c_2 - d_2 + \frac{1}{2\Delta} & c_1 - d_1 - \frac{2}{\Delta} & c_0 - d_0 - 1 - \frac{3}{2\Delta} \end{bmatrix}_{n \times n},$$

$$\boldsymbol{x} = \begin{bmatrix} a_0^1 \\ \vdots \\ a_0^n \end{bmatrix}_{n\times 1} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} f(T^1) - a_0^0[c_1 - d_1] - a_1^0[c_2 - d_2] - \cdots - a_{n-1}^0[c_n - d_n] \\ \vdots \\ f(T^{n-1}) - a_0^0[c_{n-1} - d_{n-1}] - \cdots - a_1^0[c_n - d_n] \\ f(T^n) - a_0^0[c_n - d_n] \end{bmatrix}_{n\times 1}.$$

## 4. Examples

In this section, we apply the methods derived from Section 2 to calculate the percentages of computed solutions that satisfy the infinity norm criteria. Computed solutions are obtained by randomly perturbations in each node of the initial condition but bounded within $[-\varepsilon, \varepsilon]$. The numbers in the tables indicate the percentages of successes satisfying the bounded criteria in each case. For $n =$ 1000, the number of nodes and number of test cases, three results are provided for comparison.

In all cases, $p = 0.5$. The bounded criterion of infinity norm between the computed and exact solutions is 0.1.

The following examples mainly show that for the hybrid type integro-differential equations, percentages of satisfying the infinity norm criteria between computed solutions and exact solution $x(t)$ are increasing by decreasing the perturbation bounds of initial conditions.

Example 1.

$$b(s) = 1, \phi(s) = s^2, \ f(t) = -2t^2 + 7t - \frac{5}{3}, x(t) = t^2.$$

Table 1 contains percentages that satisfy the infinity norm criteria.

Example 2.

$$b(s) = 1, \phi(s) = s, \ f(t) = -2t + \frac{7}{2}, x(t) = t.$$

Table 2 contains percentages that satisfy the infinity norm criteria.

Example 3.

$$b(s) = 1, \phi(s) = 0, \ f(t) = 1 + 2t^{0.5} - t - \frac{t^2}{2}, x(t) = t.$$

Table 3 contains percentages that satisfy the infinity norm criteria.

Example 4.

$$b(s) = s, \phi(s) = s, \ f(t) = -\frac{t}{2} + \frac{8}{3}, x(t) = t.$$

Table 4 contains percentages that satisfy the infinity norm criteria.

Example 5.

$$b(s) = s, \phi(s) = 0, \ f(t) = 1 + 2t^{0.5} - t + \frac{t^3}{6}, x(t) = t.$$

Table 5 contains percentages that satisfy the infinity norm criteria.

Example 6.

$$b(s)=s,\phi(s)=s^2,\ f(t)=-\frac{1}{12}+2t^{0.5}+\frac{7}{3}t-\frac{8}{3}t^{1.5}+\frac{t^2}{2}+\frac{t^3}{6}-\frac{t^4}{12},x(t)=t.$$

Table 6 contains percentages that satisfy the infinity norm criteria.

Table 1. Results of Example 1.

| $\varepsilon$ | 0.05 | 0.005 | 0.0005 | 0.00005 | 0.000005 |
|---|---|---|---|---|---|
| $n = 100$ | 0% | 0% | 77% | 100% | 100% |
| $n = 100$ | 0% | 0% | 70% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 90% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |

Table 2. Results of Example 2.

| $\varepsilon$ | 0.05 | 0.005 | 0.0005 | 0.00005 | 0.000005 |
|---|---|---|---|---|---|
| $n = 100$ | 0% | 0% | 69% | 100% | 100% |
| $n = 100$ | 0% | 0% | 78% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |

Table 3. Results of Example 3.

| $\varepsilon$ | 0.05 | 0.005 | 0.0005 | 0.00005 | 0.000005 |
|---|---|---|---|---|---|
| $n = 100$ | 0% | 0% | 68% | 100% | 100% |
| $n = 100$ | 0% | 0% | 57% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 100% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 99.9% | 100% | 100% |

Table 4. Results of Example 4.

| $\varepsilon$ | 0.05 | 0.005 | 0.0005 | 0.00005 | 0.000005 |
|---|---|---|---|---|---|
| $n = 100$ | 0% | 0% | 0% | 99% | 100% |
| $n = 100$ | 0% | 0% | 0% | 99% | 100% |
| $n = 1000$ | 0% | 0% | 92.5% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 90.8% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 92% | 99.9% | 100% |

Table 5. Results of Example 5.

| $\varepsilon$ | 0.05 | 0.005 | 0.0005 | 0.00005 | 0.000005 |
|---|---|---|---|---|---|
| $n = 100$ | 0% | 0% | 0% | 100% | 100% |
| $n = 100$ | 0% | 0% | 0% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 91.9% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 91.8% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 92.1% | 100% | 100% |

**Table 6.** Results of Example 6.

| $\varepsilon$ | 0.05 | 0.005 | 0.0005 | 0.00005 | 0.000005 |
|---|---|---|---|---|---|
| $n = 100$ | 0% | 0% | 0% | 100% | 100% |
| $n = 100$ | 0% | 0% | 0% | 99% | 100% |
| $n = 1000$ | 0% | 0% | 92% | 99% | 100% |
| $n = 1000$ | 0% | 0% | 92.4% | 100% | 100% |
| $n = 1000$ | 0% | 0% | 92.4% | 100% | 100% |

## 5. Conclusion

In this study, we investigated the well-posedness property of a class of hybrid integro-differential Equations by revising the numerical methods outlined in a previous study [5]. From the numerical examples, when the bounds $\varepsilon$ of perturbation of the initial conditions approach 0, the percentages of the associated solutions fall into the envelopes of 0.1 bounded criteria compared with solutions without perturbation, which increase instead.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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