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On Invertibility of Some Functional Operators with Shift

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Abstract

In this paper, we consider operators arising in the modeling of renewable systems with elements that can be in different states. These operators are functional operators with non-Carleman shifts and they act in Holder spaces with weight. The main attention was paid to non-linear equations relating coefficients to operators with a shift. The solutions of these equations were used to reduce the operators under consideration to operators with shift, the invertibility conditions for which were found in previous articles of the authors. To construct the solution of the non-linear equation, we consider the coefficient factorization problem (the homogeneous equation with a zero right-hand side) and the jump problem (the non-homogeneous equation with a unit coefficient). The solution of the general equation is represented as a composition of the solutions to these two problems.

Keywords

Operator with a Non-Carleman Shift, Inverse Operator, Non-Linear Equation, Factorization of Coefficient, Equation with Unit Coefficient

1. Introduction

In Hölder's space with weight $H_\mu^0(J,p)$, we consider operators

$$A=a(x)I+b(x)B_\alpha,\quad F=(\delta(x)I+\mu(x)B_\alpha)(\rho(x)I+\eta(x)B_\alpha),$$

$$V=k(x)I+(a_r(x)I+b_r(x)B_\alpha)(a_u(x)I+b_u(x)B_\alpha),$$

$$(\mathbb{W}v)(x)=l(x)I+s(x)B_\alpha+t(x)B_\alpha^2$$

and equations with the operators $A\varphi(x)=q(x)$, $F\varphi(x)=q(x)$, $V\varphi(x)=q(x)$, $W\varphi(x)=q(x)$.

Carrying out the modeling of renewable systems with elements that are in different states, such functional equations with shift appears in balance relations [1]

[2] [3] [4] [5]. The main method for studying these balance relations is their reduction to a system of integral equations with degenerate kernels of the second kind [6] [7]. This reduction uses inverse operators A^{-1}, V^{-1}, W^{-1}. The interest and motivation for studying such operators are growing. Since the conditions for the invertibility of operator A in the space $H_{\mu}^{0}(J,p)$ have already been calculated [8], it follows that the conditions for the invertibility of operator F are found as the intersection of the conditions for the invertibility of the factors that make up F. The question arises: when can equations $(A\varphi)(x)=q(x)$ and $(W\varphi)(x)=q(x)$ be represented as an operator F? Studying the possibility of reducing the equation $(W\varphi)(x)=q(x)$ to $(F\varphi)(x)=q(x)$, a nonlinear system of equations that describes links between coefficients arises: $\delta(x)\rho(x)=l(x)$, $\delta(x)\eta(x)+\mu(x)(B_{\alpha}\rho)(x)=s(x)$, $\mu(x)(B_{\alpha}\eta)(x)=t(x)$. Substituting $\delta(x)$ from the first equation of the system and $\mu(x)$ from the third equation into the second equation, we obtain $\varphi(x)-\dfrac{G(x)}{(B_{\alpha}\varphi)(x)}=g(x)$, where $G(x)=-\dfrac{t(x)}{l(x)}$, $g(x)=\dfrac{s(x)}{l(x)}$, $\varphi(x)=\dfrac{\eta(x)}{\rho(x)}$. Substituting $\rho(x)$ from the first equation of the system and $(B_{\alpha}\eta)(x)$ from the third equation into the second equation, we obtain a non-homogeneous and non-linear equation with an unknown function $\varphi(x)$. An analogous situation takes place when the equation $(V\varphi)(x)=q(x)$ is reduced to $(F\varphi)(x)=q(x)$.

Let $l(x)$ not vanish on the interval J. When getting connections between the coefficients of the operator W and the coefficients of the operator F, we can assume that $\delta(x)=1$. Non-linear system describing the relationship between the coefficients of operators F, W will be as follows: $\rho(x)=l(x)$, $\eta(x)+\mu(x)(B_{\alpha}l(x))(x)=s(x)$, $\mu(x)(B_{\alpha}\eta)(x)=t(x)$ and the unknown function of the non-linear equation will be the same as $\eta(x)$, $\varphi(x)=\eta(x)$. From here we find the remaining coefficients of the operator F. The conditions for the invertibility of the operator W are written out as the intersection of the conditions for the invertibility of operators $I+\mu(x)B_{\alpha}$ and $\rho(x)I+\eta(x)B_{\alpha}$.

As we can see, the solvability of non-linear equations plays an especially important role.

We will devote the third section to the solution of such non-linear equations. And now, let us recall the definition of a Hölder space with weight and formulate, obtained by us, the conditions for the invertibility of the operator A in this space.

2. On the Invertibility of a Functional Operator with Shift $A=aI+bB_{\alpha}$ in the Hölder Space with Weight

Let us recall the definition of a Hölder space with weight and formulate, obtained by us in [8], the conditions for the invertibility of operator A in this space.

A function $\varphi(x)$ that satisfies the following condition on $J=[0,1]$,

$$\left|\varphi(x_1)-\varphi(x_2)\right| \le C\left|x_1-x_2\right|^{\mu}, \quad x_1 \in J, x_2 \in J, \mu \in (0,1),$$

is called a Hölder function with exponent μ and constant C on *J*.

Let ρ be a power function which has zeros at the endpoints $x=0$, $x=1$:

$$\rho(x)=(x-0)^{\mu_0}(1-x)^{\mu_1}, \quad \mu<\mu_0<1+\mu, \mu<\mu_1<1+\mu.$$

The functions that become Hölder functions and have zero values at the points $x=0$, $x=1$, after being multiplied by $\rho(x)$, form a Banach space:

$$H_\mu^0(J,\rho), \quad J=[0,1].$$

The norm in the space $H_\mu^0(J,\rho)$ is defined by

$$\left\|f(x)\right\|_{H_\mu^0(J,\rho)}=\left\|\rho(x)f(x)\right\|_{H_\mu(J)},$$

where

$$\left\|\rho(x)f(x)\right\|_{H_\mu(J)}=\left\|\rho(x)f(x)\right\|_C+\left\|\rho(x)f(x)\right\|_\mu,$$

and

$$\left\|\rho(x)f(x)\right\|_C=\max_{x\in J}\left|\rho(x)f(x)\right|,$$

$$\left\|\rho(x)f(x)\right\|_\mu=\max_{x_1,x_2\in J,x_1\ne x_2}\left|\rho(x)f(x)\right|_\mu,$$

$$\left|\rho(x)f(x)\right|_\mu=\frac{\left|\rho(x_1)f(x_1)-\rho(x_2)f(x_2)\right|}{\left|x_1-x_2\right|^{\mu}}.$$

Let $\alpha(x)$ be a bijective orientation-preserving shift on J: if $x_1<x_2$, then $\alpha(x_1)<\alpha(x_2)$ for any $x_1\in J, x_2\in J$; and let $\alpha(x)$ have only two fixed points:

$$\alpha(0)=0, \quad \alpha(1)=1, \quad \alpha(x)\ne x, \quad \text{when } x\ne 0, x\ne 1.$$

In addition, let $\alpha(x)$ be a differentiable function with $\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)\ne 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)\in H_\mu(J)$.

The shift operator is defined by the formula $(B_\alpha\varphi)(x)=\varphi[\alpha(x)]$.

Consider the operator $A=aI-bB_\alpha$ with coefficients from the Hölder space $a(x)\in H_\mu(J)$, $b(x)\in H_\mu(J)$. Operator A acts on the Hölder space with weight described above, $H_\mu^0(J,\rho)$.

We will now formulate conditions of ivertibility for operator A in the space of Hölder class functions with weight [8].

Operator A, acting in Banach space $H_\mu^0(J,\rho)$, is invertible if the following condition is fulfilled: $\theta_\alpha\left[a(x),b(x),H_\mu^0(J,\rho)\right]\ne 0, x\in J$, where the function σ_α is defined by

$$\theta_\alpha\left[a(x),b(x),H_\mu^0(J,\rho)\right]$$
$$=\begin{cases} a(x), \text{ when } |a(0)|>\left[\alpha'(0)\right]^{-\mu_0+\mu}|b(0)| \text{ and } |a(1)|>\left[\alpha'(1)\right]^{-\mu_1+\mu}|b(1)|; \\ b(x), \text{ when } |a(0)|<\left[\alpha'(0)\right]^{-\mu_0+\mu}|b(0)| \text{ and } |a(1)|<\left[\alpha'(1)\right]^{-\mu_1+\mu}|b(1)|; \\ 0 \quad \text{ in other cases.} \end{cases} \quad (1)$$

Note that the condition for the invertibility of the operator A in Hölder space with weight $H_\mu^0(J,p)$ can be obtained in terms of convergent series based on the recurrence relation. We represent the solution of the linear operator equation $I\varphi(x)-G(x)B_\alpha\varphi(x)=g(x)$ using the recurrence relation $\varphi(x)=G(x)(B_\alpha\varphi)(x)+g(x)$:

$$\begin{aligned}\varphi(x)&=G(x)B_\alpha\left[G(x)(B_\alpha\varphi)(x)+g(x)\right]+g(x)\\&=G(x)B_\alpha G(x)B_\alpha^2\left[\varphi(x)\right]+G(x)B_\alpha g(x)+g(x)\\&=G(x)(B_\alpha G)(x)B_\alpha^2\left[G(x)(B_\alpha\varphi)(x)+g(x)\right]+G(x)B_\alpha g(x)+g(x)\\&=G(x)(B_\alpha G)(x)\left(B_\alpha^2G\right)(x)\left(B_\alpha^3\varphi\right)(x)+G(x)(B_\alpha G)(x)\left(B_\alpha^2\right)g(x)\\&\quad+G(x)B_\alpha g(x)+g(x)\end{aligned}$$

and

$$\begin{aligned}\varphi(x)&=G(x)(B_\alpha G)(x)\left(B_\alpha^2G\right)(x)\left(B_\alpha^3[\ldots]\right)+G(x)(B_\alpha G)(x)\left(B_\alpha^2\right)g(x)\\&\quad+G(x)B_\alpha g(x)+g(x).\end{aligned}\tag{2}$$

Operator A is invertible in $H_\mu^0(J,\rho)$ if the series (2) converges in this space.

Note that condition (1) is a condition for the convergence of the series (2), which is a solution to equation $I\varphi(x)-G(x)B_\alpha\varphi(x)=g(x)$.

3. Solution of the Obtained Non-Linear Equations

In Hölder's space $H_\mu(J)$, we consider the non-linear non-homogeneous equation

$$\varphi(x)-\frac{G(x)}{(B_\alpha\varphi)(x)}=g(x)\ \text{ or }\ \varphi(x)(B_\alpha\varphi)(x)-g(x)(B_\alpha\varphi)(x)=G(x),\tag{3}$$

where $G(x)\in H_\mu(J)$, $g(x)\in H_\mu(J)$ and the function $\varphi(x)$ is sought in space $H_\mu(J)$.

We will assume that all considered functions are positive.

Consider factorization problem, $G(x)\in H_\mu(J)$, $g(x)=0, x\in J$, $\nu(x)\in H_\mu(J)$,

$$\nu(x)-\frac{G(x)}{(B_\alpha\nu)(x)}=0\ \text{ or }\ G(x)=\nu(x)(B_\alpha\nu)(x).\tag{4}$$

We write the recurrent relation $\nu(x)=\dfrac{G(x)}{(B_\alpha\nu)(x)}$, from here:

$$\begin{aligned}\nu(x)&=\frac{G(x)}{B_\alpha\left[\dfrac{G(x)}{(B_\alpha\nu)(x)}\right]}=\frac{G(x)}{(B_\alpha G)(x)}\left(B_\alpha^2\nu\right)(x)=\frac{G(x)}{(B_\alpha G)(x)}\left(B_\alpha^2\left[\frac{G}{B_\alpha\nu}\right]\right)(x)\\&=\frac{G(x)}{(B_\alpha G)(x)}\frac{\left(B_\alpha^2G\right)(x)}{\left(B_\alpha^3[\nu]\right)(x)}=\frac{G(x)}{(B_\alpha G)(x)}\frac{\left(B_\alpha^2G\right)(x)}{\left(B_\alpha^3\left[\dfrac{G}{B_\alpha\nu}\right]\right)(x)}\\&=\frac{G(x)}{(B_\alpha G)(x)}\frac{\left(B_\alpha^2G\right)(x)}{\left(B_\alpha^3G\right)(x)}\left(B_\alpha^4[\nu]\right)(x)=\cdots\end{aligned}$$

The condition for the solvability of Equation (4) is the condition for the convergence of the infinite product according to Hölder norm $\left\|f(x)\right\|_{H_\mu(J)} = \max_{x\in J}\left|f(x)\right| + \max_{x_1,x_2\in J, x_1\neq x_2}\frac{\left|f(x_1)-f(x_2)\right|}{\left|x_1-x_2\right|^\mu}$ to a function from $H_\mu(J)$.

Theorem 1 *If the infinite product converges in Hölder norm to a function* $v(x)\in H_\mu(J)$, *then the only solution of the non-linear non-homogeneous* Equation (4) *will be.*

$$v(x)=\sqrt{G(1)}\frac{G(x)}{(B_\alpha G)(x)}\frac{(B_\alpha^2 G)(x)}{(B_\alpha^3 G)(x)}\frac{(B_\alpha^4 G)(x)}{(B_\alpha^5 G)(x)}\cdots \tag{5}$$

Consider jump problem: $G(x)=1, x\in J$, $h(x)\in H_\mu(J)$, $f(x)\in H_\mu(J)$,

$$f(x)-\frac{1}{(B_\alpha f)(x)}=h(x). \tag{6}$$

We write the recurrent relation $f(x)=\frac{1}{(B_\alpha f)(x)}+h(x)$, from here:

$$\begin{aligned} f(x) &= \frac{1}{\left(B_\alpha\left[\frac{1}{(B_\alpha f)(x)}+h(x)\right]\right)(x)}+h(x) \\ &= \frac{1}{\frac{1}{(B_\alpha^2[f])(x)}+(B_\alpha h)(x)}+h(x) \\ &= \frac{1}{\frac{1}{\left(B_\alpha^2\left[\frac{1}{(B_\alpha f)(x)}+h(x)\right]\right)(x)}+(B_\alpha h)(x)}+h(x)=\cdots \end{aligned}$$

The condition for the solvability of Equation (6) is the condition for the convergence of the continued infinite fraction [9] in Hölder norm to a function from $H_\mu(J)$.

Theorem 2 *If the continued infinite fraction converges in Hölder norm to a function* $f(x)\in H_\mu(J)$, *then the only solution of the non-linear homogeneous* Equation (6) *will be the function.*

$$f(x)=\frac{1}{\frac{1}{\left(B_\alpha^2\left[\frac{1}{(B_\alpha[\ldots])}+h(x)\right]\right)(x)}+(B_\alpha h)(x)}+h(x) \tag{7}$$

Let us turn to the solution of the general nonlinear equation based on the solution of the factorization problem and the jump problem. If the factorization problem is solvable and its solution is $v(x)$, then the coefficient $G(x)$ is represented by a product $v(x)(B_\alpha v)(x)$. The general non-linear equation takes on the form of a jump problem $f(x)-\frac{1}{(B_\alpha f)(x)}=h(x)$, where $h(x)=\frac{g(x)}{\nu(x)}$, $h(x)=\frac{\varphi(x)}{\nu(x)}$. If the jump problem is solvable, then its solution $f(x)$ is represented by an infinite continued fraction (7). The solution of the general nonlinear equation is determined by the formula $\varphi(x)=f(x)\nu(x)$.

Now, let's go back to Equation (3) and write down, without solving, factorization and jump problems, only on the basis of the reduction relation $\varphi(x)=\frac{G(x)}{(B_\alpha\varphi)(x)}+g(x)$, the solution of nonlinear Equation (3) in the form of an infinite continued fraction.

$$\begin{aligned}\varphi(x)&=\frac{G(x)}{B_\alpha\left[\frac{G(x)}{(B_\alpha\varphi)(x)}+g(x)\right](x)}+g(x)\\&=\frac{G(x)}{\frac{B_\alpha G(x)}{\left(\left[\frac{G(x)}{(B_\alpha^2[\ldots])}+g(x)\right]\right)}+(B_\alpha g)(x)}+g(x).\end{aligned}\tag{8}$$

To conclude, we note that because the continued fraction (8) has a more complex structure than (7), then the conditions for its convergence do not look "transparent" as the condition for the solvability of the factorization problem, that is, the condition for the convergence of the infinite product (5) and the condition for the solvability of the jump problem, that is, the condition for the convergence of the continuous function (7). On the other hand, the problem of factorization of the coefficient may turn out to be unsolvable; it remains to investigate the convergence of the continued fraction (8).

In this section, we will not touch on problems related to the description of the classes of solutions of the considered non-linear equations. We will look for solutions among the Hölder class functions $H_\mu(J)$. We will not deal with finding conditions for the convergence of the resulting infinite products and infinite continuous functions. Coefficients and free terms of non-linear equations arising in the modeling of renewable systems are real non-negative functions. Of course, this particularity and other specifics will be taken into account when applying the proposed mathematical apparatus in the analysis of the balance relations of the models.

4. Conclusion

The authors intend to generalize the method for solving nonlinear equations proposed in Section 3, which is based on the separation of the coefficient factorization problem and the jump problem. Then, we apply it to solve equations with an abstract operator defined through the description of its non-linear properties. We propose to develop a model for renewable systems with elements that are in different states, to study balance relations and find the equilibrium state, and use inverse operators constructed on the basis of solving equations with abstract nonlinear operators.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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