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From Translation to Linear and Linear Canonical Transformations

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Abstract

In order to obtain with simplicity the known and new properties of linear canonical transformations (LCTs), we show that any relation between a couple of operators (A, B) having commutator identical to unity, called dual couple in this work, is valuable for any other dual couple, so that from the known translation operator $\exp(a\partial_x)$ one may obtain the explicit form and properties of a category of linear and linear canonical transformations in 2*N*-phase spaces. Moreover, other forms of LCTs are also obtained in this work as so as the transforms by them of functions by integrations as so as by derivations. In this way, different kinds of LCTs such as Fast Fourier, Fourier, Laplace, Xin Ma and Rhodes, Baker-Campbell-Haussdorf, Bargman transforms are found again.

Keywords

Dual Operators, Fundamental Law of Operator Calculus, Newtonian Binomial and Translation, Linear and Linear Canonical Transforms, From Fourier to Gauss and LCTs' Transforms

1. Introduction

Linear canonical transforms (LCTs), probably first studied by Moshinsky and Quesne in 1971 [1], are the transformations in a 2*N*-dimensional phase space which leave invariant the Hamiltonian and the Poisson brackets of coordinates and momenta. Their studies permit to calculate the unitary representations of each of these transformations, realizing by a parametrized operator $S \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $B\tilde{A} \equiv A\tilde{B}$, $C\tilde{D} \equiv D\tilde{C}$, $D\tilde{A} - C\tilde{B} \equiv I$, then calculate the transform by *S* of a wavefunction into a new one. Afterward there are the works of Stern [2], Wolf

[3], etc. The method utilized in [1] [3] for studying LCTs is based on the methods of symplectic group and the “2 + 1” Lorentz group which are not very well-known by many readers. This work, following a recent work [4] on the role of the Fourier transform, which is a special case of LCTs, in quantum mechanics, proposes another approach for studying LCTs based on the fundamental law of operator calculus [5] saying that any relation between a said dual couple of operators (A,B), *i.e.*, operators such that $[A,B] \equiv \tilde{A}B - \tilde{B}A \equiv I$, is valuable also for any other dual couple. With this law, from the translation operator $e^{a\partial_x}$ which transforms x into $(x+a)$ we arrive to get the dilatation operator $e^{\lambda BA}$ which transforms (A,B) into $(e^{-\lambda I}A, e^{\lambda I}B)$ then a LCT which transforms the dual couple (A,B) into the dual couple $(aA+bB, cA+dB)$ if $\tilde{a}c = \tilde{c}a$, $\tilde{b}d = \tilde{d}b$, $\tilde{a}d - \tilde{c}b \equiv I$. We obtain also that products of three operators of the forms $e^{\tilde{A}\Phi A}, e^{\tilde{B}\Phi B}, e^{\tilde{A}\Phi B}$ form different types of LCTs. To get the transforms of functions by integration as so as by differentiation we firstly search for a method for obtaining the Gaussian transforms of Dirac delta and unity functions. This may be done departing from the common definition of the Fourier transform. By this method we arrive to find again known results on different kinds of LCTs such as Fast Fourier, Fourier, Laplace, Xin Ma and Rhodes, Baker-Campbell-Haussdorf, Bargman transforms. Details of reasoning and calculations in this work are presented in the following paragraphs. Section 2 exposed the method of operator calculus. Section 3 illustrated the special case of generalized translations. Section 4 discussed the LCTs in 2N-phase spaces formed by products of three Gaussian operators. Section 5 devoted to the research of a method leaned on the Fourier transform for calculating the LCT transforms of functions. Section 6 presented the way we calculate the transforms of functions by integrations. Section 7 exposed some cases of LCTs. Section 8 showed how to calculate the LCT transforms of functions by differentiations.

2. Method of Operator Calculus

2.1. The Fundamental Law

In a one-dimensional space of functions, consider the derivative operator D_x

$$D_x f(x) = f'(x) \tag{1}$$

and the Eckaert operator $\hat{X}$ which consists in “multiply by the variable x” [6]

$$\hat{X} f(x) = x f(x) \tag{2}$$

we get the identity

$$D_x \hat{X} \equiv \hat{X} D_x + I \tag{3}$$

by applying both members of it on any derivable function $f(x)$, I being the unitary operator.

More generally, let A and B be two operators constructed from D_x and $\hat{X}$ respecting the following condition that we will be called duality in this work

$$[A,B] \equiv AB - BA \equiv I \tag{4}$$

From (4) we deduce that

$$A^m B \equiv BA^m + mA^{m-1} \tag{5}$$

$$A^{m+1} B \equiv BA^{m+1} + (m+1) A^m \tag{6}$$

so that let

$$f(x) = \sum_{m=0}^{\infty} a_m x^m \tag{7}$$

be an entire function and $f'(x)$ its derivative function then because

$$f(A) = \sum_{m=0}^{\infty} a_m A^m \tag{8}$$

we get from (5) the identity in Operator Calculus [5]

$$f(A) B \equiv B f(A) + f'(A)$$

$$f(A) B f^{-1}(A) \equiv B + f'(A) f^{-1}(A) \tag{9}$$

which shows the way that $f(A)$ transforms B if (A,B) formed a dual couple.

We remark that as the identity (9) applies for any couple of dual operators then consequently from one known relation between a dual couple of operators (A,B) we may deduce another relation simply by replacing (A,B) with another dual couple. We dare say that this affirmation is a fundamental law in operator calculus because it gives us a powerful tool in mathematics and quantum mechanics as we may see in this work.

2.2. The Simplest Transforms of Operators

From (9) we deduce the identities

- $$\mathrm{e}^{aA} B \mathrm{e}^{-aA} \equiv B + aI \tag{10}$$

which means that the operator e^{aA} transforms or more precisely translates the operator B into $(B + aI)$.

For curiosity we remark that the translation operator may be obtained from and at the same time leads to the Newton's binomial formula

$$(x+a)^n = \sum_{m=0}^{n} \binom{n}{m} a^m x^{n-m} = \sum_{m=0}^{n} \frac{1}{m!} a^m D_x^m x^n = \mathrm{e}^{aD_x} x^n = (x+a)^n \tag{11}$$

The formula (10) may be generalized by replacing the dual couple $(D_x, \hat{X})$ with the dual couple $\frac{1}{\sqrt{2}}(D_x - \hat{X}, D_x + \hat{X})$ for example

$$\mathrm{e}^{\frac{\alpha}{\sqrt{2}}(D_x - \hat{X})} (D_x + \hat{X}) \mathrm{e}^{-\frac{\alpha}{\sqrt{2}}(D_x - \hat{X})} \equiv (D_x + \hat{X}) + \alpha\sqrt{2} I$$

and so all.

- $$\mathrm{e}^{aA^2} B \mathrm{e}^{-aA^2} \equiv B + 2aA \tag{12}$$

Association of (10), (12) generate the very interesting formula

$$e^{(B+2aA)} \equiv e^{aA^2} e^{B} e^{-aA^2} e^{-B} e^{B} \equiv e^{aA^2} e^{-a(A-1)^2} e^{B} \equiv e^{-a} e^{2aA} e^{B}$$

$$e^{(bB+aA)} \equiv e^{\frac{a}{2b}A^2} e^{bB} e^{-\frac{a}{2b}A^2} e^{-bB} e^{bB} \equiv e^{\frac{a}{2b}A^2} e^{-\frac{a}{2b}(A-b)^2} e^{bB}$$

$$\bullet \quad e^{(aA+bB)} \equiv e^{-\frac{1}{2}ab} e^{aA} e^{bB} \equiv e^{\frac{1}{2}ab} e^{bB} e^{aA} \tag{13}$$

Identity (13) was proven by Stone-von Neumann in 1930's utilizing the Baker-Campbell-Hausdorff formula as we can find easily on the net. From (13) we see that within a scalar factor the operator e^{aD_x} permute with the operator $e^{b\hat{X}}$ and that the exponential of creation and annihilation operators in quantum mechanics may each be disentangled into two simple operators

$$\bullet \quad e^{a^{\pm}} \equiv e^{\frac{1}{\sqrt{2}}\left(\mp D_x + \hat{X}\right)} \equiv e^{\pm\frac{1}{4}} e^{\mp\frac{1}{\sqrt{2}}D_x} e^{\frac{1}{\sqrt{2}}\hat{X}} \equiv e^{\mp\frac{1}{4}} e^{\frac{1}{\sqrt{2}}\hat{X}} e^{\mp\frac{1}{\sqrt{2}}D_x} \tag{14}$$

3. Operators Realizing Linear Transforms in Phase Spaces

3.1. In one Dimensional Space

Joint (11) with the evident identity

$$A^{aA^2} A A^{-aA^2} \equiv A$$

we may write down the important properties of the linear transforms e^{aA^2} and e^{bB^2} under matrix forms where an element of the phase space is presented by the matrix $\begin{pmatrix} x \\ p \end{pmatrix}$ instead of the vector (x, p) as in [1]

$$\bullet \quad e^{aA^2} \begin{pmatrix} A \\ B \end{pmatrix} e^{-aA^2} \equiv \begin{pmatrix} e^{aA^2} A e^{-aA^2} \\ e^{aA^2} B e^{-aA^2} \end{pmatrix} \equiv \begin{pmatrix} A \\ B + 2aA \end{pmatrix}$$

$$i.e. \quad e^{aA^2} \begin{pmatrix} A \\ B \end{pmatrix} e^{-aA^2} \equiv \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{15}$$

$$\bullet \quad e^{bB^2} \begin{pmatrix} A \\ B \end{pmatrix} e^{-bB^2} \equiv \begin{pmatrix} 1 & -2b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{16}$$

Moreover, because of the fact

$$\left[\hat{X}D_x, \ln \hat{X}\right] \equiv \hat{X}\left[D_x, \ln \hat{X}\right] + \left[\hat{X}, \ln \hat{X}\right] D_x \equiv I \tag{17}$$

we get miraculously from (10)

$$e^{a\hat{X}D_x} \ln \hat{X} e^{-a\hat{X}D_x} \equiv \ln \hat{X} + aI$$

$$e^{a\hat{X}D_x} \hat{X} e^{-a\hat{X}D_x} \equiv e^{\ln \hat{X} + aI} \equiv e^{aI} \hat{X} \tag{18}$$

and, if the couple (A, B) is a dual couple as is $\left(D_x, \hat{X}\right)$, the remarkable identity

$$\bullet \quad e^{aBA} B e^{-aBA} \equiv e^{a} B \tag{19}$$

Afterward, by substituting in (19) the dual couple (A, B) with another dual couple $(-B, A)$

$$e^{-aBA} A e^{aBA} \equiv e^{a} A \tag{20}$$

we arrive to get the very interesting realization of the dilatation by a hyperdifferential operator written under the proposed matrix form

- $$e^{aBA}\begin{pmatrix}A\\B\end{pmatrix}e^{-aBA}\equiv\begin{pmatrix}e^{aBA}Ae^{-aBA}\\e^{aBA}Be^{-aBA}\end{pmatrix}\equiv\begin{pmatrix}e^{-a}A\\e^{a}B\end{pmatrix}$$

$$\text{i.e. } e^{aBA}\begin{pmatrix}A\\B\end{pmatrix}e^{-aBA}\equiv\begin{pmatrix}e^{-aI}&0\\0&e^{aI}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}\tag{21}$$

Combining (15) and (16) we may write down the matrix formula for products of exponential operators

- $$e^{cA^2}e^{bB^2}e^{aA^2}\begin{pmatrix}A\\B\end{pmatrix}e^{-aA^2}e^{-bB^2}e^{-cA^2}\equiv\begin{pmatrix}1&0\\2a&1\end{pmatrix}\begin{pmatrix}1&-2b\\0&1\end{pmatrix}\begin{pmatrix}1&0\\2c&1\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$$

$$e^{cA^2}e^{bB^2}e^{aA^2}\begin{pmatrix}A\\B\end{pmatrix}e^{-aA^2}e^{-bB^2}e^{-cA^2}\equiv\begin{pmatrix}1-4bc&-2b\\2a-8abc+2c&1-4ab\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}\tag{22}$$

Equivalently, with

$$\alpha=1-4bc,\quad \beta=-2b,\quad 1-4ab=\delta,\quad \alpha\delta-\beta\gamma=1\tag{23}$$

$$e^{\frac{\alpha-1}{2\beta}A^2}e^{-\frac{\beta}{2}B^2}e^{\frac{\delta-1}{2\beta}A^2}\begin{pmatrix}A\\B\end{pmatrix}e^{-\frac{\delta-1}{2\beta}A^2}e^{\frac{\beta}{2}B^2}e^{-\frac{\alpha-1}{2\beta}A^2}\equiv\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}\tag{24}$$

The formula (24) means that in a phase space within a multiplicative constant λ the hyperdifferential operator

$$S(\alpha,\beta,\gamma,\delta)\equiv\lambda e^{\frac{\alpha-1}{2\beta}A^2}e^{-\frac{\beta}{2}B^2}e^{\frac{\delta-1}{2\beta}A^2}\tag{25}$$

realizes the linear transformation of a dual couple of operators (A,B) into the dual couple $(\alpha A+\beta B,\gamma A+\delta B)$.

Hereinlater we will extend the theorem (25) for $2N$-dimensional phases spaces.

3.2. In a Two-Dimensional Phase Space

In the case where the operator A is a set of two operators A_1, A_2 and B of two operators B_1, B_2 we may write for examples

$$e^{a_1A_1}\begin{pmatrix}B_1\\B_2\end{pmatrix}e^{-a_1A_1}\equiv\begin{pmatrix}B_1+a_1I\\B_2\end{pmatrix}\tag{26}$$

$$e^{a_1B_1}\begin{pmatrix}A_1\\A_2\end{pmatrix}e^{-a_1B_1}\equiv\begin{pmatrix}A_1-a_1I\\A_2\end{pmatrix}\tag{27}$$

$$e^{a_1A_1+a_2A_2}\begin{pmatrix}B_1\\B_2\end{pmatrix}e^{-a_2A_2-a_1A_1}\equiv\begin{pmatrix}B_1+a_1I\\B_2+a_2I\end{pmatrix}\tag{28}$$

$$e^{b_1B_1^2+b_2B_2^2}\begin{pmatrix}A_1\\A_2\\B_1\\B_2\end{pmatrix}e^{-b_2B_2^2-b_1B_1^2}\equiv\begin{pmatrix}1&0&-2b_1&0\\0&1&0&-2b_2\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}A_1\\A_2\\B_1\\B_2\end{pmatrix}\tag{29}$$

$$\mathrm{e}^{a_1 B_1 A_1 + a_2 B_2 A_2} \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} \mathrm{e}^{-a_2 B_2 A_2 - a_1 B_1 A_1} \equiv \begin{pmatrix} \exp(-a_1) & 0 & 0 & 0 \\ 0 & \exp(-a_2) & 0 & 0 \\ 0 & 0 & \exp(a_1) & 0 \\ 0 & 0 & 0 & \exp(a_2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} \tag{30}$$

3.3. In a 2*N*-Dimensional Phase Space

With bold letters A, B designed Nx1 column matrices, Φ a $N \times N$ matrix and tilde sign the transposition of a matrix, we have from the operational relations (15), (16), (21) the following formulae

- $$\mathrm{e}^{\tilde{A}\Phi A} \begin{pmatrix} A \\ B \end{pmatrix} \mathrm{e}^{-\tilde{A}\Phi A} \equiv \begin{pmatrix} A \\ B + 2\Phi A \end{pmatrix} \equiv \begin{pmatrix} I & 0I \\ 2\Phi & I \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{31}$$
- $$\mathrm{e}^{\tilde{B}\Phi B} \begin{pmatrix} A \\ B \end{pmatrix} \mathrm{e}^{-\tilde{B}\Phi B} \equiv \begin{pmatrix} A - 2\Phi B \\ B \end{pmatrix} \equiv \begin{pmatrix} I & -2\Phi \\ 0 & I \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{32}$$
- $$\mathrm{e}^{\tilde{A}\Phi B} \begin{pmatrix} A \\ B \end{pmatrix} \mathrm{e}^{-\tilde{A}\Phi B} \equiv \begin{pmatrix} \mathrm{e}^{-\Phi} A \\ \mathrm{e}^{\Phi} B \end{pmatrix} \equiv \begin{pmatrix} \mathrm{e}^{-\Phi} & 0 \\ 0 & \mathrm{e}^{\Phi} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{33}$$

where for simplicity we suppose that Φ is a $N \times N$ symmetric matrix, *i.e.*, $\Phi \equiv \tilde{\Phi}$.

3.4. Case Study of Linear Transformations Deriving from Translations

Let us define the duality of a couple of operators (A, B) in a multidimensional space by the identity

$$[A, B] \equiv \tilde{A}B - \tilde{B}A \equiv I \tag{34}$$

As

$$\begin{aligned} &[aA + bB, cA + dB] \\ &\equiv \tilde{A}(\tilde{a}c - \tilde{c}a)A + \tilde{B}(\tilde{b}d - \tilde{d}b)B + \tilde{A}(\tilde{a}d - \tilde{c}b)B + \tilde{B}(\tilde{b}c - \tilde{d}a)A \end{aligned} \tag{35}$$

we may affirm that if

$$\tilde{a}c \equiv \tilde{c}a, \quad \tilde{b}d \equiv \tilde{d}b, \quad \tilde{a}d - \tilde{c}b \equiv I \tag{36}$$

then

$$[aA + bB, cA + dB] \equiv [A, B]$$

i.e., that $(aA + bB, cA + dB)$ is a dual couple as being (A, B).

In this case we may write, according to the fundamental property of operator calculus saying that any identity between a dual couple of operators is valuable for any other dual couple,

- $$\mathrm{e}^{(\tilde{A}\tilde{a} + \tilde{B}\tilde{b})\Phi(cA + dB)} \begin{pmatrix} aA + bB \\ cA + dB \end{pmatrix} \mathrm{e}^{-(\tilde{A}\tilde{a} + \tilde{B}\tilde{b})\Phi(cA + dB)} \equiv \begin{pmatrix} \mathrm{e}^{-\Phi} & 0 \\ 0 & \mathrm{e}^{\Phi} \end{pmatrix} \begin{pmatrix} aA + bB \\ cA + dB \end{pmatrix}$$

i.e., $$\mathrm{e}^{(\tilde{A}\tilde{a} + \tilde{B}\tilde{b})\Phi(cA + dB)} \begin{pmatrix} A \\ B \end{pmatrix} \mathrm{e}^{-(\tilde{A}\tilde{a} + \tilde{B}\tilde{b})\Phi(cA + dB)} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{e}^{-\Phi} & 0 \\ 0 & \mathrm{e}^{\Phi} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{37}$$

• In the cases where the set $\left(a,b,c,d\right)$ does not verify the condition $\tilde{a}d-\tilde{c}b\equiv I$ we introduce the matrix

$$\Phi\equiv\tilde{a}d-\tilde{c}b \tag{38}$$

which leads if $\Phi\neq 0$ to

$$\tilde{a}d\Phi^{-1}-\tilde{c}b\Phi^{-1}\equiv\Phi\Phi^{-1}\equiv I \tag{39}$$

and see that the set $\left(a,b\Phi^{-1},c,d\Phi^{-1}\right)$ verifies the said condition.

We can then write, in accordance with (36), (37)

$$\begin{aligned}
&\mathrm{e}^{\left(\tilde{A}\tilde{a}+\tilde{B}\tilde{\Phi}^{-1}\tilde{b}\right)\Phi\left(cA+d\Phi^{-1}B\right)}\begin{pmatrix}A\\B\end{pmatrix}\mathrm{e}^{-\left(\tilde{A}\tilde{a}+\tilde{B}\tilde{\Phi}^{-1}\tilde{b}\right)\Phi\left(cA+d\Phi^{-1}B\right)}\\
&\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d} & -\tilde{\Phi}^{-1}\tilde{b}\\ -\tilde{c} & \tilde{a}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\Phi} & 0I\\ 0I & \mathrm{e}^{\Phi}\end{pmatrix}\begin{pmatrix}a & b\Phi^{-1}\\ c & d\Phi^{-1}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}\\
&\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d} & -\tilde{\Phi}^{-1}\tilde{b}\\ -\tilde{c} & \tilde{a}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\Phi}a & \mathrm{e}^{-\Phi}b\Phi^{-1}\\ \mathrm{e}^{\Phi}c & \mathrm{e}^{\Phi}d\Phi^{-1}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}\\
&\equiv\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}
\end{aligned} \tag{40}$$

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d}\mathrm{e}^{-\tilde{\Phi}}a-\tilde{\Phi}^{-1}\tilde{b}\mathrm{e}^{\tilde{\Phi}}c & \tilde{\Phi}^{-1}\tilde{d}\mathrm{e}^{-\Phi}b\Phi^{-1}-\tilde{\Phi}^{-1}\tilde{b}\mathrm{e}^{\tilde{\Phi}}d\tilde{\Phi}^{-1}\\ -\tilde{c}\mathrm{e}^{-\Phi}a+\tilde{a}\mathrm{e}^{\Phi}c & -\tilde{c}\mathrm{e}^{-\tilde{\Phi}}b\Phi^{-1}+\tilde{a}\mathrm{e}^{\tilde{\Phi}}d\Phi^{-1}\end{pmatrix} \tag{41}$$

It is easy to verify that in (41) we indeed have the relation $\alpha\tilde{\delta}-\beta\tilde{\gamma}\equiv I$.

As example consider the case

$$\bullet\quad a\equiv-b\equiv i\frac{\theta}{2}I,\quad c\equiv d\equiv I,\quad\Rightarrow\Phi\equiv i\theta I \tag{42}$$

We have according to (41)

$$\begin{aligned}
&\mathrm{e}^{i\frac{\theta}{2}\left(D_x-\hat{X}\right)\left(D_x+\hat{X}\right)}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}\mathrm{e}^{-i\frac{\theta}{2}\left(D_x-\hat{X}\right)\left(D_x+\hat{X}\right)}\equiv\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}\\
&\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d} & -\tilde{\Phi}^{-1}\tilde{b}\\ -\tilde{c} & \tilde{a}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\Phi} & 0I\\ 0I & \mathrm{e}^{\Phi}\end{pmatrix}\begin{pmatrix}a & b\Phi^{-1}\\ c & d\Phi^{-1}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}\\
&\equiv\begin{pmatrix}\frac{1}{2}\left(\mathrm{e}^{-i\theta}I+\mathrm{e}^{i\theta}I\right) & \frac{1}{2i\theta}\left(-\mathrm{e}^{-i\theta}I+\mathrm{e}^{i\theta}I\right)\\ \frac{i\theta}{2}\left(-\mathrm{e}^{-i\theta}I+\mathrm{e}^{i\theta}I\right) & \frac{1}{2}\left(\mathrm{e}^{-i\theta}I+\mathrm{e}^{i\theta}I\right)\end{pmatrix}
\end{aligned}$$

so that

$$\mathrm{e}^{i\frac{\theta}{2}\left(D_x^2-\hat{X}^2\right)}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}\mathrm{e}^{-i\frac{\theta}{2}\left(D_x^2-\hat{X}^2\right)}\equiv\begin{pmatrix}\cos\theta I & \theta^{-1}\sin\theta I\\ -\theta\sin\theta I & \cos\theta I\end{pmatrix}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix} \tag{43}$$

In the circumstance that

$$\bullet\quad\tilde{c}b\equiv\tilde{b}c \tag{44}$$

we may replace (39) with

$$\tilde{a}d\Phi^{-1}-\tilde{b}c\Phi^{-1}\equiv\Phi\Phi^{-1}\equiv I \tag{45}$$

and see that (43) becomes

$$e^{i\frac{\theta}{2}(D_x-\hat{X})(D_x+\hat{X})}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}e^{-i\frac{\theta}{2}(D_x-\hat{X})(D_x+\hat{X})}\equiv\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}$$

$$\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d} & -\tilde{b}\\ -\tilde{\Phi}^{-1}\tilde{c} & \tilde{a}\end{pmatrix}\begin{pmatrix}e^{-\Phi} & 0I\\ 0I & e^{\Phi}\end{pmatrix}\begin{pmatrix}a & b\\ c\Phi^{-1} & d\Phi^{-1}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}$$

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}\frac{1}{2}\left(e^{-i\theta}I+e^{i\theta}I\right) & \frac{1}{2}\left(-e^{-i\theta}I+e^{i\theta}I\right)\\ \frac{1}{2}\left(-e^{-i\theta}I+e^{i\theta}I\right) & \frac{1}{2}\left(e^{-i\theta}I+e^{i\theta}I\right)\end{pmatrix}\equiv\begin{pmatrix}\cos\theta I & i\sin\theta I\\ i\sin\theta I & \cos\theta I\end{pmatrix}\quad(46a)$$

We have to remark that if $\tilde{c}b\equiv\tilde{b}c$ then (43) and (46a) are all valuable. This, we think, may explain the difference between parameters in the integral representation of LCTs of Wolf [3] versus that of de Bruijn [7].

Formula (46a) shows that $\exp i\frac{\theta}{2}\left(D_x^2-\hat{X}^2\right)$ realizes the Fractional Fourier transform.

We find again also the result of Wolf [8] showing that $\exp i\frac{\pi}{4}\left(D_x^2-\hat{X}^2\right)$ realizes the Fourier transformation.

Another example is

- $a=c=\frac{\theta}{2}I$, $b=-d\Rightarrow\Phi=-\theta b$

$$e^{\left(\frac{\theta}{2}D_x+b\hat{X}\right)\left(\frac{\theta}{2}D_x-b\hat{X}\right)}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}e^{-\left(\frac{\theta}{2}D_x+b\hat{X}\right)\left(\frac{\theta}{2}D_x-b\hat{X}\right)}\equiv\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}$$

$$\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d} & -\tilde{\Phi}^{-1}\tilde{b}\\ -\tilde{c} & \tilde{a}\end{pmatrix}\begin{pmatrix}e^{-\Phi} & 0I\\ 0I & e^{\Phi}\end{pmatrix}\begin{pmatrix}a & b\Phi^{-1}\\ c & d\Phi^{-1}\end{pmatrix}\begin{pmatrix}D_x\\ \hat{X}\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}\frac{1}{2}\left(e^{ib\theta}+e^{-ib\theta}\right) & \frac{1}{\theta^2}\left(e^{ib\theta}-e^{-ib\theta}\right)\\ \frac{\theta^2}{4}\left(e^{ib\theta}-e^{-ib\theta}\right) & \frac{1}{2}\left(e^{ib\theta}+e^{-ib\theta}\right)\end{pmatrix}\equiv\begin{pmatrix}\cos\theta b & \frac{2i}{\theta^2}\sin\theta b\\ \frac{i\theta^2}{2}\sin\theta b & \cos\theta b\end{pmatrix}\quad(46b)$$

The second solution according to (46a) is

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}\equiv\begin{pmatrix}\tilde{\Phi}^{-1}\tilde{d} & -\tilde{b}\\ -\tilde{\Phi}^{-1}\tilde{c} & \tilde{a}\end{pmatrix}\begin{pmatrix}e^{-\Phi} & 0I\\ 0I & e^{\Phi}\end{pmatrix}\begin{pmatrix}a & b\\ c\Phi^{-1} & d\Phi^{-1}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}$$

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\equiv\begin{pmatrix}\cosh\theta b & -2\theta^{-1}b\sinh\theta b\\ -\frac{\theta}{2}b^{-1}\sinh\theta b & \cosh\theta b\end{pmatrix}\quad(46c)$$

and is found to correspond to the multimode squeeze operators of Xin Ma and Rhodes [8] who work with the dual couple $\left(a,a^{+}\right)$ that as we have seen may be generalized for any other dual couple.

4. General Linear Transforms of Operators in Phase Spaces

Apart from the operators realizing one category of linear transforms in phase space developed in (41) we have also the followings.

4.1. The General Cases

In the $2N$-dimensional phase space scanned by $\begin{pmatrix} D_x \\ \hat{X} \end{pmatrix}$ or $\begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix}$ we have from the formulae (31), (32), (33)

- $$e^{\tilde{\hat{X}}\frac{\Phi_1}{2}\hat{X}} e^{\tilde{D}_x\frac{\Phi_2}{2}D_x} e^{\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} e^{-\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} e^{-\tilde{D}_x\frac{\Phi_2}{2}D_x} e^{-\tilde{\hat{X}}\frac{\Phi_1}{2}\hat{X}} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} \quad (47a)$$

with

$$\alpha = I - \Phi_2\Phi_1, \beta = -\Phi_1 - \Phi_3 + \Phi_3\Phi_2\Phi_1$$

$$\gamma = \Phi_2, \delta = I - \Phi_3\Phi_2$$

- $$e^{\tilde{D}_x\frac{\Phi_1}{2}D_x} e^{\tilde{D}_x\frac{\Phi_2}{2}\hat{X}} e^{\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} e^{-\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} e^{-\tilde{D}_x\frac{\Phi_2}{2}\hat{X}} e^{-\tilde{D}_x\frac{\Phi_1}{2}D_x}$$
$$\equiv \begin{pmatrix} I & -\Phi_3 \\ 0I & I \end{pmatrix} \begin{pmatrix} e^{-\Phi_2} & 0I \\ 0I & e^{\Phi_2} \end{pmatrix} \begin{pmatrix} I & 0I \\ \Phi_1 & I \end{pmatrix} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} \quad (47b)$$
$$\equiv \begin{pmatrix} e^{-\Phi_2} - \Phi_3 e^{\Phi_2}\Phi_1 & -\Phi_3 e^{\Phi_2} \\ e^{\Phi_2}\Phi_1 & e^{\Phi_2} \end{pmatrix} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix}$$

- $$e^{\tilde{\hat{X}}\frac{\Phi_1}{2}\hat{X}} e^{\tilde{D}_x\frac{\Phi_2}{2}\hat{X}} e^{\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} e^{-\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} e^{-\tilde{D}_x\frac{\Phi_2}{2}\hat{X}} e^{-\tilde{\hat{X}}\frac{\Phi_1}{2}\hat{X}}$$
$$\equiv \begin{pmatrix} e^{-\Phi_2} & -e^{-\Phi_2}\Phi_1 - \Phi_3 e^{\Phi_2} \\ 0 & e^{\Phi_2} \end{pmatrix} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} \quad (47c)$$

4.2. Study Case on One Type of Linear Transforms of Operators in Phase Space

The identity (44) and the relation (45) lead to the conclusion "In a $2N$-dimensional phase space $\begin{pmatrix} D_x \\ \hat{X} \end{pmatrix}$ or $\begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix}$, within a scalar factor λ, the hyperdifferential operator

$$S(\alpha,\beta,\gamma,\delta) \equiv \lambda^N e^{\tilde{\hat{X}}\frac{\Phi_1}{2}\hat{X}} e^{\tilde{D}_x\frac{\Phi_2}{2}D_x} e^{\tilde{\hat{X}}\frac{\Phi_3}{2}\hat{X}} \quad (48)$$

with

$$\alpha\tilde{\delta} - \beta\tilde{\gamma} \equiv I$$

$$\Phi_1 \equiv \gamma^{-1}(I-\alpha), \Phi_2 \equiv \gamma, \Phi_3 \equiv \gamma^{-1}(I-\delta) \quad (49)$$

realizes the linear transformation of

$$\begin{pmatrix} D_x \\ \hat{X} \end{pmatrix} \text{ into } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} D_x \\ \hat{X} \end{pmatrix}\text{''} \quad (50)$$

Thanks to the following formula coming from (49)

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \tilde{\delta} & -\tilde{\beta} \\ -\tilde{\gamma} & \tilde{\alpha} \end{pmatrix} \equiv \begin{pmatrix} I & 0I \\ 0I & I \end{pmatrix}$$

we get

$$S(\alpha,\beta,\gamma,\delta)\tilde{S}(\delta,-\beta,-\gamma,\alpha)\equiv I \tag{51}$$

and see that the inverse of $S(\alpha,\beta,\gamma,\delta)$ is

$$S(\delta,-\beta,-\gamma,\alpha)\equiv \mathrm{e}^{\tilde{\hat{X}}\frac{-\Phi_3}{2}\hat{X}}\mathrm{e}^{\tilde{D}_x\frac{-\Phi_2}{2}D_x}\mathrm{e}^{\tilde{\hat{X}}\frac{-\Phi_1}{2}\hat{X}} \tag{52}$$

It is foreseen that there are equivalent theorems concerning other forms of transforms shown in (47a), (47b).

5. Roles of Fourier and Gauss Transforms in LCTs

As we shall see the Gauss transforms are related to the Fourier transformation as discussed hereafter.

5.1. Useful Properties of the Fourier Transform

Let us adopt the convention that the Fourier transform of a function, if it exists, in one dimensional space has the definition

$$F(x)\equiv FTf(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-ixx_0}f(x_0)\mathrm{d}x_0 \tag{53}$$

with this choice we obtain the properties

$$FT\delta(x)=\frac{1}{\sqrt{2\pi}} \tag{54}$$

$$FTxf(x)=iD_xFTf(x) \tag{55}$$

$$FTf'(x)=ixFTf(x) \tag{56}$$

and conclude that FT transforms the operator D_x into the operator $i\hat{X}$ and vice-versa $\hat{X}$ into iD_x

$$FTD_xFT^{-1}\equiv i\hat{X} \tag{57}$$

$$FT\hat{X}FT^{-1}\equiv iD_x \tag{58}$$

According to (15), (16) we see that the operator

$$\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\mathrm{e}^{\frac{i}{2}D_x^2}\mathrm{e}^{-\frac{i}{2}\hat{X}^2} \tag{59}$$

has these properties as shown hereinafter

$$\left(\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\mathrm{e}^{\frac{i}{2}D_x^2}\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\right)\hat{X}\left(\mathrm{e}^{\frac{i}{2}\hat{X}^2}\mathrm{e}^{-\frac{i}{2}D_x^2}\mathrm{e}^{\frac{i}{2}\hat{X}^2}\right)\equiv \mathrm{e}^{-\frac{i}{2}\hat{X}^2}\left(\mathrm{e}^{\frac{i}{2}D_x^2}\hat{X}\,\mathrm{e}^{-\frac{i}{2}D_x^2}\right)\mathrm{e}^{\frac{i}{2}\hat{X}^2}$$

$$\equiv \mathrm{e}^{-\frac{i}{2}\hat{X}^2}\left(\hat{X}+iD_x\right)\mathrm{e}^{\frac{i}{2}\hat{X}^2}\equiv \hat{X}+i\left(D_x+i\hat{X}\right)\equiv iD_x$$

$$\left(\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\mathrm{e}^{\frac{i}{2}D_x^2}\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\right)D_x\left(\mathrm{e}^{\frac{i}{2}\hat{X}^2}\mathrm{e}^{-\frac{i}{2}D_x^2}\mathrm{e}^{\frac{i}{2}\hat{X}^2}\right)\equiv i\hat{X}$$

so that we may write

$$FT\equiv\lambda\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\mathrm{e}^{\frac{i}{2}D_x^2}\mathrm{e}^{-\frac{i}{2}\hat{X}^2} \tag{60}$$

$$FT\mathrm{e}^{\frac{i}{2}x^2}=\lambda\mathrm{e}^{-\frac{i}{2}\hat{X}^2} \tag{61}$$

Remarking now that because

$$\left(D_x \pm i\hat{X}\right)e^{\mp\frac{i}{2}\hat{X}^2} = 0 \tag{62}$$

$$FT\left(D_x + i\hat{X}\right)e^{-\frac{i}{2}\hat{X}^2} = -i\left(D_x - i\hat{X}\right)FTe^{+\frac{i}{2}\hat{X}^2} = 0 \tag{63}$$

we get

$$FTe^{\frac{i}{2}\hat{X}^2} = C_- e^{-\frac{i}{2}\hat{X}^2} \tag{64}$$

$$FTe^{-\frac{i}{2}\hat{X}^2} = C_+ e^{\frac{i}{2}\hat{X}^2} \tag{65}$$

so that

$$FTFTe^{-\frac{i}{2}\hat{X}^2} = C_+C_- e^{-\frac{i}{2}\hat{X}^2} \Rightarrow C_+C_- = 1 \tag{66}$$

Combination of (64) with the property

$$FTf^*(x) = \left(F(-x)\right)^* \Rightarrow FTe^{\frac{i}{2}x^2} = \left(C_+ e^{\frac{i}{2}x^2}\right)^* = C_+^* e^{-\frac{i}{2}x^2} \tag{67}$$

gives the relation

$$C_+^* = C_- = e^{\pm i\frac{\pi}{4}}$$

In this work we adopt the choice

$$C_+^* = C_- = e^{i\frac{\pi}{4}} \tag{68}$$

although the other choice is equally valuable as seemingly affirmed Moshinsky and Quesne [1].

On the other hand, by comparing (61) with (68)

$$FTe^{\frac{i}{2}x^2} = \lambda e^{-\frac{i}{2}x^2} e^{\frac{i}{2}D_x^2} e^{-\frac{i}{2}x^2} e^{\frac{i}{2}x^2} = \lambda e^{-\frac{i}{2}x^2} = C_- e^{-\frac{i}{2}x^2} \tag{69}$$

we get finally the waited value of λ

$$\lambda = C_- = e^{i\frac{\pi}{4}} \tag{70}$$

with this choice we get the hyperdifferential realization of the Fourier transformation

$$FT \equiv e^{i\frac{\pi}{4}} e^{-\frac{i}{2}\hat{X}^2} e^{\frac{i}{2}D_x^2} e^{-\frac{i}{2}\hat{X}^2} \tag{71}$$

The above formula together with the formula

$$FT \equiv e^{i\frac{\pi}{4}} e^{\frac{i}{2}\left(D_x^2 - \hat{X}^2\right)} \tag{72}$$

was obtained by Wolf by another method [9].

It is interesting to note that the Fourier transform of itself is itself

$$FTFTFT^{-1} \equiv FT$$

so that it has another hyperdifferential realization

$$FT \equiv \mathrm{e}^{i\frac{\pi}{4}} \mathrm{e}^{\frac{i}{2}D_x^2} \mathrm{e}^{-\frac{i}{2}\hat{X}^2} \mathrm{e}^{\frac{i}{2}D_x^2} \tag{73}$$

From the formula (54) we may calculate the FT of the unity function as followed

$$FTFT\delta(x) = FT\frac{1}{\sqrt{2\pi}} = \delta(-x) = \delta(x)$$

$$FT1 = \sqrt{2\pi}\delta(x) \tag{74}$$

5.2. The Gaussian Transformation in One-Dimensional Space

From the equation

$$\left(D_x \pm 2a\hat{X}\right)\mathrm{e}^{\mp a\hat{X}^2} = 0 \tag{75}$$

and the properties (57), (58) of the Fourier transformation we get

$$FT\left(D_x + 2a\hat{X}\right)\mathrm{e}^{-a\hat{X}^2} = i\left(2aD_x + \hat{X}\right)FT\mathrm{e}^{-a\hat{X}^2} = 0$$

$$FT\mathrm{e}^{-ax^2} = C(a)\mathrm{e}^{-\frac{1}{4a}x^2},\ a > 0 \tag{76}$$

Similarly

$$FT\left(2aD_x + \hat{X}\right)\mathrm{e}^{-\frac{1}{4a}\hat{X}^2} = i\left(D_x + 2a\hat{X}\right)FT\mathrm{e}^{-\frac{1}{4a}\hat{X}^2} = 0$$

$$FT\mathrm{e}^{-\frac{1}{4a}x^2} = C\left(\frac{1}{4a}\right)\mathrm{e}^{-ax^2} \tag{77}$$

so that

$$C(a)C\left(\frac{1}{4a}\right) = 1 \Rightarrow C(a) = (2a)^{-m} \tag{78}$$

Now, because

$$\lim_{a\to 0} FT\mathrm{e}^{-ax^2} = \lim_{a\to 0}\frac{\mathrm{e}^{-x^2/4a}}{(2a)^m} = \lim_{a\to 0}\frac{\mathrm{e}^{-x^2/4a}\left(\frac{x^2}{4a^2}\right)}{2m(2a)^{m-1}} = \lim_{a\to 0}\frac{x^2\mathrm{e}^{-x^2/4a}}{2m(2a)^{m+1}} = 0 \Rightarrow m = -1$$

we obtain that

$$FT\mathrm{e}^{-ax^2} = (2a)^{-1}\mathrm{e}^{-\frac{1}{4a}x^2} \tag{79}$$

$$FT\mathrm{e}^{-a\hat{X}^2}1 = \mathrm{e}^{aD_x^3}FT1 = (2a)^{-1}\mathrm{e}^{-\frac{1}{4a}x^2}$$

$$\mathrm{e}^{-aD_x^2}\mathrm{e}^{-\frac{1}{4a}x^2} = \sqrt{4\pi a}\delta(x),\ a > 0 \tag{80}$$

Concerning Gaussians with imaginary parameter a we utilize the definition formula of FT and get

$$FT\delta(x) \equiv \mathrm{e}^{i\frac{\pi}{4}}\mathrm{e}^{-\frac{i}{2}\hat{X}^2}\mathrm{e}^{-\frac{i}{2}D_x^2}\delta(x) = \frac{1}{\sqrt{2\pi}} \tag{81}$$

$$\mathrm{e}^{\frac{i}{2}D_x^2}\mathrm{e}^{\frac{i}{2}x^2} = \sqrt{2\pi i}\delta(x)$$

$$e^{-iaD_x^2} e^{\frac{i}{4a}x^2} = \sqrt{2\pi i a}\,\delta(x) \tag{82}$$

Joining (80) and (82) we get the Gaussian transform of the corresponding Gaussian function

$$e^{-aD_x^2} e^{-\frac{1}{4a}x^2} = \sqrt{4\pi a}\,\delta(x),\ a > 0 \text{ or } a = i\theta \tag{83}$$

and inversely of the Dirac delta function

$$e^{aD_x^2} \delta(x) = \frac{1}{\sqrt{4\pi a}} e^{-\frac{1}{4a}x^2},\ a > 0 \text{ or } a = i\theta \tag{84}$$

5.3. Gaussian Transforms in 2*N*-Dimensional Phase Space

In this case we have

$$FT \equiv e^{iN\frac{\pi}{4}} e^{-\tilde{r}\frac{i}{2}\hat{r}} e^{\tilde{\nabla}\frac{i}{2}\nabla} e^{-\tilde{r}\frac{i}{2}\hat{r}} \tag{85}$$

and, according to the generalization of (84) for the dual couple $\left(\Phi^{\frac{1}{2}}\nabla, \Phi^{-\frac{1}{2}}\vec{r}\right)$, the formula

$$e^{\tilde{\nabla}\frac{\Phi}{2}\nabla} \delta\left(\Phi^{-1/2} r\right) = (2\pi)^{-N/2} e^{-\tilde{r}\frac{\Phi^{-1}}{2} r} \tag{86}$$

In purpose to calculate $\delta\left(\Phi^{-1/2} r\right)$ let us propose that there exists a function $\Gamma(\Phi)$ such that

$$\delta(\Phi r) = \Gamma(\Phi)\delta(r) \tag{87}$$

Under this hypothesis we have

$$\delta(\Phi\Psi r) = \Gamma(\Phi)\Gamma(\Psi)\delta(r) \tag{88}$$

so that for a matrix Φ diagonalizable into the diagonal matrix Δ

$$U\Phi U^{-1} = \Delta \tag{89}$$

we have the relation

$$\delta(\Phi r) = \delta\left(U\Phi U^{-1} r\right) = \delta(\Delta r) \tag{90}$$

which leads to

$$\Gamma(\Phi) = \Gamma(\Delta) \tag{91}$$

But between Φ and Δ the only relation is

$$\det\Phi = \det\Delta \tag{92}$$

so that we can take for Φ real

$$\Gamma(\Phi) = \Gamma(\Delta) = |\det\Delta| = |\det\Phi| \tag{93}$$

and get from (86) and the property $\delta(-r) = \delta(r)$ the generalization

- $$e^{-\tilde{\nabla}\frac{\Phi}{2}\nabla} e^{-\tilde{r}\frac{\Phi^{-1}}{2} r} = (2\pi)^{N/2} |\det\Phi|^{1/2}\,\delta(r) \tag{94}$$

From (94) we deduce that

$$\mathrm{e}^{-\tilde{\nabla}\frac{\Phi}{2}\nabla}\mathrm{e}^{-\tilde{r}\frac{\Phi'}{2}r}=\mathrm{e}^{-\tilde{\nabla}\frac{\Phi-\Phi'^{-1}}{2}\nabla}\mathrm{e}^{-\tilde{\nabla}\frac{\Phi'^{-1}}{2}\nabla}\mathrm{e}^{-\tilde{r}\frac{\Phi'}{2}r}$$

- $$=(2\pi)^{\frac{N}{2}}\left|\det\Phi'^{-1/2}\right|(2\pi)^{-\frac{N}{2}}\left|\det\left(-\left(\Phi-\Phi'^{-1}\right)\right)^{-1/2}\right|\mathrm{e}^{\tilde{r}\frac{\left(\Phi-\Phi'^{-1}\right)^{-1}}{2}r} \tag{95}$$

$$=\left|\det\left(I-\Phi'\Phi\right)\right|^{-1/2}\mathrm{e}^{\tilde{r}\frac{\left(\Phi-\Phi'^{-1}\right)^{-1}}{2}r},\ \Phi\neq\Phi'^{-1}$$

- $$\mathrm{e}^{-\tilde{\nabla}\frac{i\Phi}{2}\nabla}\mathrm{e}^{-\tilde{r}\frac{\Phi'}{2i}r}=\left|\det\left(I-\Phi'\Phi\right)\right|^{-1/2}\mathrm{e}^{\tilde{r}\frac{i\left(\Phi-\Phi'^{-1}\right)^{-1}}{2}r},\ \Phi\neq\Phi'^{-1} \tag{96}$$

5.4. Gaussian Transforms of Functions

Consider the differential equation of Hermite polynomials

$$\left(D_x^2-2xD_x+2n\right)y=0 \tag{97}$$

Thank to (10) we may write

$$\mathrm{e}^{D_x^2/4}2\hat{X}\equiv\left(2\hat{X}+D_x\right)\mathrm{e}^{D_x^2/4}$$

$$\mathrm{e}^{D_x^2/4}\left(D_x^2-2xD_x+2n\right)y=\left(-2xD_x+2n\right)\mathrm{e}^{D_x^2/4}y=0$$

and get

$$y=H_n(x)=\mathrm{e}^{-D_x^2/4}c_nx^n \tag{98}$$

On the other hand, from the factorizations of a linear operator in $\left(D_x,\hat{X}\right)$

$$\left(D_x-2\hat{X}\right)^n\equiv\mathrm{e}^{-\frac{1}{4}D_x^2}\left(-2\hat{X}\right)^n\mathrm{e}^{\frac{1}{4}D_x^2}\equiv\mathrm{e}^{\frac{1}{4}\hat{X}^2}D_x^n\mathrm{e}^{-\frac{1}{4}\hat{X}^2} \tag{99}$$

we get the Rodrigues formula for Hermite polynomials

$$H_n(x)=(-1)^n\mathrm{e}^{x^2/4}D_x^n\mathrm{e}^{-x^2/4} \tag{100}$$

which leads to the formula

$$H_n(x)=\mathrm{e}^{-D_x^2/4}(2x)^n \tag{101}$$

From (101) we see that the Gaussian transform of an entire function $f(x)$ may be put under the symbolic form

$$\mathrm{e}^{-D_x^2/4}f(x)=\mathrm{e}^{-D_x^2/4}\sum_{n=0}^{\infty}f_nx^n=\sum_{n=0}^{\infty}\frac{1}{2^n}f_nH_n(x)=:f\left(\frac{H}{2}\right) \tag{102}$$

where H^n is replaced with $H_n(x)$.

6. Integral Realization of Linear and Linear Canonical Transforms

Consider the hyperdifferential operator realizing a linear transformation in a $2N$-phase space

$$S(\alpha,\beta,\gamma,\delta)\equiv\mathrm{e}^{\tilde{r}\frac{\Phi_1}{2}\hat{r}}\mathrm{e}^{\tilde{\nabla}\frac{\Phi_2}{2}\nabla}\mathrm{e}^{\tilde{r}\frac{\Phi_3}{2}\hat{r}}$$

or, by (49)

$$\equiv \mathrm{e}^{\tilde{\hat{r}}\frac{\gamma^{-1}(I-\alpha)}{2}\hat{r}}\mathrm{e}^{\tilde{\nabla}\frac{\gamma}{2}\nabla}\mathrm{e}^{\tilde{\hat{r}}\frac{\gamma^{-1}(I-\delta)}{2}\hat{r}} \tag{103}$$

Remarking that in the case $\Phi_2 \equiv 0I$, $S(\alpha,\beta,\gamma,\delta)$ is reduced to $\mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1+\Phi_3}{2}\hat{r}}$ so that in the following we suppose that $\Phi_2 \equiv \gamma \neq 0I$.

We may write

$$S(\alpha,\beta,\gamma,\delta)\, f(r) = \int_{R^N} S(\alpha,\beta,\gamma,\delta)\, f(r_0)\,\delta(r-r_0)\,\mathrm{d}r_0 \tag{104}$$

But thanks to (95)

$$\begin{aligned} S(\alpha,\beta,\gamma,\delta)\,\delta(r-r_0) &= \mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1}{2}\hat{r}}\mathrm{e}^{\tilde{\nabla}\frac{\Phi_2}{2}\nabla}\mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_3}{2}\hat{r}}\delta(r-r_0) \\ &= \mathrm{e}^{\tilde{\hat{r}}_0\frac{\Phi_3}{2}\hat{r}_0}\mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1}{2}\hat{r}}\mathrm{e}^{\tilde{\nabla}\frac{\Phi_2}{2}\nabla}\delta(r-r_0) \\ &= (2\pi)^{-N/2}\left|\det\Phi_2^{-1/2}\right|\mathrm{e}^{\tilde{\hat{r}}_0\frac{\Phi_3}{2}\hat{r}_0}\mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1}{2}\hat{r}}\mathrm{e}^{-\left(\tilde{\hat{r}}-\tilde{\hat{r}}_0\right)\frac{\Phi_2^{-1}}{2}(\hat{r}-\hat{r}_0)} \end{aligned} \tag{105}$$

so that

$$S(\alpha,\beta,\gamma,\delta)\, f(r) = (2\pi)^{-N/2}\left|\det\Phi_2\right|^{-1/2}\mathrm{e}^{\frac{1}{2}\tilde{\hat{r}}\left(\Phi_1-\Phi_2^{-1}\right)\hat{r}}\int_{R^N} f(r_0)\,\mathrm{e}^{\tilde{\hat{r}}\Phi_2^{-1}\hat{r}_0}\mathrm{e}^{-\frac{1}{2}\hat{r}_0\left(\Phi_2^{-1}-\Phi_3\right)\hat{r}_0}\mathrm{d}r_0 \tag{106}$$

From the relations coming from (49)

$$\Phi_1 = \gamma^{-1}(I-\alpha),\ \Phi_2 = \gamma,\ \Phi_3 = (I-\delta)\gamma^{-1}$$

$$\Phi_1 - \Phi_2^{-1} \equiv -\alpha\gamma^{-1},\ \Phi_3 - \Phi_2^{-1} \equiv -\delta\gamma^{-1},\ \Phi_2 \equiv \gamma,\ \Phi_3^{-1}\delta = \Phi_3^{-1} - \Phi_2$$

we may also write

$$S(\alpha,\beta,\gamma,\delta)\, f(r) = (2\pi)^{-N/2}\left|\det\gamma\right|^{-1/2}\int_{R^N} f(r_0)\,\mathrm{e}^{-\frac{1}{2}\left(\tilde{\hat{r}}\alpha\gamma^{-1}\hat{r}-2\tilde{\hat{r}}\gamma^{-1}\hat{r}_0+\tilde{\hat{r}}_0\delta\gamma^{-1}\hat{r}_0\right)}\mathrm{d}r_0 \tag{107}$$

From (103) we see that

$$\begin{aligned} S(\delta,-i\gamma,i\beta,\alpha)\, f(r) &= \mathrm{e}^{\tilde{\hat{r}}\frac{-i\beta^{-1}(\mathrm{I}-\delta)}{2}\hat{r}}\mathrm{e}^{\tilde{\nabla}\frac{i\beta}{2}\nabla}\mathrm{e}^{\tilde{\hat{r}}\frac{-i\beta^{-1}(\mathrm{I}-\alpha)}{2}\hat{r}} f(r) \\ \bullet\quad &= (2\pi i)^{-N/2}\left|\det\beta\right|^{-1/2}\mathrm{e}^{\tilde{\hat{r}}\frac{i\delta\beta^{-1}}{2}\hat{r}}\int_{R^N} f(r_0)\,\mathrm{e}^{-\frac{i}{2}\left(2\tilde{\hat{r}}\beta^{-1}\hat{r}_0-\tilde{\hat{r}}_0\alpha\beta^{-1}\hat{r}_0\right)}\mathrm{d}r_0 \\ &= (2\pi)^{-N/2}\left|\det i\beta\right|^{-1/2}\int_{R^N} f(r_0)\,\mathrm{e}^{\frac{i}{2}\left(\tilde{\hat{r}}\delta\beta^{-1}\hat{r}-2\tilde{\hat{r}}\beta^{-1}\hat{r}_0+\tilde{\hat{r}}_0\alpha\beta^{-1}\hat{r}_0\right)}\mathrm{d}r_0 \end{aligned} \tag{108}$$

In one-dimensional space the formula (108) is identical with the formula on the integral representation of a canonical transform given by Wolf [3] in his work "A Top-Down Account of Linear Canonical Transforms" as so as with that of Stern [2].

Resuming the results, we may state that:

Principal Theorem on LCT: The linear canonical transformation represented by the hyperdifferential operator

$$S_{\text{canonical}}(\alpha,\beta,\gamma,\delta) \equiv S(\delta,-i\gamma,i\beta,\alpha) \equiv \mathrm{e}^{\tilde{\hat{r}}\frac{-i\beta^{-1}(I-\delta)}{2}\hat{r}}\mathrm{e}^{\tilde{\nabla}\frac{i\beta}{2}\nabla}\mathrm{e}^{\tilde{\hat{r}}\frac{-i\beta^{-1}(I-\alpha)}{2}\hat{r}} \tag{109}$$

transforms operators as followed

$$\begin{pmatrix} \hat{r} \\ i\nabla \end{pmatrix} \text{ into } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \hat{r} \\ i\nabla \end{pmatrix}, \textit{ i.e.}, \begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix} \text{ into } \begin{pmatrix} \delta & -i\gamma \\ i\beta & \alpha \end{pmatrix} \begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix}$$

and transforms functions according to the integral formula

$$\begin{aligned} & S_{\text{canonical}}(\alpha,\beta,\gamma,\delta) f(r) = S(\delta,-i\gamma,i\beta,\alpha) f(r) \\ & = (2\pi i)^{-N/2} |\det\beta|^{-1/2} \mathrm{e}^{\tilde{\hat{r}}\frac{i\delta\beta^{-1}}{2}\hat{r}} \int_{R^N} f(r_0) \mathrm{e}^{-i\tilde{\hat{r}}\beta^{-1}\hat{r}_0} \mathrm{e}^{\tilde{\hat{r}}_0 i\frac{\alpha\beta^{-1}}{2}\hat{r}_0} \mathrm{d}r_0 \end{aligned} \tag{110}$$

or, equivalently, according to a Gaussian function multiplied with a Gaussian transform of a Fourier transform

$$\begin{aligned} & S_{\text{canonical}}(\alpha,\beta,\gamma,\delta) f(r) \\ & = (2\pi i)^{-N/2} |\det\beta|^{-1/2} \mathrm{e}^{\tilde{\hat{r}}\frac{i\delta\beta^{-1}}{2}\hat{r}} \mathrm{e}^{-i\tilde{\nabla}\frac{\alpha\beta}{2}\nabla} \int_{R^N} f(r_0) \mathrm{e}^{-i\tilde{\hat{r}}\beta^{-1}\hat{r}_0} \mathrm{d}r_0 \\ & = \mathrm{e}^{\tilde{\hat{r}}\frac{i\delta\beta^{-1}}{2}\hat{r}} |\det\beta|^{-1/2} \mathrm{e}^{-i\tilde{\nabla}\frac{\alpha\beta}{2}\nabla} \tilde{f}(\beta^{-1}r) \end{aligned} \tag{111}$$

7. Examples of Canonical Transforms

7.1. The Fractional Fourier Transform

Consider the case

$$\alpha = \cos\theta I,\ \beta = i\sin\theta I,\ \gamma = i\sin\theta I,\ = \cos\theta I \tag{112}$$

Thank to (49)

$$\Phi_1 = \frac{1}{i\sin\theta}(1-\cos\theta) I = -i\tan\frac{\theta}{2} I$$

$$\Phi_2 = i\sin\theta I$$

$$\Phi_3 = \frac{1}{i\sin\theta}(1-\cos\theta) I = -i\tan\frac{\theta}{2} I$$

we see that this case corresponds to

The Fractional Fourier transformation

$$FT^{(\theta)} \equiv \mathrm{e}^{-N\frac{i\pi}{4}} \mathrm{e}^{-\tilde{\hat{r}}\frac{i}{2}\tan\frac{\theta}{2}\hat{r}} \mathrm{e}^{\tilde{\nabla}\frac{i}{2}\sin\theta\nabla} \mathrm{e}^{-\tilde{\hat{r}}\frac{i}{2}\tan\frac{\theta}{2}\hat{r}} \tag{113}$$

which transforms

$$\begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix} \text{ into } \begin{pmatrix} \cos\theta I & i\sin\theta I \\ i\sin\theta I & \cos\theta I \end{pmatrix} \begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix} \tag{114}$$

and canonically transforms

$$\begin{pmatrix} \hat{r} \\ i\nabla \end{pmatrix} \text{ into } \begin{pmatrix} \cos\theta I & \sin\theta I \\ -\sin\theta I & \cos\theta I \end{pmatrix} \begin{pmatrix} \hat{r} \\ i\nabla \end{pmatrix} \tag{115}$$

It is seen that the special case $\theta = \pi/2$ corresponds to the Fourier transformation.

7.2. The Laplace Transformation LT

We know that in a Laplace transformation $f'(x) = D_x f(x)$ is transformed into

$\hat{X}F(x) - f(0)$

$$\hat{X}f(x) = xf(x) \text{ is transformed into } \left(-D_x F(x)\right) \tag{116}$$

so that in the space of homogeneous derivable functions we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \; \Phi_1 = -\frac{I}{2}, \Phi_2 = -\frac{I}{2}, \Phi_3 = -\frac{I}{2} \tag{117}$$

and see that the Laplace transformation is realized by λ^N times the operator

$$S(\alpha,\beta,\gamma,\delta) \equiv \mathrm{e}^{-\frac{1}{2}\tilde{\hat{r}}\hat{r}} \mathrm{e}^{-\frac{1}{2}\tilde{\nabla}\nabla} \mathrm{e}^{-\frac{1}{2}\tilde{\hat{r}}\hat{r}} \tag{118}$$

which transforms

$$\begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix} \text{ into } \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \nabla \\ \hat{r} \end{pmatrix}$$

and canonically

$$\begin{pmatrix} \hat{r} \\ i\nabla \end{pmatrix} \text{ into } \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ i\nabla \end{pmatrix} \tag{119}$$

8. Obtaining Canonical Transforms of Functions

For calculating the transform of a function by the operator $S(\alpha,\beta,\gamma,\delta)$ we write

$$\begin{aligned} S(\alpha,\beta,\gamma,\delta) f(r) &= S(\alpha,\beta,\gamma,\delta) f(\hat{r}) S^{-1}(\alpha,\beta,\gamma,\delta) S(\alpha,\beta,\gamma,\delta) I \\ &= f(\gamma\nabla + \delta\hat{r}) S(\alpha,\beta,\gamma,\delta) I \end{aligned} \tag{120}$$

and see that we must calculate firstly that of the unity function $u(r) = I$

$$S(\alpha,\beta,\gamma,\delta) I \equiv \mathrm{e}^{\tilde{r}\frac{\Phi_1}{2}r} \mathrm{e}^{\tilde{\nabla}\frac{\Phi_2}{2}\nabla} \mathrm{e}^{\tilde{r}\frac{\Phi_3}{2}r} I \tag{121}$$

and afterward utilize the formulae coming from (14)

$$f(\delta\nabla + \gamma\hat{r}) \equiv \mathrm{e}^{\tilde{\nabla}\frac{\gamma\delta^{-1}}{2}\nabla} f(\delta r) \mathrm{e}^{-\tilde{\nabla}\frac{\gamma\delta^{-1}}{2}\nabla} \equiv \mathrm{e}^{-\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} f(\gamma\nabla) \mathrm{e}^{\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} \tag{122}$$

8.1. Obtaining $S(\alpha,\beta,\gamma,\delta)I$

Concerning the transform

- $$S_1(\alpha,\beta,\gamma,\delta) I = \mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1}{2}\hat{r}} \mathrm{e}^{\tilde{\nabla}\frac{\Phi_2}{2}\hat{r}} \mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_3}{2}\hat{r}} I \tag{123}$$

we get immediately according to (21)

$$S_1(\alpha,\beta,\gamma,\delta) I = \mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1}{2}\hat{r}} \mathrm{e}^{\tilde{\hat{r}}\mathrm{e}^{\frac{\tilde{\Phi}_2}{2}}\frac{\Phi_3}{2}\mathrm{e}^{\frac{\Phi_2}{2}}\hat{r}} \mathrm{e}^{\frac{\Phi_2}{2}} \tag{124}$$

As for

- $$S(\alpha,\beta,\gamma,\delta) I = \mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_1}{2}\hat{r}} \mathrm{e}^{\tilde{\nabla}\frac{\Phi_2}{2}\nabla} \mathrm{e}^{\tilde{\hat{r}}\frac{\Phi_3}{2}\hat{r}} I \tag{125}$$

we utilize the formulae (92), (94) as so as (49) to get

$$\Phi_1 = \gamma^{-1}(I - \alpha), \Phi_2 = \gamma, \Phi_3 = (I - \delta)\gamma^{-1}$$

$$S(\alpha,\beta,\gamma,\delta)I = \mathrm{e}^{\tilde{\hat r}\frac{\Phi_1}{2}\hat r}\mathrm{e}^{\tilde\nabla\left(\frac{\Phi_2}{2}-\frac{\Phi_3^{-1}}{2}\right)\nabla}\mathrm{e}^{\tilde\nabla\frac{\Phi_3^{-1}}{2}\nabla}\mathrm{e}^{\tilde{\hat r}\frac{\Phi_3}{2}\hat r}I$$
$$= \mathrm{e}^{\tilde{\hat r}\frac{\Phi_1}{2}\hat r}\mathrm{e}^{\tilde\nabla\left(\frac{\Phi_2}{2}-\frac{\Phi_3^{-1}}{2}\right)\nabla}\delta(r)$$
$$= (2\pi)^{N/2}\left|\det(-\Phi_3)\right|^{-1/2}\mathrm{e}^{\tilde{\hat r}\frac{\Phi_1}{2}\hat r}(2\pi)^{-\frac{N}{2}}\left|\det\left(\Phi_3^{-1}-\Phi_2\right)\right|^{-1/2}\mathrm{e}^{-\tilde{\hat r}\left(\frac{\Phi_2}{2}-\frac{\Phi_3^{-1}}{2}\right)\hat r}$$
$$= \left|\det\left(I-\Phi_3\Phi_2\right)\right|^{-1/2}\mathrm{e}^{\tilde{\hat r}\frac{\left(\Phi_1-\Phi_2+\Phi_3^{-1}\right)^{-1}}{2}\hat r}$$
$$= \left|\det\delta\right|^{-1/2}\mathrm{e}^{\tilde{\hat r}\left(\frac{\Phi_1+\delta^{-1}\Phi_3}{2}\right)\hat r}\quad \text{if}\quad \Phi_2\neq\Phi_3^{-1} \tag{126}$$
$$= (2\pi)^{\frac{N}{2}}\left|\det\left(-\gamma^{-1}\right)\right|^{-1/2}\delta(r)\quad\text{if}\quad \Phi_2=\Phi_3^{-1} \tag{127}$$

8.2. Obtaining $S(\alpha,\beta,\gamma,\delta)f(r)$ by Gaussian Transforms

According to (120) and (49) we have

$$S(\alpha,\beta,\gamma,\delta)f(r) = f\left(\gamma\nabla+\delta\hat r\right)S(\alpha,\beta,\gamma,\delta)I$$
$$= \mathrm{e}^{\tilde\nabla\frac{\gamma\delta^{-1}}{2}\nabla}f(\delta r)\mathrm{e}^{-\tilde\nabla\frac{\gamma\delta^{-1}}{2}\nabla}\left|\det\delta\right|^{-1/2}\mathrm{e}^{\tilde{\hat r}\left(\frac{\Phi_1+\delta^{-1}\Phi_3}{2}\right)\hat r},\ \delta\neq 0$$

- $$S(\alpha,\beta,\gamma,\delta)f(r) = \mathrm{e}^{\tilde\nabla\frac{\gamma\delta^{-1}}{2}\nabla}f(\delta r)\mathrm{e}^{-\tilde\nabla\frac{\gamma\delta^{-1}}{2}\nabla}\left|\det\delta^{-1/2}\right|\mathrm{e}^{-\tilde{\hat r}\frac{\gamma^{-1}\alpha-\delta^{-1}\gamma^{-1}}{2}\hat r}$$

and, by (93)

$$S(\alpha,\beta,\gamma,\delta)f(r)$$
$$= \mathrm{e}^{\tilde\nabla\frac{\gamma\delta^{-1}}{2}\nabla}f(\delta r)\left|\det\left(\delta-\left(\delta\gamma^{-1}\alpha\gamma-I\right)\delta^{-1}\right)\right|^{-1/2}\mathrm{e}^{\tilde{\hat r}\frac{\left(\gamma\delta^{-1}-\left(\gamma^{-1}\alpha-\delta^{-1}\gamma^{-1}\right)^{-1}\right)^{-1}}{2}\hat r} \tag{128}$$

- In the case where $\alpha\gamma=\gamma\alpha$ and $\alpha\delta=\delta\alpha$ we have the formula

$$S(\alpha,\beta,\gamma,\delta)f(r) = \mathrm{e}^{\tilde\nabla\frac{\gamma\delta^{-1}}{2}\nabla}f(\delta r)\left|\det\left(\delta-\beta\gamma\delta^{-1}\right)\right|^{-1/2}\mathrm{e}^{\tilde{\hat r}\frac{\left(\gamma\delta^{-1}-\beta^{-1}\delta\right)^{-1}}{2}\hat r} \tag{129}$$

For example

- For the dual couple

$$(A,B)\equiv\frac{1}{(2i)^{1/2}}\left(D_x-i\hat X, D_x+i\hat X\right) \tag{130}$$

we get from (32)

$$\mathrm{e}^{\theta\frac{1}{2i}\left(D_x^2+\hat X^2\right)}\begin{pmatrix} I & -iI\\ I & iI\end{pmatrix}\begin{pmatrix} D_x\\ \hat X\end{pmatrix}\mathrm{e}^{-\theta\frac{1}{2i}\left(D_x^2+\hat X^2\right)}\equiv\begin{pmatrix}\mathrm{e}^{-\theta}I & 0I\\ 0I & \mathrm{e}^{\theta}I\end{pmatrix}\begin{pmatrix} I & -iI\\ I & iI\end{pmatrix}\begin{pmatrix} D_x\\ \hat X\end{pmatrix}$$
$$\mathrm{e}^{\theta\frac{1}{i}\left(D_x^2+\hat X^2\right)}\begin{pmatrix} D_x\\ \hat X\end{pmatrix}\mathrm{e}^{-\theta\frac{1}{i}\left(D_x^2+\hat X^2\right)}\equiv\begin{pmatrix} I/2 & I/2\\ iI/2 & -iI/2\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-2\theta}I & 0I\\ 0I & \mathrm{e}^{2\theta}I\end{pmatrix}\begin{pmatrix} I & -iI\\ I & iI\end{pmatrix}\begin{pmatrix} D_x\\ \hat X\end{pmatrix}$$
$$\mathrm{e}^{\theta\left(D_x^2+\hat X^2\right)}\begin{pmatrix} D_x\\ \hat X\end{pmatrix}\mathrm{e}^{-\theta\left(D_x^2+\hat X^2\right)}\equiv\begin{pmatrix}\cos 2\theta I & -\sin 2\theta I\\ \sin 2\theta I & \cos 2\theta I\end{pmatrix}\begin{pmatrix} D_x\\ \hat X\end{pmatrix} \tag{131}$$

and from (129)

$$e^{\theta\left(D^2+X^2\right)} f\left(x\right)=\left|\cos 2\theta\right|^{1/2} e^{\frac{1}{2}D_x^2 \operatorname{tg} 2\theta} f\left(x\cos 2\theta\right) e^{\frac{1}{4}x^2 \sin 4\theta} \tag{132}$$

$$e^{\frac{\pi}{8}\left(D^2+X^2\right)} f\left(x\right)=2^{-1/4} e^{\frac{1}{2}D_x^2} f\left(2^{-1/2}x\right) e^{\frac{1}{4}x^2} \tag{133}$$

The formulae (129), (130) are the Baker-Campbell-Hausdorff and Bargman formulae that we can find in [9].

8.3. Obtaining $S\left(\alpha,\beta,\gamma,\delta\right) f\left(r\right)$ by Differentiation

According to (120) and (49) we have

$$S\left(\alpha,\beta,\gamma,\delta\right) f\left(r\right)= f\left(\gamma\nabla+\delta\hat{r}\right) S\left(\alpha,\beta,\gamma,\delta\right) I$$

$$\Phi_1 \equiv \gamma^{-1}\left(I-\alpha\right), \Phi_2 \equiv \gamma, \Phi_3 \equiv \gamma^{-1}\left(I-\delta\right)$$

so that by (126), (127)

- For $\gamma \neq 0$ and $\gamma \equiv \Phi_2 \neq \Phi_3^{-1} \Rightarrow \delta \neq 0$

$$\begin{aligned} S\left(\alpha,\beta,\gamma,\delta\right) f\left(r\right) &= e^{-\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} f\left(\gamma\nabla\right) e^{\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} \left|\det\delta\right|^{-1/2} e^{\tilde{\hat{r}}\frac{\Phi_1+\delta^{-1}\Phi_3}{2}\hat{r}} \\ &= e^{-\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} f\left(\gamma\nabla\right)\left|\det\delta\right|^{-1/2} e^{\tilde{\hat{r}}\frac{\gamma^{-1}\left(I-\alpha+\delta\right)+\delta^{-1}\gamma^{-1}\left(I-\delta\right)}{2}\hat{r}} \end{aligned} \tag{134}$$

- For $\gamma \neq 0$ and $\gamma \equiv \Phi_2 \equiv \Phi_3^{-1}$

$$\begin{aligned} S\left(\alpha,\beta,\gamma,\delta\right) f\left(r\right) &= \left(2\pi\right)^{\frac{N}{2}} \left|\det\left(-\gamma^{-1}\right)\right|^{-1/2} e^{-\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} f\left(\gamma\nabla\right) e^{\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} \delta\left(r\right) \\ &= \left(2\pi\right)^{\frac{N}{2}} \left|\det\left(-\gamma^{-1}\right)\right|^{-1/2} e^{-\tilde{\hat{r}}\frac{\gamma^{-1}\delta}{2}\hat{r}} f\left(\gamma\nabla\right)\delta\left(r\right) \end{aligned} \tag{135}$$

The Fourier transform corresponds to this case.

9. Remarks and Conclusions

Firstly, we think that this work is interesting by its simplicity because no knowledge of Lie groups' method is necessary. Is necessary only the use of couples of dual operators $\left(A,B\right)$ obeying $\left(\tilde{A}B-\tilde{B}A\right)\equiv I$ such as $\left(\nabla,\hat{r}\right)$, $\left(D_x,\hat{X}\right)$, $\left(a,a^+\right)$, $\left(\hat{X}D_x,\ln\hat{X}\right)$, etc. together with the fundamental law affirming that any relation between two dual operators is applicable to any other dual couple. Secondly from the Newtonian formula for a binomial $\left(x+a\right)^n$ we get immediately the translation operator $\exp\left(aD_x\right)$ then, thanks to the said fundamental law, the dilatation operator $\exp\left(a\hat{X}D_x\right)$, is not yet well-known until now. Always from the fundamental law, we get then the precious operator $\exp\left(\left(aD_x+b\hat{X}\right)\left(cD_x+d\hat{X}\right)\right)$ which transforms the dual couple $\left(D_x,\hat{X}\right)$ into $\left(\alpha D_x+\beta\hat{X},\gamma D_x+\delta\hat{X}\right)$ where $\left(\alpha,\beta,\gamma,\delta\right)$ is easily calculable from $\left(a,b,c,d\right)$. Thirdly, by taking the products of three operators having the forms $\exp\left(\tilde{\hat{X}}\Phi\hat{X}\right)$, $\exp\left(\tilde{D}_x\Phi\hat{X}\right)$, $\exp\left(\tilde{D}_x\Phi D_x\right)$ we obtain other operators realizing linear and linear canonical transforms. From these we may calculate the linear transforms and LCTs of operators and of functions by an integral as so as by Gaussian transformation. From the formula representing the said integral realization we get a clear relation between linear and linear canonical transforms. Many examples of LCTs are given for showing the simplicity of the method.

We are conscientious that the approach for studying LCTs in this work is too simple with respect to the work of Wolf [3]. Nevertheless, we think that it may be a useful initiation to the subject. Closing this work on LCTs we predict as Quesne [1] that a similar study on nonlinear canonical transforms is conceivable if we utilize couples of dual operators (A, B) of order higher than two such as $\left[-\hat{X}^2 D_x, \hat{X}^{-1}\right] \equiv I$, etc.

We hope that in the future we may study the properties of LCTs and NLCTs and their applications in quantum mechanics, signal processing, optics and mechanics, etc., following the present work, for comparison in simplicity with those given by Bastiaans, Alieva [10] and Ranaivoson *et al.* [11] etc. utilizing group methods.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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