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History

Expression of Concern:

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X no

Correction:

☐ yes, date: yyyy-mm-dd

X no

Comment:

The author checked the published article in detail and found that the proof of "periodicity" is not strict. The author voluntarily withdraws the manuscript.

This article has been retracted to straighten the academic record. In making this decision the Editorial Board follows COPE's Retraction Guidelines. Aim is to promote the circulation of scientific research by offering an ideal research publication platform with due consideration of internationally accepted standards on publication ethics. The Editorial Board would like to extend its sincere apologies for any inconvenience this retraction may have caused.

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The Proof of Riemann Hypothesis, the Key to the Door Is the Periodicity

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Abstract

The Riemann hypothesis is a well-known mathematical problem that has been in suspense for 162 years. Its difficulty lies in the fact that it is involved in an infinite integral which includes infinite series with complex variables. To detour this is in vain, since all the messages are hid in it. To unscramble them, there is a totally new idea, that is, the "periodicity"! By investigating the numerical approximate values of zero points, an explicit distribution law on the critical line was found. To accord with this, a periodic form for the real part of Xi function was constructed and rigidly proved. The Riemann hypothesis can be divided into three progressive propositions. The first proposition (the number of zero points in the critical strip satisfies a certain estimation) had been proved in 1905. The second proposition (the number of zero points on the critical line satisfies the same estimation as in the critical strip) is ever in suspense. It can be solved perfectly with the newly found "periodicity". The third proposition (all the nontrivial zero points are on the critical line), that is, the Riemann hypothesis, is also true. The proof is a combination of the symmetry, monotonicity, periodicity of the Xi function and the extremum principle of the harmonic functions. It is the moment to draw full stop for this suspending problem.

Keywords

Riemann Hypothesis, Riemann Zeta Function, Distribution Law of Zero Point, Periodicity, Monotonicity, Extremum Principle

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1. Introduction

The Riemann hypothesis is a well-known mathematical problem. It had ever been a global hot topic when British mathematician Michael F. Atiyah (1929-2019) reported his proof in Heidelberg Laureate Forum on Sep. 24, 2018. Unfortunately, his approach does not work and now this problem is still in suspense. To crack it, just as reported by New Scientist in [1], "*unless with a totally new idea*." To read through the monographs in [2] [3] [4] and the popular readings in [5] [6], one can find the difficulty of Riemann hypothesis lies in the complexity of the Zeta function. Essentially, it is involved in an infinite integral which includes infinite series with complex variables. To detour this infinite integral is in vain, since all the messages are hid in it. To unscramble them, in my brain many ideas appeared and died during the past three years. Among them there is only one which survives and could be seen as "a totally new idea". This is the "periodicity"! The proof will be a combination of the symmetry, monotonicity, periodicity and extremum principle. Notice that the present article may become the terminator of this problem, a whole story will be told.

This problem was left by German mathematician Bernhard Riemann (1826-1866) in 1859 [4] [7] [8]. He found that the distribution of prime numbers among all natural numbers is very closely related to the analytic continuation of the following infinite series on complex plane $\mathbb{C}$:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots \tag{1}$$

with $s=\sigma+it$, which is usually called *Riemann Zeta* function. This series converges only for the case $\sigma>1$, so all the mentioned $\zeta(s)$ below should be understood in the sense of analytic continuation. If there is a s_0 which satisfies $\zeta(s_0)=0$, then we call it one zero point of $\zeta(s)$. This function has real zero points at the negative even integers $-2,-4,-6,\cdots$ and one refers to them as the *trivial zero points*. Relatively, the other complex zero points of it are called the *nontrivial zero points*.

Riemann hypothesis: *All the nontrivial zero points of* $\zeta(s)$ *have real part* $\sigma=1/2$.

Is it true? In the official millennium problem description [9], E. Bombieri had reviewed that, in 1986 the first 1.5 × 10^9 nontrivial zero points of $\zeta(s)$ (arranged by increasing positive imaginary part) had been checked by J. Lune *et al* with numerical approach in [10], and the result showed that they are simple and all possess real part $\sigma=1/2$. More zero points had been checked by the followers, the same thing occurred. Till now the known record was set by X. Gourdon in 2004 [11], 10^{13} zero points. So the Riemann hypothesis is very likely true. What lacks is the theoretical proof.

The analytic continuation for $\sum_{n=1}^{\infty}n^{-s}$ (defined on $\sigma>1$) is not unique. Among them, the most popular one is given by:

$$\zeta(s)=\frac{\pi^{s/2}\xi(s)}{(s-1)\Gamma(s/2+1)}, \tag{2}$$

which is meromorphic on $\mathbb{C}$ with a unique pole at $s=1$ (its residual is 1) [2] [3] [4]. Here

$$\Gamma(s/2+1)=\int_0^{\infty}x^{s/2}\mathrm{e}^{-x}\mathrm{d}x \tag{3}$$

with a default requirement $\sigma/2+1>0$. Its analytic continuation to the negative direction is done by $\Gamma(s/2)=\Gamma(s/2+1)/(s/2)$ step by step. In this sense, $-2,-4,-6,\cdots$ are the zero points of $1/\Gamma(s/2+1)$, and this is why they are the trivial zero points of $\zeta(s)$. Naturally, the function $\xi(s)$ satisfies the inverse relation which was written down by Riemann:

$$\begin{aligned}\xi(s)&=(s-1)\pi^{-\frac{s}{2}}\Gamma(s/2+1)\zeta(s)\\&=\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s)\\&=\frac{s(s-1)}{2}\left[\frac{1}{s(s-1)}+\int_1^\infty\psi(x)\left(x^{\frac{s}{2}-1}+x^{-\frac{s+1}{2}}\right)\mathrm{d}x\right]\\&=\frac{1}{2}-\frac{s(1-s)}{2}\int_1^\infty\psi(x)\left(x^{\frac{s}{2}-1}+x^{-\frac{s+1}{2}}\right)\mathrm{d}x\end{aligned}\tag{4}$$

with $\psi(x)=\sum_{n=1}^{\infty}\mathrm{e}^{-\pi n^2 x}$, which satisfies a peculiar relation bellow (see the deduction in [12] [13]):

$$2\psi(x)+1=x^{-1/2}\left[2\psi(x^{-1})+1\right]\tag{5}$$

Notice that $\psi(x)$ is positive and exponentially decreasing on the interval $[1,\infty)$, the integral in Equation (4) is finite and hence $\xi(s)$ is analytic on $\mathbb{C}$. Particularly, the zero points of $\xi(s)$ coincide with the nontrivial zero points of $\zeta(s)$, and the exploring of Riemann hypothesis can be converted to considering $\xi(s)$. To substitute s by $1-s$ the expression maintains unchanged, so $\xi(s)=\xi(1-s)$. This symmetric property implies the most important relation between the function $\zeta(s)$ and the *critical line* $\sigma=1/2$. Correspondingly, the complex region with $0\le\sigma\le1$ is called the *critical strip* which includes all the zero points of $\xi(s)$.

In fact, $\xi(s)$ has no zero points outside the critical strip. With the aid of Euler formula:

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}=\prod_p\left(1-p^{-s}\right)^{-1}\tag{6}$$

(where the last expression is a product respect to all the prime numbers $p=2,3,5,\cdots$), one can easily checked that $|\zeta(s)|\neq0$, that is $|\xi(s)|\neq0$, for the case $\sigma>1$ (see [5]). Furthermore, the relation $\xi(s)=\xi(1-s)$ indicates that the same thing is true for $\sigma<0$.

We note that, as the *critical strip* concerned, the two lines $\sigma=1$ and $\sigma=0$ can be gotten rid of. As reviewed in [4] [13], the proof given by de la Vallée Poussion in 1899 for this is very complex. Yet, as our approach concerned, it doesn't matter. Any chosen strip with $1/2-a\le\sigma\le1/2+a$ and $a\ge1/2$ can fulfil the request.

To denote $\xi(\sigma+it)=U(\sigma,t)+iV(\sigma,t)$, then it follows from the well-known Reflection Principle $\overline{\xi(s)}=\xi(\bar{s})$ of complex conjugate that $U(\sigma,t)-iV(\sigma,t)=U(\sigma,-t)+iV(\sigma,-t)$. Meanwhile, the relation $\xi(s)=\xi(1-s)$ yields $U(\sigma,t)+iV(\sigma,t)=U(1-\sigma,-t)+iV(1-\sigma,-t)$. Based on these two relations, the symmetric properties of $\xi(s)$ are clarified as:

Lemma 1. *To denote* $\xi(\sigma+it)=U(\sigma,t)+iV(\sigma,t)$*, then its real part and imaginary part separately satisfy the symmetries and anti-symmetries bellow:*

$$U(\sigma,-t)=U(\sigma,t),\quad V(\sigma,-t)=-V(\sigma,t),$$
$$U(1-\sigma,t)=U(\sigma,t),\quad V(1-\sigma,t)=-V(\sigma,t).$$

For the particular cases with $t=0$ and $\sigma=1/2$, the relations for V read $V(\sigma,0)=-V(\sigma,0)$ and $V(1/2,t)=-V(1/2,t)$. So $V(\sigma,0)=V(1/2,t)=0$. This indicates that the values of $\xi(\sigma+it)$ are real on the lines $t=0$ and $\sigma=1/2$. This lemma can be understood as: *U and V are symmetric and anti-symmetric about the two lines* $t=0$ *and* $\sigma=1/2$*, respectively*. That is why most of the arguments are about the upper quarter strip with $1/2\le\sigma\le1$ and $t>0$. Particularly, due to the direct relationship with the Riemann hypothesis, the line $\sigma=1/2$ has drawn much attention.

It is enlightening to divide Riemann hypothesis into three progressive propositions (see [5], pages 22-23):

Proposition 1. *In the region bounded by* $0\le\sigma\le1$ *and* $0<t<T$*, the number of zero points of* $\xi(s)$ *is about* $T/2\pi\log(T/2\pi e)$*.*

Proposition 2. *On the critical line* $\sigma=1/2$ *with* $0<t<T$*, the number of zero points of* $\xi(s)$ *is also about* $T/2\pi\log(T/2\pi e)$*.*

Proposition 3. *All the zero points of* $\xi(s)$ *are on the critical line* $\sigma=1/2$*.*

Till now only the first proposition had been proved. It was finished by German mathematician Hans von Mangoldt in 1905 [4] [5] [14]. His estimation for the zero-points number of $\xi(s)$ is

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log(T)),\tag{7}$$

which is understood as taking the integer part. We note that, $O(\log(T))$ does not imply the existence of a term $C\log(T)$ for some constant C. Its meaning is that $|N(T)-(T/2\pi)\log(T/2\pi e)|\le C\log(T)$. In fact, the previously mentioned checks are done according to this theoretical result, and the numerical approximate values for the first four zero points are

$$1/2+14.1347251i,\quad 1/2+21.0220396i,$$
$$1/2+25.0108575i,\quad 1/2+30.4248761i.$$

On the aspect of theoretical study, in 1914 Hardy firstly proved that, *there are infinitely many zero points of* $\xi(s)$ *on the critical line* $\sigma=1/2$ (see the review in [3]). Due to the efforts of Selberg [15], Levinson [16] and Conrey [17], the ratio was lifted step by step. Now the known optimal estimation is that *more than* 40% *of zero points of* $\xi(s)$ *are on the critical line* $\sigma=1/2$ (they are also simple ones). These indicate that Proposition 2 is far from settled. Our approach is a direct attack to it. In addition, as reviewed in [4], "*All the zero points of* $\xi(s)$ *are simple ones*" already became an accompanying conjecture to Riemann hypothesis. Yet, under our approach, this is not a thing.

The above survey is about the known knowledge of Riemann hypothesis, and the next is about our new findings. We begin with investigating the distribution law of zero points on the critical line $\sigma = 1/2$ in *Section* 2; The third section is about the main content, which includes 3 subsections. In the last subsection the periodicity of $U(\sigma,t)$ is detailed illustrated. During the argument processes of periodicity, the monotonicity of the related functions become the preconditions. With the aid of explicit expression of $U(\sigma,t)$, the distribution law of zero points is verified. This indicates Proposition 2 given by Riemann holds true. The proof of Proposition 3, that is the Riemann hypothesis, is given in *Section* 4. The conclusions and related remarks are given in the last.

2. Distribution Law for the Zero Points on the Critical Line

On the critical line $\sigma = 1/2$, the values of $\xi(s)$ are real (ensured by Lemma 1), that is, only the real part $U(1/2,t)$ of $\xi(1/2+it)$ is left. To recall the expression of $\xi(s)$ in (4), it follows from [3] [4] that

$$\begin{aligned} U(1/2,t) &= (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s)\Big|_{s=1/2+it} \\ &= -\frac{1+4t^2}{8}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\Big|_{s=1/2+it} \\ &= -a(t)e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right), \end{aligned} \tag{8}$$

where ($\Re$ and $\Im$ mean the taking of real and imaginary parts, respectively)

$$a(t) = \frac{1+4t^2}{8}\pi^{-1/4}\exp\left[\Re\log\Gamma\left(\frac{1+2ti}{4}\right)\right], \tag{9}$$

$$\begin{aligned} \vartheta(t) &= \Im\log\Gamma\left(\frac{1+2ti}{4}\right) - \frac{t}{2}\log\pi \\ &= \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \cdots. \end{aligned} \tag{10}$$

Notice that $a(t) > 0$, the zero points on the critical line are only determined by the sign shift of $Z(t) = e^{i\vartheta(t)}\zeta(1/2+it)$. Furthermore, as reviewed in [4], the inverse expression can be employed to make the estimation [the zero points of $\zeta(1/2+it)$ coincide with that of $\xi(1/2+it)$]:

$$\zeta(1/2+it) = e^{-i\vartheta(t)}Z(t) = Z(t)\cos\vartheta(t) - iZ(t)\sin\vartheta(t). \tag{11}$$

According to this, for the first time, Gram calculated the first 15 zero points in 1903. His method is an estimation approach based on $\sin\vartheta(t) = 0$, which yields the known *Gram points* at $\vartheta(t_k) = k\pi$ with an approximation:

$$\frac{t_k}{2}\log\frac{t_k}{2\pi e} - \frac{\pi}{8} = k\pi, \quad k = 1, 2, \cdots. \tag{12}$$

Certainly, to estimate the location of a zero point, the change of $Z(t)$ should be also considered. This adds the complexity. By the way, 29 years later, the approximate approach for $Z(t)$ was improved. That is the known *Riemann-Siegel formula*, which was discovered by Siegel in 1932 among Riemann's private papers (see [4]).

Though the method given by Gram is so rough, it is enlightening. The distribution of zero points on the critical line $\sigma = 1/2$ may possess some kind of periodicity. In fact, for some given $t = T$, the argument $\vartheta(T)$ obtained on the critical line and the zero-points number $N(T)$ in the region $0 \le \sigma \le 1$ and $0 < t < T$ have close relationship:

$$\vartheta(T) = \frac{T}{2}\log\frac{T}{2\pi e} - \frac{\pi}{8} + \cdots,$$

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + \cdots.$$

Exactly, as reviewed in [3] (page 132-134), in 1918 Backlund had revealed that, for all $T \ge 2$,

$$\left|N(T) - \pi^{-1}\vartheta(T) - 1\right| < 4.350 + 0.137\log T + 0.443\log\log T.$$

Enlightened by this, we have a new idea. Proposition 2 holds true provided that $U(1/2, t) = \xi(1/2 + it)$ can be expressed in the following form:

$$U(1/2, t) = A(t)\sin\left(\frac{t}{2}\log\frac{t}{2\pi e} + \alpha + \varepsilon(t)\right), \tag{13}$$

where $A(t)$, α and $\varepsilon(t)$ are the amplitude (>0), initial phase angle and small perturbation function. Certainly, for the case with $t = 2\pi$ we have $(t/2)\log(t/2\pi e) = -\pi$. Hence, a suitable phase angle α should be chosen to accord with the fact that $\xi(1/2 + 2\pi i) = U(1/2, 2\pi) > 0$ (since $\xi(0) = -\zeta(0) = 1/2$ and there is no zero points for $0 \le t \le 14$, see [3], page 31). If $U(1/2, t)$ is indeed in such a periodic form, the thing becomes very simple. All the zero points on the critical line $\sigma = 1/2$ can be solved one by one with the following formula:

$$\frac{t_k}{2}\log\frac{t_k}{2\pi e} + \alpha + \varepsilon(t_k) = k\pi, \quad k = 1, 2, \cdots. \tag{14}$$

By the way, to count the number of zero points it can be used in a reverse way, that is,

$$N = \frac{t_N}{2\pi}\log\frac{t_N}{2\pi e} + \frac{\alpha}{\pi} + \frac{\varepsilon(t_N)}{\pi},$$

which accords with Proposition 2. Relative to Gram's approximation in Equation (12), this formula is closer to the truth. In the following we give some numerical evidences for this. The theoretical proof will be given in the next sections.

To neglect the perturbation term $\varepsilon(t_k)$ and make fitting with the known numerical approximate values t_k^* to the zero points (provided by Odlyzko in [18]), it leads to the results in **Figure 1**. The first 50 zero points fitting yields an optimal phase angle $\alpha = 4\pi/3$, that is, the k-th zero point on the critical line is very close to the solution of

$$\frac{t_k}{2}\log\frac{t_k}{2\pi e} + \frac{4}{3}\pi = k\pi, \quad k = 1, 2, \cdots. \tag{15}$$

We call this newly found formula as the distribution law of zero points.

The check in **Figure 2** with 10,000 zero points shows that the above distribution

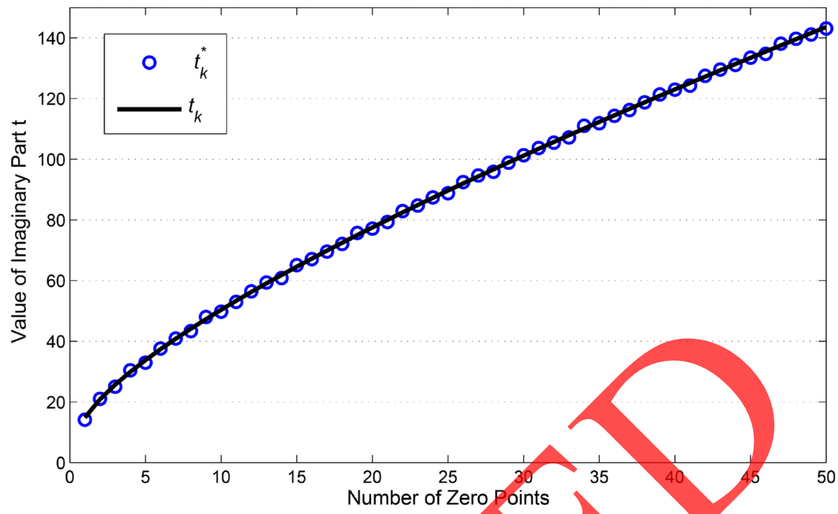

Figure 1. The fitting curve with the known numerical approximate values t_k^* of the first 50 zero points, according to the formula in Equation (14) for the case with $\varepsilon(t_k)=0$.

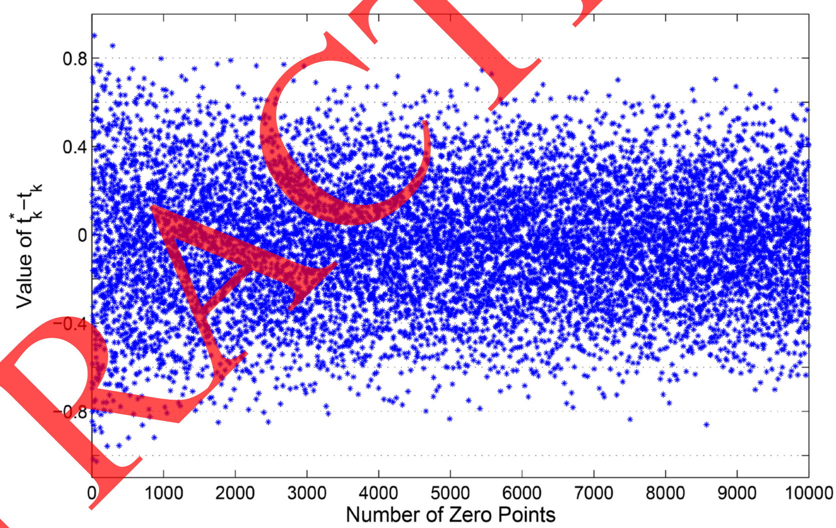

Figure 2. The variation of the error $t_k^*-t_k$ with respect to the first 10,000 zero points.

law is also obeyed. The error is small. Though the value of t_k^* is up to 9877, except the 14-th and 64-th zero points, all the corresponding errors satisfy $\left|t_k-t_k^*\right|<1$.

In addition to the perturbation term $\varepsilon(t_k)$, these errors may also partly ascribe to the numerical scheme for t_k^*, though its calculation accuracy is up to 10^{-9}. Not forget that, essentially, the calculation of $\xi(s)$ is involved in an infinite integral which includes infinite series with complex variables. At least, the approximate values for the 14-th and 64-th zero points are questionable. To take the 14-th zero as an example. $t_{14}^*=60.831778525$, $t_{15}^*=65.112544048$ and hence

$$\frac{t_{15}^*}{2}\log\frac{t_{15}^*}{2\pi e}-\frac{t_{14}^*}{2}\log\frac{t_{14}^*}{2\pi e}\approx 43.5681-38.6353=4.9328>\pi. \qquad (16)$$

Yet, it follows from $N(T)\approx\pi^{-1}(T/2)\log(T/2\pi e)$ that, to add one π means to add a zero point. This indicates that 4.9328 is a too big difference between two adjacent zero points, it should be understood as a calculation error. With this understanding, the departure from the newly-found distribution law caused by the perturbation term $\varepsilon(t_k)$ may be smaller than the difference of $\left|t_k^* - t_k\right|$ depicted here.

Further check is given in **Figure 3**. This law is also obeyed for the zero points numbered from 99,900 to 100,000.

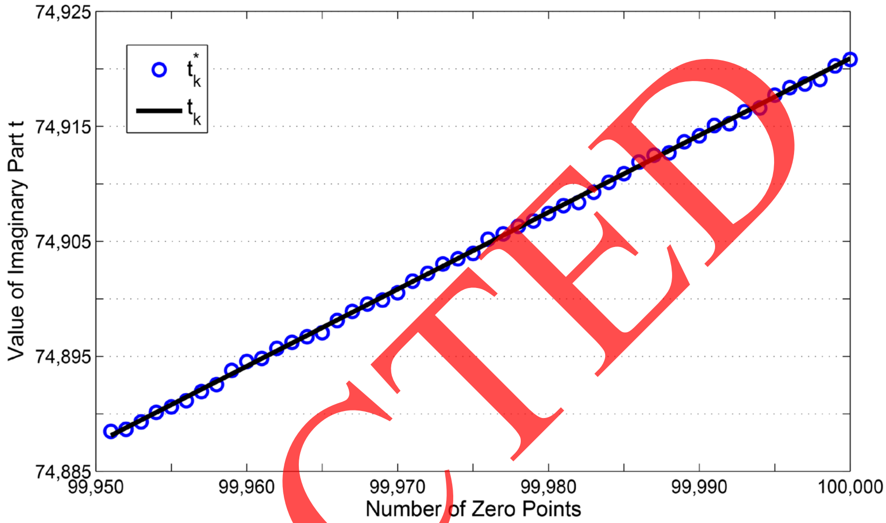

Figure 3. The comparison between the solutions of Equation (15) and the approximate values t_k^* of the zero points numbered from 99,900 to 100,000.

3. To Prove the Periodicity of $U(\sigma,t)$

Can $U(1/2,t) = \xi(1/2+it)$ be expressed in the periodic form of Equation (13)? It needs a theoretical proof. Our theme is not limited by this, and in order to solve the Riemann hypothesis the periodicity of $U(\sigma,t)$ with $1/2 \le \sigma \le 1$ and $t > 0$ will be considered in a unified way.

Firstly, the 2-terms expression for $\xi(s)$ in Equation (4) is awkward for use. In the following we adopt the derived form (see [3], pages 16-17):

$$\begin{aligned}\xi(s) &= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s+1}{2}}\right)\mathrm{d}x \\ &= 4\int_1^\infty \left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}} \cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\log x\right]\mathrm{d}x \\ &= 4\int_1^\infty \left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}} \cdot \frac{1}{2}\left[\mathrm{e}^{\frac{1}{2}\left(\sigma+it-\frac{1}{2}\right)\log x} + \mathrm{e}^{-\frac{1}{2}\left(\sigma+it-\frac{1}{2}\right)\log x}\right]\mathrm{d}x \qquad (17) \\ &= 2\int_1^\infty \left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}} \left[x^{\frac{1}{2}\left(\sigma-\frac{1}{2}\right)} + x^{-\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}\right]\cos\left(\frac{t}{2}\log x\right)\mathrm{d}x \\ &\quad + 2i\int_1^\infty \left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}} \left[x^{\frac{1}{2}\left(\sigma-\frac{1}{2}\right)} - x^{-\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}\right]\sin\left(\frac{t}{2}\log x\right)\mathrm{d}x,\end{aligned}$$

here the superscript "′" denotes the taking of conventional derivative. Hence,

$$U(\sigma,t)=2\int_1^{\infty}\left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}}\left[x^{\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}+x^{-\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}\right]\cos\left(\frac{t}{2}\log x\right)\mathrm{d}x,$$

$$V(\sigma,t)=2\int_1^{\infty}\left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}}\left[x^{\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}-x^{-\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}\right]\sin\left(\frac{t}{2}\log x\right)\mathrm{d}x. \tag{18}$$

In the following we mainly concern the variation of $U(\sigma,t)$.

To denote

$$f(\sigma,x)=2\left[x^{\frac{3}{2}}\psi'(x)\right]' x^{-\frac{1}{4}}\left[x^{\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}+x^{-\frac{1}{2}\left(\sigma-\frac{1}{2}\right)}\right],$$

$$\theta(t)=\frac{t}{4}\log\frac{t}{2\pi\mathrm{e}}+\frac{\pi}{6}, \tag{19}$$

with a substitution of

$$x=\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}y,$$

it follows from the first formula in Equation (18) that

$$\begin{aligned}U(\sigma,t)&=\int_1^{\infty}f(\sigma,x)\cos\left(\frac{t}{2}\log x\right)\mathrm{d}x\\&=\int_{\left(\frac{t}{2\pi\mathrm{e}}\right)^{1/2}\mathrm{e}^{\pi/3t}}^{\infty}f\left[\sigma,\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}y\right]\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\\&\quad\cdot\cos\left\{\frac{t}{2}\log\left[\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}y\right]\right\}\mathrm{d}y\\&=\int_{\left(\frac{t}{2\pi\mathrm{e}}\right)^{1/2}\mathrm{e}^{\pi/3t}}^{\infty}f\left[\sigma,\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}y\right]\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\\&\quad\cdot\cos\left[\frac{t}{2}\log y-\theta(t)\right]\mathrm{d}y\\&=\frac{2}{t}\left[I_1(\sigma,t)\cos\theta(t)+I_2(\sigma,t)\sin\theta(t)\right],\end{aligned} \tag{20}$$

where, under another substitution $y=\mathrm{e}^{2z/t}$,

$$\begin{aligned}I_1(\sigma,t)&=\frac{t}{2}\int_{\left(\frac{t}{2\pi\mathrm{e}}\right)^{1/2}\mathrm{e}^{\pi/3t}}^{\infty}f\left[\sigma,\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\mathrm{e}^{2z/t}\right]\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\cos\left(\frac{t}{2}\log y\right)\mathrm{d}y\\&=\int_{\theta(t)}^{\infty}f\left[\sigma,\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\mathrm{e}^{2z/t}\right]\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\mathrm{e}^{2z/t}\cos z\mathrm{d}z\\&=\int_{\theta(t)}^{\infty}F\left(\sigma,x(z,t)\right)\cos z\mathrm{d}z,\end{aligned} \tag{21}$$

$$\begin{aligned}I_2(\sigma,t)&=\frac{t}{2}\int_{\left(\frac{t}{2\pi\mathrm{e}}\right)^{1/2}\mathrm{e}^{\pi/3t}}^{\infty}f\left[\sigma,\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}y\right]\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\sin\left(\frac{t}{2}\log y\right)\mathrm{d}y\\&=\int_{\theta(t)}^{\infty}F\left(\sigma,x(z,t)\right)\sin z\mathrm{d}z,\end{aligned} \tag{22}$$

in which $F(\sigma,x)=f(\sigma,x)x$ with

$$x(z,t)=\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}y=\left(\frac{2\pi\mathrm{e}}{t}\right)^{1/2}\mathrm{e}^{-\pi/3t}\cdot\mathrm{e}^{2z/t}.$$

3.1. To Prove the Monotonicity of $F(\sigma,x)$ about x

To recall that $\psi(x)=\sum_{n=1}^{\infty}\mathrm{e}^{-\pi n^2 x}$, with a denotation $\delta=(\sigma-1/2)/2$, the definition of $f(\sigma,x)$ in (19) leads to

$$\begin{aligned}F(\sigma,x)&=2\left[x^{3/2}\psi'(x)\right]'x^{-1/4}\left(x^{\delta}+x^{-\delta}\right)\cdot x\\&=2\left[\frac{3}{2}x^{1/2}\psi'(x)+x^{3/2}\psi''(x)\right]x^{3/4}\left(x^{\delta}+x^{-\delta}\right)\\&=2\left[\frac{3}{2}\psi'(x)+x\psi''(x)\right]x^{5/4}\left(x^{\delta}+x^{-\delta}\right)\\&=2\left[\frac{3}{2}\sum_{n=1}^{\infty}\left(-\pi n^2\right)\mathrm{e}^{-\pi n^2 x}+x\sum_{n=1}^{\infty}\left(-\pi n^2\right)^2\mathrm{e}^{-\pi n^2 x}\right]x^{5/4}\left(x^{\delta}+x^{-\delta}\right)\\&=2x^{5/4}\left(x^{\delta}+x^{-\delta}\right)\sum_{n=1}^{\infty}a_n\left(a_n x-\frac{3}{2}\right)\mathrm{e}^{-a_n x}\end{aligned}\tag{23}$$

with $a_n=\pi n^2$. Since $\pi n^2 x-3/2>0$ for $n\geq 1$ and $x\geq 1$, we have $F(\sigma,x)>0$. In addition, to recall that $\delta=(\sigma-1/2)/2$, we have

$$\frac{\partial}{\partial\sigma}\left(x^{\delta}+x^{-\delta}\right)=\frac{\log x}{2}\left(x^{\delta}-x^{-\delta}\right)\geq 0$$

for $x\geq 1$, and hence $F(\sigma,x)$ increases along with the increasing of σ on $[1/2,1]$. By the way, $f(\sigma,x)>0$ also holds and it follows from Equation (18) that

$$\xi(1/2)=U(1/2,0)=\int_1^{\infty}f(1/2,x)\mathrm{d}x>0,$$

which accords with the documented result $\xi(1/2)=-\zeta(1/2)>0$ in [3] (see page 122).

To take the partial derivative of $F(\sigma,x)$ with respect to x, it yields

$$\begin{aligned}F_x(\sigma,x)&=2\sum_{n=1}^{\infty}a_n\frac{\partial}{\partial x}\left[\left(x^{\delta}+x^{-\delta}\right)\left(a_n x^{9/4}-\frac{3}{2}x^{5/4}\right)\mathrm{e}^{-a_n x}\right]\\&=2\sum_{n=1}^{\infty}a_n\delta\left(x^{\delta-1}-x^{-\delta-1}\right)\left(a_n x^{9/4}-\frac{3}{2}x^{5/4}\right)\mathrm{e}^{-a_n x}\\&\quad+2\sum_{n=1}^{\infty}a_n\left(x^{\delta}+x^{-\delta}\right)\left(\frac{9}{4}a_n x^{5/4}-\frac{15}{8}x^{1/4}\right)\mathrm{e}^{-a_n x}\\&\quad-2\sum_{n=1}^{\infty}a_n^2\left(x^{\delta}+x^{-\delta}\right)\left(a_n x^{9/4}-\frac{3}{2}x^{5/4}\right)\mathrm{e}^{-a_n x}\\&=2\delta\left(x^{\delta}-x^{-\delta}\right)x^{1/4}\sum_{n=1}^{\infty}a_n\left(a_n x-\frac{3}{2}\right)\mathrm{e}^{-a_n x}\\&\quad+2\left(x^{\delta}+x^{-\delta}\right)x^{1/4}\sum_{n=1}^{\infty}a_n\left(-a_n^2x^2+\frac{15}{4}a_n x-\frac{15}{8}\right)\mathrm{e}^{-a_n x}\\&=2\left(x^{\delta}+x^{-\delta}\right)x^{1/4}\sum_{n=1}^{\infty}a_n\left[-a_n^2x^2+\frac{15}{4}a_n x-\frac{15}{8}+\delta\frac{x^{2\delta}-1}{x^{2\delta}+1}\left(a_n x-\frac{3}{2}\right)\right]\mathrm{e}^{-a_n x}\end{aligned}$$

$$
\begin{aligned}
&\le 2\left(x^{\delta}+x^{-\delta}\right)x^{1/4}\sum_{n=1}^{\infty}a_n\left[-a_n^2x^2+\frac{15}{4}a_nx-\frac{15}{8}\right.\\
&\left.+\frac{1}{4}\frac{x-1}{1+1}\left(a_nx-\frac{3}{2}\right)\right]\mathrm{e}^{-a_nx}\\
&=-2\left(x^{\delta}+x^{-\delta}\right)x^{1/4}\sum_{n=1}^{\infty}a_n\left[a_n\left(a_n-\frac{1}{8}\right)x^2\right.\\
&\left.-\left(\frac{29}{8}a_n-\frac{3}{16}\right)x+\frac{27}{16}\right]\mathrm{e}^{-a_nx}
\end{aligned}
\tag{24}
$$

due to the fact that for $x\ge 1$, $n\ge 1$ and $0\le\delta\le 1/4$ it holds $a_nx-3/2=\pi n^2x-3/2>0$ and $1\le x^{2\delta}\le x$.

Let

$$h(n,x)=a_n\left(a_n-\frac{1}{8}\right)x^2-\left(\frac{29}{8}a_n-\frac{3}{16}\right)x+\frac{27}{16},$$

to solve the equation $h(n,x)=0$ it yields two roots

$$x_{1,2}=\frac{\left[(29/8)a_n-3/16\right]\pm\sqrt{\Delta}}{2a_n\left(a_n-1/8\right)},$$

in which x_2 stands for the bigger one. Here

$$
\begin{aligned}
\Delta&=\left(\frac{29}{8}a_n-\frac{3}{16}\right)^2-4a_n\left(a_n-\frac{1}{8}\right)\cdot\frac{27}{16}\\
&=\frac{1}{64}\left[a_n\left(409a_n-33\right)+\frac{9}{4}\right]>0,
\end{aligned}
$$

due to the fact that $409a_n-33=409\pi n^2-33>0$ for $n\ge 1$.

For $n=1,2,3,\cdots$, the values of x_2 are about $1.0047, 0.2464, 0.1091,\cdots$. In view of $x\ge 1$, we have $h(n,x)>0$ for $n\ge 2$. In case $n=1$, the inequality $h(1,x)>0$ also holds for $x>x_2\approx 1.0047$. On $[1,x_2]$, $h(1,x)\le 0$ and to ensure $F_x(\sigma,x)<0$ it suffices to consider the variation of the summation on this interval. In fact, on this interval,

$$
\begin{aligned}
&\sum_{n=1}^{\infty}a_n\left[a_n\left(a_n-\frac{1}{8}\right)x^2-\left(\frac{29}{8}a_n-\frac{3}{16}\right)x+\frac{27}{16}\right]\mathrm{e}^{-a_nx}\\
&>\sum_{n=1}^{3}a_n\left[a_n\left(a_n-\frac{1}{8}\right)x^2-\left(\frac{29}{8}a_n-\frac{3}{16}\right)x+\frac{27}{16}\right]\mathrm{e}^{-a_nx}
\end{aligned}
$$

whose value increases from 8.1862×10^{-17} to 0.0040. Hence, for all $\sigma\in[1/2,1]$, $F_x(\sigma,x)<0$ always holds and $F(\sigma,x)$ is monotone decreasing about x on $[1,\infty)$.

3.2. To Prove $I_1^2(\sigma,t)+I_2^2(\sigma,t)>0$ and Estimate Its Upper Bound

In order to combine the two terms $I_1(\sigma,t)\cos\theta(t)+I_2(\sigma,t)\sin\theta(t)$ into a single periodic function, it requires $\sqrt{I_1^2(\sigma,t)+I_2^2(\sigma,t)}>0$, that is $I_1^2(\sigma,t)+I_2^2(\sigma,t)>0$. To split z into $z=z^*+\theta(t)$, then

$$x(z,t)=\left(\frac{2\pi e}{t}\right)^{1/2} e^{-\pi/3t} e^{2z/t} = \left(\frac{2\pi e}{t}\right)^{1/2} e^{-\pi/3t} e^{2\left(z^*+\theta(t)\right)/t} = e^{2z^*/t}. \tag{25}$$

It follows from Equation (21) and Equation (22) that

$$\begin{aligned} & I_1^2(\sigma,t)+I_2^2(\sigma,t) \\ &= \int_{\theta(t)}^{\infty} F(\sigma,x(z,t))\cos z\mathrm{d}z \cdot \int_{\theta(t)}^{\infty} F(\sigma,x(w,t))\cos w\mathrm{d}w \\ &\quad + \int_{\theta(t)}^{\infty} F(\sigma,x(z,t))\sin z\mathrm{d}z \cdot \int_{\theta(t)}^{\infty} F(\sigma,x(w,t))\sin w\mathrm{d}w \\ &= \int_{\theta(t)}^{\infty}\int_{\theta(t)}^{\infty} F(\sigma,x(z,t))F(\sigma,x(w,t))\cos(z-w)\mathrm{d}w\mathrm{d}z \\ &= \int_0^{\infty}\int_0^{\infty} F(\sigma,x(z^*+\theta(t),t))F(\sigma,x(w^*+\theta(t),t))\cos(z^*-w^*)\mathrm{d}w^*\mathrm{d}z^* \\ &= \int_0^{\infty}\int_0^{\infty} F\left(\sigma,e^{2z^*/t}\right)F\left(\sigma,e^{2w^*/t}\right)\cos(z^*-w^*)\mathrm{d}w^*\mathrm{d}z^* \\ &= \int_0^{\infty} F\left(\sigma,e^{2z^*/t}\right)\cos(z^*)\mathrm{d}z^* \cdot \int_0^{\infty} F\left(\sigma,e^{2w^*/t}\right)\cos(w^*)\mathrm{d}w^* \\ &\quad + \int_0^{\infty} F\left(\sigma,e^{2z^*/t}\right)\sin(z^*)\mathrm{d}z^* \cdot \int_0^{\infty} F\left(\sigma,e^{2w^*/t}\right)\sin(w^*)\mathrm{d}w^* \\ &= \left[\int_0^{\infty} F\left(\sigma,e^{2z/t}\right)\cos(z)\mathrm{d}z\right]^2 + \left[\int_0^{\infty} F\left(\sigma,e^{2z/t}\right)\sin(z)\mathrm{d}z\right]^2. \end{aligned} \tag{26}$$

For a fixed $t(>0)$, the variations of $F\left(\sigma,e^{2z/t}\right), F\left(\sigma,e^{2z/t}\right)\cos(z)$ and $F\left(\sigma,e^{2z/t}\right)\sin(z)$ for the case with $\sigma=1/2$ are simulated in **Figure 4**.

Notice that $F_x(\sigma,x)<0$ for $x\geq 1$ we have

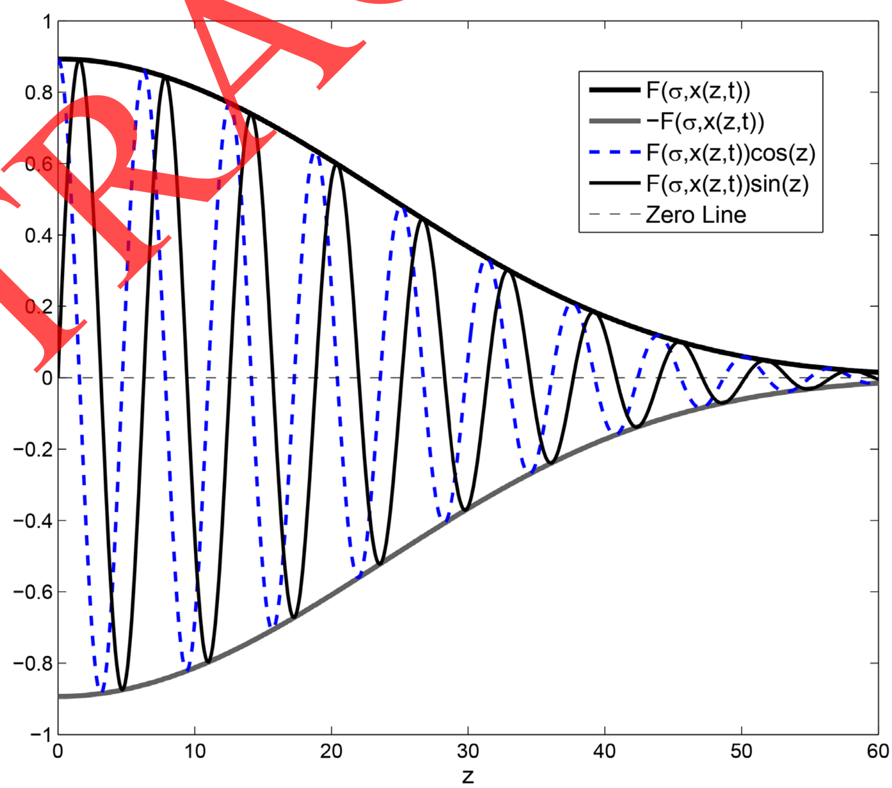

Figure 4. The variations of $F\left(\sigma,e^{2z/t}\right), F\left(\sigma,e^{2z/t}\right)\cos(z)$ and $F\left(\sigma,e^{2z/t}\right)\sin(z)$ along with the increasing of *z*, for the case with $\sigma=1/2$ and $t=100$.

$$\frac{\partial}{\partial z}F\left(\sigma,x\left(z,t\right)\right)=F_x\left(\sigma,x\right)\frac{\partial x}{\partial z}=F_x\left(\sigma,x\right)\frac{2}{t}\mathrm{e}^{2z/t}<0$$

for the case with $x=\mathrm{e}^{2z/t}$. Hence, for fixed $t>0$, $F\left(\sigma,\mathrm{e}^{2z/t}\right)$ decreases along with the increasing of z (see **Figure 4**). In view of this,

$$\begin{aligned}
&\int_0^{\infty}F\left(\sigma,\mathrm{e}^{2z/t}\right)\sin\left(z\right)\mathrm{d}z\\
&=\sum_{k=0}^{\infty}\int_{2k\pi}^{2(k+1)\pi}F\left(\sigma,\mathrm{e}^{2z/t}\right)\sin\left(z\right)\mathrm{d}z\\
&=\sum_{k=0}^{\infty}\int_0^{2\pi}F\left(\sigma,\mathrm{e}^{2\left(z^*+2k\pi\right)/t}\right)\sin\left(z^*\right)\mathrm{d}z^*\\
&=\sum_{k=0}^{\infty}\int_0^{\pi}\left[F\left(\sigma,\mathrm{e}^{2\left(z^*+2k\pi\right)/t}\right)-F\left(\sigma,\mathrm{e}^{2\left(z^*+2k\pi+\pi\right)/t}\right)\right]\sin\left(z^*\right)\mathrm{d}z^*\\
&=\sum_{k=0}^{\infty}\left[F\left(\sigma,\mathrm{e}^{2(z'_k+2k\pi)/t}\right)-F\left(\sigma,\mathrm{e}^{2(z'_k+2k\pi+\pi)/t}\right)\right]\int_0^{\pi}\sin\left(z^*\right)\mathrm{d}z^*\\
&=2\sum_{k=0}^{\infty}\left[F\left(\sigma,\mathrm{e}^{2(z'_k+2k\pi)/t}\right)-F\left(\sigma,\mathrm{e}^{2(z'_k+2k\pi+\pi)/t}\right)\right]>0
\end{aligned}\tag{27}$$

with $0\le z'_k\le\pi$, here the mean-value theorem of integrals is used.

To combine Equation (26) and Equation (27) it leads to $I_1^2\left(\sigma,t\right)+I_2^2\left(\sigma,t\right)>0$. The assertion is proved. In addition, $I_1^2\left(\sigma,t\right)+I_2^2\left(\sigma,t\right)$ is also upper bounded.

In fact, on the one hand, it follows from Equation (27) that

$$\begin{aligned}
&\int_0^{\infty}F\left(\sigma,\mathrm{e}^{2z/t}\right)\sin\left(z\right)\mathrm{d}z\\
&=2F\left(\sigma,\mathrm{e}^{2z_0/t}\right)-2\sum_{k=1}^{\infty}\left[F\left(\sigma,\mathrm{e}^{2(z'_k+2k\pi-\pi)/t}\right)-F\left(\sigma,\mathrm{e}^{2(z'_k+2k\pi)/t}\right)\right]\\
&<2F\left(\sigma,\mathrm{e}^{2z_0/t}\right)\le 2F\left(\sigma,1\right)=2F_0\left(1\right)
\end{aligned}\tag{28}$$

with

$$F_0\left(1\right)=4\sum_{n=1}^{\infty}a_n\left(a_n-\frac{3}{2}\right)\mathrm{e}^{-a_n},\tag{29}$$

which does not rely on σ [see the expression in Equation (23)]. On the other hand,

$$\begin{aligned}
&\int_0^{\infty}F\left(\sigma,\mathrm{e}^{2z/t}\right)\cos\left(z\right)\mathrm{d}z\\
&=\int_0^{\pi/2}F\left(\sigma,\mathrm{e}^{2z/t}\right)\cos\left(z\right)\mathrm{d}z+\sum_{k=0}^{\infty}\int_{(4k+1)\pi/2}^{(4k+5)\pi/2}F\left(\sigma,\mathrm{e}^{2z/t}\right)\cos\left(z\right)\mathrm{d}z\\
&=F\left(\sigma,\mathrm{e}^{2z_0/t}\right)-\sum_{k=0}^{\infty}\int_0^{2\pi}F\left(\sigma,\mathrm{e}^{\left[2z^*+(4k+1)\pi\right]/t}\right)\sin\left(z^*\right)\mathrm{d}z^*\\
&=F\left(\sigma,\mathrm{e}^{2z_0/t}\right)-\sum_{k=0}^{\infty}\int_0^{\pi}\left[F\left(\sigma,\mathrm{e}^{\left[2z^*+(4k+1)\pi\right]/t}\right)\right.\\
&\qquad\left.-F\left(\sigma,\mathrm{e}^{\left[2z^*+(4k+3)\pi\right]/t}\right)\right]\sin\left(z^*\right)\mathrm{d}z^*\\
&=F\left(\sigma,\mathrm{e}^{2z_0/t}\right)-2\sum_{k=0}^{\infty}\left[F\left(\sigma,\mathrm{e}^{\left[2z'_k+(4k+1)\pi\right]/t}\right)-F\left(\sigma,\mathrm{e}^{\left[2z'_k+(4k+3)\pi\right]/t}\right)\right]
\end{aligned}\tag{30}$$

with $0 \le z_0 \le \pi/2$ and $0 \le z_k' \le \pi$. Its upper bound is $F\left(\sigma, e^{2z_0/t}\right) \le F_0(1)$, and lower bound is

$F\left(\sigma, e^{2z_0/t}\right) - 2F\left(\sigma, e^{(2z_0'+\pi)/t}\right) \ge F\left(\sigma, e^{\pi/t}\right) - 2F\left(\sigma, e^{\pi/t}\right) = -F\left(\sigma, e^{\pi/t}\right) > -F_0(1)$.

Based on the above estimations, we have the final result:

$$0 < \sqrt{I_1^2(\sigma,t) + I_2^2(\sigma,t)} < \sqrt{F_0^2(1) + \left[2F_0(1)\right]^2} = \sqrt{5}F_0(1). \tag{31}$$

3.3. Detailed Arguments on the Periodicity of $U(\sigma,t)$

Since $I_1^2(\sigma,t) + I_2^2(\sigma,t)$, the real part of $\xi(\sigma+it)$ can be combined into a single periodic one:

$$\begin{aligned} U(\sigma,t) &= \frac{2}{t}\left[I_1(\sigma,t)\cos\theta(t) + I_2(\sigma,t)\sin\theta(t)\right] \\ &= \frac{2}{t}A(\sigma,t)\left[\sin\phi(\sigma,t)\cos\theta(t) + \cos\phi(\sigma,t)\sin\theta(t)\right] \\ &= \frac{2}{t}A(\sigma,t)\sin\left[\phi(\sigma,t) + \theta(t)\right], \end{aligned} \tag{32}$$

in which $A(\sigma,t) = \sqrt{I_1^2(\sigma,t) + I_2^2(\sigma,t)}$, $\theta(t) = (t/4)\log(t/2\pi e) + \pi/6$ and

$$\sin\phi(\sigma,t) = \frac{I_1(\sigma,t)}{A(\sigma,t)}, \quad \cos\phi(\sigma,t) = \frac{I_2(\sigma,t)}{A(\sigma,t)}. \tag{33}$$

Notice that $A(\sigma,t) > 0$, their signs are determined by $I_1(\sigma,t)$ and $I_2(\sigma,t)$, respectively. In the following we take the first one as the research object.

Firstly, we make certain the variation of $\theta(t) = (t/4)\log(t/2\pi e) + \pi/6$. As the function $(t/4)\log(t/2\pi e)$ concerned, in case $0 < t < 2\pi e$ it is negative; in case $t > 2\pi e$ it is positive. To take the derivative of $\theta(t)$, it leads to a unique extreme point at $t = 2\pi$. Notice that the first zero point of $U(1/2,t) = \xi(1/2+it)$ is at about $t = 14.1347251$, we only consider the case with $t \in [2\pi, \infty)$ on which $\theta(t)$ is increasing with value ≥-π/3.

There is a fact that, along with the increasing of $\theta(t)$, for every given $\sigma \in [1/2, 1]$, the function $I_1(\sigma,t)$ oscillates in a periodic manner about *t*. Explicitly, for the case with $\theta(\tilde{t}_m) = 2m\pi + \pi/2$ ($m = 0,1,2,\cdots$), it follows from Equation (21), Equation (26) and Equation (27) that

$$\begin{aligned} I_1(\sigma,\tilde{t}_m) &= \int_{\theta(\tilde{t}_m)}^{\infty} F\left(\sigma, x(z,\tilde{t}_m)\right)\cos z\mathrm{d}z \\ &= \int_0^{\infty} F\left(\sigma, e^{2z^*/\tilde{t}_m}\right)\cos\left[z^* + \theta(\tilde{t}_m)\right]\mathrm{d}z^* \\ &= \int_0^{\infty} F\left(\sigma, e^{2z^*/\tilde{t}_m}\right)\cos\left(z^* + 2m\pi + \pi/2\right)\mathrm{d}z^* \\ &= -\int_0^{\infty}\left(\sigma, e^{2z^*/\tilde{t}_m}\right)\sin z^*\mathrm{d}z^* < 0. \end{aligned} \tag{34}$$

Similarly, we have $I_1(\sigma,\hat{t}_m) > 0$ for the case with $\theta(\hat{t}_m) = \theta(\tilde{t}_m) + \pi = 2m\pi + 3\pi/2$ ($m = 0,1,2,\cdots$).

The above results accord with the fact that $\sin\phi(\sigma,t)$ shifts its sign when $\phi(\sigma,t)$ is added by π. This consistency implies that $\theta(t)$ contributes to the variation of $\sin\phi(\sigma,t)$, and is possibly the main part of $\phi(\sigma,t)$. In the following we verify this by considering the detailed variation of $I_1(\sigma,t)$ on the interval $[\tilde{t}_m,\hat{t}_m]$ (together with another half period $[\hat{t}_m,\tilde{t}_{m+1}]$).

For any given $t_0(\geq 2\pi)$, in case $t>t_0$ we have

$$\begin{aligned}
&I_1(\sigma,t)-I_1(\sigma,t_0)\\
&=\int_{\theta(t)}^{\infty}F(\sigma,x(z,t))\cos zdz-\int_{\theta(t_0)}^{\infty}F(\sigma,x(z,t_0))\cos zdz\\
&=-\int_{\theta(t_0)}^{\theta(t)}F(\sigma,x(z,t_0))\cos zdz+\int_{\theta(t)}^{\infty}\left[F(\sigma,x(z,t))-F(\sigma,x(z,t_0))\right]\cos zdz.
\end{aligned}\tag{35}$$

Relative to $\theta(t_0)=2m\pi+\pi/2$ (for a given m), to verify that $\theta(t)$ is the main part of $\phi(\sigma,t)$, it suffices to show that the monotone increasing term $-\int_{\theta(t_0)}^{\theta(t)}F(\sigma,x(z,t_0))\cos zdz$ dominates the relative change of $I_1(\sigma,t)$ with respect to $\theta(t)\in[2m\pi+\pi/2,2m\pi+3\pi/2]$ [that is, for $t\in(\tilde{t}_m,\hat{t}_m)$]. In other word, the contribution from the additional term is relatively small. In the following we estimate them in detail.

For the first term, in view of the monotone decreasing of $F(\sigma,x(z,t))$ about *z*, with a denotation $\Delta\theta=\theta(t)-\theta(t_0)$ and a substitution $z=z^*+\theta(t_0)$, we have

$$\begin{aligned}
&-\int_{\theta(t_0)}^{\theta(t)}F(\sigma,x(z,t_0))\cos zdz\\
&=-\int_0^{\Delta\theta}F\left(\sigma,e^{2z^*/t_0}\right)\cos\left(z^*+2m\pi+\pi/2\right)dz^*\\
&=\int_0^{\Delta\theta}F\left(\sigma,e^{2z^*/t_0}\right)\sin z^*dz^*.
\end{aligned}\tag{36}$$

3.3.1. To Estimate the Additional Term

The second term is an additional one which is very complex. We estimate it step by step. In the neighbourhood of t_0, to make the Taylor expansion of $F(\sigma,x(z,t))$ it leads to

$$F(\sigma,x(z,t))=F(\sigma,x(z,t_0))+F_t(\sigma,x(z,t_0))\Delta t+\cdots,$$

where $\Delta t=t-t_0$, the subscript "*t*" means the taking of partial derivative with respect to *t*.

Recall that $x(z,t)=(2\pi e/t)^{1/2}e^{-\pi/3t}e^{2z/t}$, to replace *z* by $z^*+\theta(t)=z^*+(t/4)\log(t/2\pi e)+\pi/6$, it leads to

$$\begin{aligned}
\left.\frac{\partial x}{\partial t}\right|_{t=t_0}&=-\frac{1}{t}\left[\frac{1}{2}+\frac{2(z-\pi/6)}{t}\right]\left(\frac{2\pi e}{t}\right)^{1/2}e^{-\pi/3t}e^{2z/t}\Big|_{t=t_0}\\
&=-\frac{1}{t}\left[\frac{1}{2}+\frac{2(z^*+\theta(t)-\pi/6)}{t}\right]\left(\frac{2\pi e}{t}\right)^{1/2}e^{-\pi/3t}e^{2(z^*+\theta(t))/t}\Big|_{t=t_0}\\
&=-\frac{1}{t_0}\left(\frac{2z^*}{t_0}+\frac{1}{2}\log\frac{t_0}{2\pi}\right)e^{2z^*/t_0}.
\end{aligned}\tag{37}$$

Furthermore, notice that $x(z,t_0)=x(z^*+\theta(t_0),t_0)=e^{2z^*/t_0}$, to recall the expression of $F_x(\sigma,x)$ in Equation (24), we get

$$
\begin{aligned}
&F_t\left(\sigma, x\left(z, t_0\right)\right) \\
&= F_x\left(\sigma, x\right)\Big|_{z=z^*+\theta(t_0)} \cdot \frac{\partial x}{\partial t}\Big|_{t=t_0} \\
&= \sum_{n=1}^{\infty} a_n \left[-a_n^2 x^2 + \frac{15}{4} a_n x - \frac{15}{8} + \delta \frac{x^{2\delta}-1}{x^{2\delta}+1}\left(a_n x - \frac{3}{2}\right)\right] e^{-a_n x} \\
&\quad \cdot 2\left(x^{\delta} + x^{-\delta}\right) x^{1/4} \cdot \left[-\frac{1}{t_0}\left(\frac{2z^*}{t_0} + \frac{1}{2}\log\frac{t_0}{2\pi}\right) x \right] \\
&= \frac{2}{t_0}\left(\log x + \frac{1}{2}\log\frac{t_0}{2\pi}\right)\left(x^{\delta} + x^{-\delta}\right) x^{5/4} \\
&\quad \cdot \sum_{n=1}^{\infty} a_n \left[a_n^2 x^2 - \frac{15}{4} a_n x + \frac{15}{8} - \delta \frac{x^{2\delta}-1}{x^{2\delta}+1}\left(a_n x - \frac{3}{2}\right)\right] e^{-a_n x} \\
&=: \frac{1}{t_0} G_1\left(\sigma, x\right) + \frac{1}{2t_0}\log\frac{t_0}{2\pi} G_2\left(\sigma, x\right)
\end{aligned}
\tag{38}
$$

with $x\left(z^*, t_0\right) = e^{2z^*/t_0}$, $a_n = \pi n^2$ and $\delta = \left(\sigma - 1/2\right)/2$.

It follows from the proofs in *Section* 3.1 that $G_1\left(\sigma, e^{2z^*/t_0}\right) > 0$ and $G_2\left(\sigma, e^{2z^*/t_0}\right) > 0$ for $z^* \ge 0$, $t_0 \ge 2\pi$ and $1/2 \le \sigma \le 1$. With respect to the single variable $w = 2z^*/t_0$, each of them has a unique extreme point (maximum one), say w_1 and w_2 respectively, which can be estimated by their dominate terms with $n = 1$. Due to the existence of strong decaying factor $e^{-a_n e^w}$, the effect from σ is very small, see **Figure 5**. In the following we mainly consider the case with $\sigma = 1/2$.

The approximate values for the maximum extreme points of $G_1\left(1/2, e^w\right)$ and

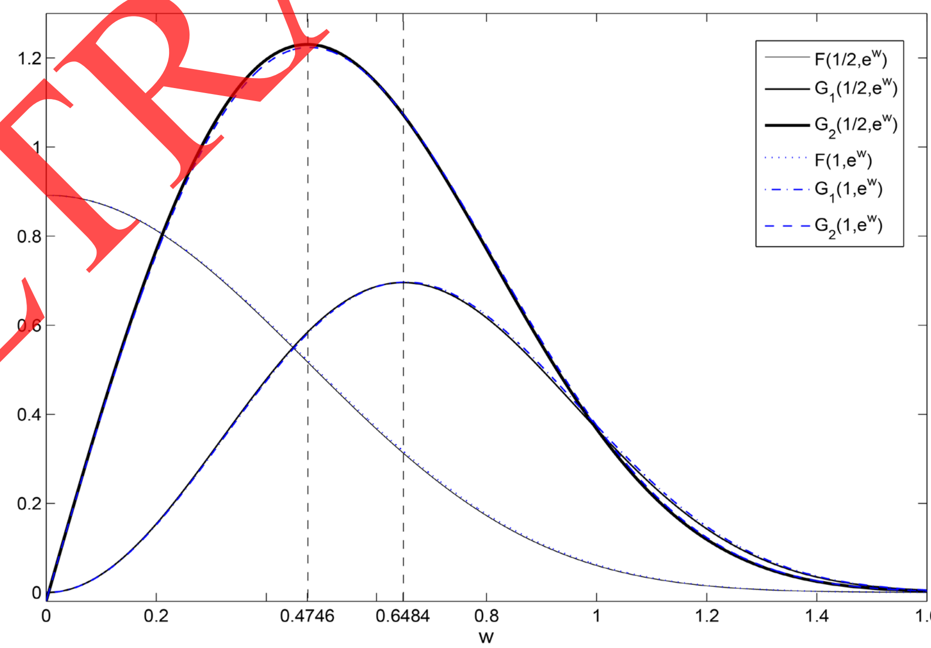

Figure 5. The changes of $G_1\left(\sigma, e^w\right)$ (only the first term), $G_2\left(\sigma, e^w\right)$ (only the first term) and $F\left(\sigma, e^w\right)$ (only the first 3 terms) along with the increasing of *w*, for the cases with $\sigma = 1/2$ and $\sigma = 1$.

$G_2\left(1/2, e^w\right)$ are at about $(0.6484, 0.6957)$ and $(0.4746, 1.2304)$, respectively. Explicitly, $w_1 \approx 0.6484$ and $w_2 \approx 0.4746$ satisfy separately the equations $\partial G_1\left(1/2, e^w\right)/\partial w = 0$ and $\partial G_2\left(1/2, e^w\right)/\partial w = 0$ (for each of them only the first term is concerned), which read:

$$\left(y_1^3 - 7y_1^2 + \frac{165}{16}y_1 - \frac{75}{32}\right)\log\left(\frac{y_1}{\pi}\right) = y_1^2 - \frac{15}{4}y_1 + \frac{15}{8},$$

$$y_2^3 - 7y_2^2 + \frac{165}{16}y_2 - \frac{75}{32} = 0$$

with $y_1 = \pi e^{w_1}$ and $y_2 = \pi e^{w_2}$. By the way, the maximum extreme point of $F\left(1/2, e^w\right)$ is at about $(0, 0.8934)$ (to ensure the monotonicity, the first 3 terms are considered), which will be used in the related estimation.

When σ varies in $[1/2, 1]$, the locations of the maximum extreme points of $G_1\left(\sigma, e^w\right)$ and $G_2\left(\sigma, e^w\right)$ have small shifts. Explicitly, when $\sigma = 1/2$ the location of $G_1\left(\sigma, e^w\right)$ is at about $(0.6484, 0.6957)$, and along with the increasing of σ its value increases a little. At the right end $\sigma = 1$, the location is shifted to about $(0.6538, 0.6967)$. Similarly, that of $G_2\left(\sigma, e^w\right)$ is shifted from about $(0.4746, 1.2304)$ to about $(0.4820, 1.2237)$. These indicate that the effect of σ to the maximum values of G_1 and G_2 can be neglected, yet the effects of σ to the horizontal shifting can not be neglected, since $z^* = w_1 t_0/2$ and $z^* = w_2 t_0/2$ may magnify the small differences along with the increasing of t_0 and result in considerable impacts to the corresponding integrals. By the way, we note that w_1 and w_2 depend on the variable σ, and should be in the forms $w_1(\sigma)$ and $w_2(\sigma)$, to neglect the variable is just for simplicity.

Based on the above analysis, we estimate the second term of Equation (35) in the following:

$$\begin{aligned}
&\int_{\theta(t)}^{\infty}\left[F\left(\sigma, x(z,t)\right) - F\left(\sigma, x(z,t_0)\right)\right]\cos z\mathrm{d}z \\
&\approx \int_{\theta(t)}^{\infty} F_t\left(\sigma, x(z,t_0)\right)\Delta t\cos z\mathrm{d}z \\
&= \Delta t\int_{\Delta\theta}^{\infty} F_t\left(\sigma, x\left(z^* + \theta(t_0), t_0\right)\right)\cos\left(z^* + \theta(t_0)\right)\mathrm{d}z^* \\
&= \Delta t\int_{\Delta\theta}^{\infty}\left[\frac{1}{t_0}G_1\left(\sigma, e^{2z^*/t_0}\right) + \frac{1}{2t_0}\log\frac{t_0}{2\pi}G_2\left(\sigma, e^{2z^*/t_0}\right)\right]\cos\left(z^* + 2m\pi + \pi/2\right)\mathrm{d}z^* \\
&= -\frac{\Delta t}{t_0}\int_{\Delta\theta}^{\infty} G_1\left(\sigma, e^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* - \frac{\Delta t}{2t_0}\log\frac{t_0}{2\pi}\int_{\Delta\theta}^{\infty} G_2\left(\sigma, e^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*.
\end{aligned} \tag{39}$$

In the following we give some illustrations for the case $\sigma = 1/2$. In view of **Figure 5**, $G_1\left(1/2, e^{2z^*/t_0}\right)$ has maximum value at $2z^*/t_0 = w_1$, that is, at about $z^* = t_0 w_1/2 \approx 0.3242 t_0$. Similarly, $G_2\left(1/2, e^{2z^*/t_0}\right)$ has maximum value at about $z^* = t_0 w_2/2 \approx 0.2373 t_0$. For example, for the cases with $m = 0, 1$ and 2, to solve

$$\theta(t_0) = \frac{t_0}{4}\log\left(\frac{t_0}{2\pi e}\right) + \frac{\pi}{6} = 2m\pi + \pi/2,$$

it yields $t_0 = 20.8747, 37.4043$ and 50.3595, respectively. The corresponding maximum extreme points of $G_1\left(1/2, e^{2z^*/t_0}\right)$ are shifted to $(6.7676, 0.6957)$,

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$(12.1265, 0.6957)$ and $(16.3265, 0.6957)$, respectively. Similarly, those for $G_2\left(1/2, e^{2z^*/t_0}\right)$ are shifted to $(4.9536, 1.2304)$, $(8.8760, 1.2304)$ and $(11.9503, 1.2304)$, respectively (see **Figure 6**). What should be mentioned is that, in the neighborhoods of these two maximum extreme points, $G_1\left(1/2, e^{2z^*/t_0}\right)\sin z^*$ and $G_2\left(1/2, e^{2z^*/t_0}\right)\sin z^*$ may possess their peak absolute values separately, which dominate the integrals. Yet, since $t_0 w_1/2 \approx 0.3242 t_0$ and $t_0 w_2/2 \approx 0.2373 t_0$, the changes of $G_1\left(1/2, e^{2z^*/t_0}\right)$ and $G_2\left(1/2, e^{2z^*/t_0}\right)$ are not synchronous with that of $\theta(t)$. This adds complexity to the additional term.

Notice that $w = w_1$ is the unique extreme point of $G_1\left(\sigma, e^{w}\right)$ at which it possesses the maximum value, we get an increasing interval $[0, t_0 w_1/2]$ and a decreasing interval $[t_0 w_1/2, \infty)$ for $G_1\left(\sigma, e^{2z^*/t_0}\right)$ with respect to the variable z^*. For $m \geq 0$, there must be an integer $k_0 \geq 1$ such that $z^* = t_0 w_1/2 \in \left[k_0\pi, (k_0+1)\pi\right)$ (see **Figure 6**). Furthermore, in case $k_0 = 2j_0$ with $j_0 \geq 1$, we have

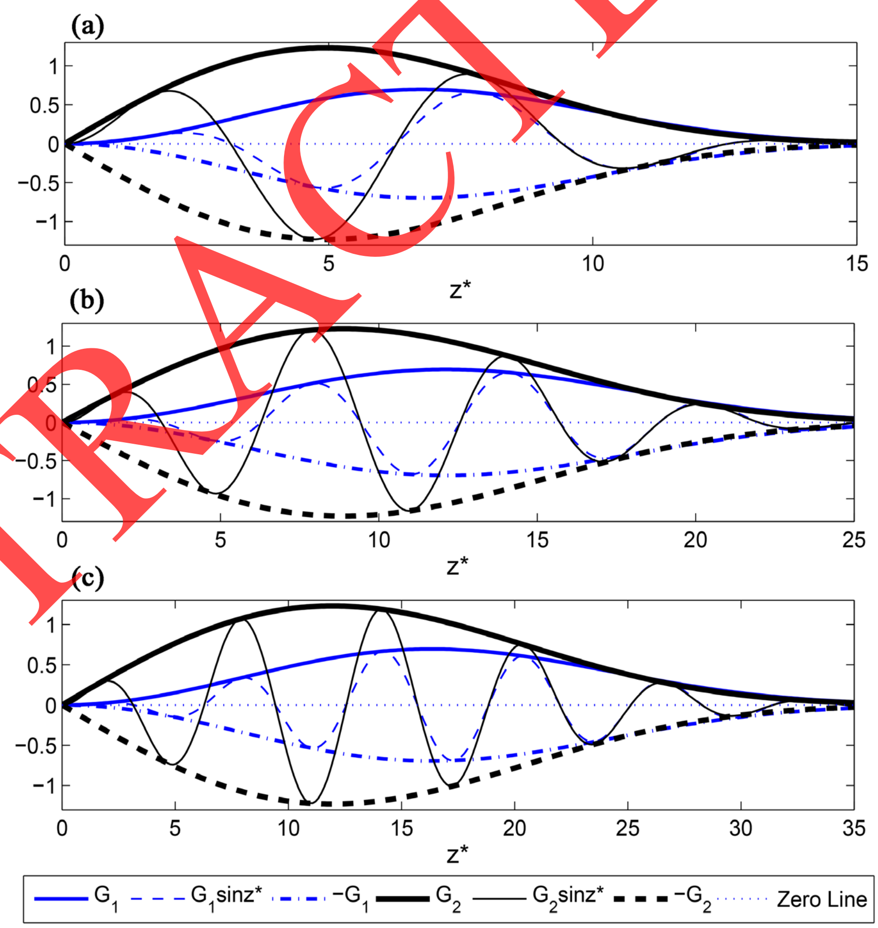

Figure 6. The changes of $G_1\left(1/2, e^{2z^*/t_0}\right)$, $G_2\left(1/2, e^{2z^*/t_0}\right)$, $G_1\left(1/2, e^{2z^*/t_0}\right)\sin z^*$ and $G_2\left(1/2, e^{2z^*/t_0}\right)\sin z^*$ along with the increasing of z^* for the cases with a: $m = 0$; b: $m = 1$ and c: $m = 2$ with respect to $\theta(t_0) = 2m\pi + \pi/2$ (that is, for $t_0 = 20.8747, 37.4043$ and 50.3595 separately).

$$
\begin{aligned}
&\int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* \\
&= -\int_0^{\Delta\theta} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* + \sum_{k=0}^{\infty}(-1)^k \int_0^{\pi} G_1\left(\sigma, \mathrm{e}^{2(z+k\pi)/t_0}\right)\sin z\mathrm{d}z \\
&= -\int_0^{\Delta\theta} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* + \int_0^{\pi} G_1\left(\sigma, \mathrm{e}^{2(z+2j_0\pi)/t_0}\right)\sin z\mathrm{d}z \\
&\quad -\sum_{j=0}^{j_0-1}\int_0^{\pi}\left[G_1\left(\sigma, \mathrm{e}^{2(z+(2j+1)\pi)/t_0}\right) - G_1\left(\sigma, e^{2(z+2j\pi)/t_0}\right)\right]\sin z\mathrm{d}z \\
&\quad -\sum_{j=j_0+1}^{\infty}\int_0^{\pi}\left[G_1\left(\sigma, \mathrm{e}^{2(z+(2j-1)\pi)/t_0}\right) - G_1\left(\sigma, \mathrm{e}^{2(z+2j\pi)/t_0}\right)\right]\sin z\mathrm{d}z \\
&< \int_0^{\pi} G_1\left(\sigma, \mathrm{e}^{2(z+2j_0\pi)/t_0}\right)\sin z\mathrm{d}z < 2G_1\left(\sigma, \mathrm{e}^{w_1}\right) \le 2G_1\left(1, \mathrm{e}^{w_1}\right).
\end{aligned}
\tag{40}
$$

Here the approximation $0.6957 \approx G_1\left(1/2, \mathrm{e}^{w_1}\right) \le G_1\left(\sigma, \mathrm{e}^{w_1}\right) \le G_1\left(1, \mathrm{e}^{w_1}\right) \approx 0.6967$ is used.

Similarly, in case $k_0 = 2j_0 - 1$ with $j_0 \ge 1$, we have

$$
\begin{aligned}
&\int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* \\
&= \int_{\Delta\theta}^{\pi} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* + \sum_{k=1}^{\infty}(-1)^k \int_0^{\pi} G_1\left(\sigma, e^{2(z+k\pi)/t_0}\right)\sin z\mathrm{d}z \\
&= \int_{\Delta\theta}^{\pi} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* - \int_0^{\pi} G_1\left(\sigma, \mathrm{e}^{2(z+(2j_0-1)\pi)/t_0}\right)\sin z\mathrm{d}z \\
&\quad +\sum_{j=1}^{j_0-1}\int_0^{\pi}\left[G_1\left(\sigma, \mathrm{e}^{2(z+2j\pi)/t_0}\right) - G_1\left(\sigma, \mathrm{e}^{2(z+(2j-1)\pi)/t_0}\right)\right]\sin z\mathrm{d}z \\
&\quad +\sum_{j=j_0}^{\infty}\int_0^{\pi}\left[G_1\left(\sigma, \mathrm{e}^{2(z+2j\pi)/t_0}\right) - G_1\left(\sigma, \mathrm{e}^{2(z+(2j+1)\pi)/t_0}\right)\right]\sin z\mathrm{d}z \\
&> -\int_0^{\pi} G_1\left(\sigma, \mathrm{e}^{2(z+(2j_0-1)\pi)/t_0}\right)\sin z\mathrm{d}z > -2G_1\left(\sigma, \mathrm{e}^{w_1}\right) \ge -2G_1\left(1, \mathrm{e}^{w_1}\right).
\end{aligned}
\tag{41}
$$

In a word, with respect to $\theta(t_0) = 2m\pi + \pi/2$, for all $m \ge 0$ it holds the estimation bellow:

$$
\left|\int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*\right| < 2G_1\left(1, \mathrm{e}^{w_1}\right) \approx 2\times 0.6967 = 1.3934. \tag{42}
$$

When $G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)$ is substituted by $G_2\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)$, with the same approach we get a similar estimation:

$$
\left|\int_{\Delta\theta}^{\infty} G_2\left(\sigma, \mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*\right| < 2G_2\left(1/2, \mathrm{e}^{w_2}\right) \approx 2\times 1.2304 = 2.4608, \tag{43}
$$

since for this case $1.2237 \approx G_2\left(1, \mathrm{e}^{w_2}\right) \le G_2\left(\sigma, \mathrm{e}^{w_2}\right) \le G_2\left(1/2, \mathrm{e}^{w_2}\right) \approx 1.2304$.

3.3.2. To Estimate the Change of $I_1(\sigma, t)$ with Respect to $\theta(t)$

With the help of the above results, in the following we estimate the relative change $I_1(\sigma,t) - I_1(\sigma,t_0)$ with respect to $\theta(t) \in (2m\pi + \pi/2, 2m\pi + 3\pi/2)$ ($m \ge 0$).

Relative to $\theta(t_0) = 2m\pi + \pi/2$, there is a Taylor expansion for $\theta(t)$:

$$
\Delta\theta = \theta(t) - \theta(t_0) = \theta'(t)\Delta t + \theta''(t)\frac{(\Delta t)^2}{2!} + \cdots = \frac{1}{4}\log\frac{t_0}{2\pi}\Delta t + \frac{1}{8t_0}(\Delta t)^2 + \cdots. \tag{44}
$$

Hence, $\Delta t \approx 4\Delta\theta / \log(t_0/2\pi)$. It follows from Equations (35), (36), (39), (42) and (43) that

$$
\begin{aligned}
& I_1(\sigma,t) - I_1(\sigma,t_0) \\
& = -\int_{\theta(t_0)}^{\theta(t)} F(\sigma, x(z,t_0)) \cos z \mathrm{d}z + \int_{\theta(t)}^{\infty} \left[F(\sigma, x(z,t)) - F(\sigma, x(z,t_0)) \right] \cos z \mathrm{d}z \\
& \approx \int_0^{\Delta\theta} F\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* - \frac{\Delta t}{t_0} \int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \\
& \quad - \frac{\Delta t}{2t_0} \log \frac{t_0}{2\pi} \int_{\Delta\theta}^{\infty} G_2\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \\
& \approx \int_0^{\Delta\theta} F\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* - \frac{4\Delta\theta}{t_0 \log(t_0/2\pi)} \int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \\
& \quad - \frac{2\Delta\theta}{t_0} \int_{\Delta\theta}^{\infty} G_2\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \\
& =: R(\sigma, \Delta\theta).
\end{aligned}
\tag{45}
$$

Furthermore, in view of $0 < \Delta\theta < \pi$, to take the partial derivative of $R(\sigma,\Delta\theta)$ with respect to $\Delta\theta$, it reads

$$
\begin{aligned}
\frac{\partial R}{\partial \Delta\theta} & = F\left(\sigma, \mathrm{e}^{2\Delta\theta/t_0}\right) \sin \Delta\theta - \frac{4}{t_0 \log(t_0/2\pi)} \int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \\
& \quad - \frac{2}{t_0} \int_{\Delta\theta}^{\infty} G_2\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \\
& \quad + \frac{2\Delta\theta}{t_0} \left[\frac{2}{\log(t_0/2\pi)} G_1\left(\sigma, \mathrm{e}^{2\Delta\theta/t_0}\right) + G_2\left(\sigma, \mathrm{e}^{2\Delta\theta/t_0}\right) \right] \sin \Delta\theta \\
& > F\left(\sigma, \mathrm{e}^{2\pi/t_0}\right) \sin \Delta\theta - \frac{4}{t_0 \log(t_0/2\pi)} \left| \int_{\Delta\theta}^{\infty} G_1\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \right| \\
& \quad - \frac{2}{t_0} \left| \int_{\Delta\theta}^{\infty} G_2\left(\sigma, \mathrm{e}^{2z^*/t_0}\right) \sin z^* \mathrm{d}z^* \right| \\
& > F\left(1/2, \mathrm{e}^{2\pi/t_0}\right) \sin \Delta\theta - \frac{4}{t_0} \left[\frac{2G_1\left(1, \mathrm{e}^{w_1}\right)}{\log(t_0/2\pi)} + G_2\left(1/2, \mathrm{e}^{w_2}\right) \right] \\
& \approx F\left(1/2, \mathrm{e}^{2\pi/t_0}\right) \Delta\theta - \frac{4}{t_0} \left[\frac{2G_1\left(1, \mathrm{e}^{w_1}\right)}{\log(t_0/2\pi)} + G_2\left(1/2, \mathrm{e}^{w_2}\right) \right] \\
& > F\left(1/2, \mathrm{e}^{2\pi/t_0}\right) \Delta\theta - \frac{8G_2\left(1/2, \mathrm{e}^{w_2}\right)}{t_0} > 0,
\end{aligned}
\tag{46}
$$

provided that

$$
\Delta\theta > \frac{8G_2\left(1/2, \mathrm{e}^{w_2}\right)}{t_0 F\left(1/2, \mathrm{e}^{2\pi/t_0}\right)}.
$$

Here the inequality $2G_1\left(1, \mathrm{e}^{w_1}\right) / \log(t_0/2\pi) < G_2\left(1/2, \mathrm{e}^{w_2}\right)$ is also used. In fact, to solve $\theta(t_0) = \pi/2$ it yields $t_0 \approx 20.8747$. To recall that $G_1\left(1, \mathrm{e}^{w_1}\right) \approx 0.6967$ and $G_2\left(1/2, \mathrm{e}^{w_2}\right) \approx 1.2304$, we get

$$
\frac{2G_1\left(1, \mathrm{e}^{w_1}\right)}{\log(t_0/2\pi) G_2\left(1/2, \mathrm{e}^{w_2}\right)} \approx 0.9432 < 1.
$$

Along with the increasing of m, that is t_0, the ratio gets smaller and smaller, and the inequality always holds.

For the case with $m = 0$ and $t_0 \approx 20.8747$, the approximate value of $F\left(1/2, e^{2\pi/t_0}\right)$ is about 0.7205. At this time,

$$\frac{8G_2\left(1/2, e^{w_2}\right)}{t_0 F\left(1/2, e^{2\pi/t_0}\right)} \approx \frac{8\times 1.2304}{20.8747\times 0.7205} \approx 0.2083\pi.$$

This indicates that, for the case with $\theta(t)\in(\pi/2, 3\pi/2)$, no matter the additional term

$$\int_{\theta(t)}^{\infty}\left[F\left(\sigma, x(z,t)\right) - F\left(\sigma, x(z,t_0)\right)\right]\cos z\mathrm{d}z$$

is positive or not, the increasing term $-\int_{\theta(t_0)}^{\theta(t)} F\left(\sigma, x(z,t_0)\right)\cos z\mathrm{d}z$ dominates the relative change of $I_1(\sigma,t)$ for all $\sigma\in[1/2,1]$, at least, with respect to $\Delta\theta$ on the interval $(0.2083\pi, \pi)$. In another word, $I_1(\sigma,t)$ increases along with the increasing of $\theta(t)$ on $(0.7083\pi, 3\pi/2)$ (ensured by $\partial R/\partial\Delta\theta > 0$). For the other cases with $m\ge 1$, the interval may become wider than this. To summarize the results, we get a theorem below:

Theorem 1. *For the case with*

$$\theta(t) = \frac{t}{4}\log\left(\frac{t}{2\pi e}\right) + \frac{\pi}{6} \in \left[2m\pi + \pi/2, 2m\pi + 3\pi/2\right)$$

(m is a non-negative integer), let $\theta(t_0) = 2m\pi + \pi/2$, *then for all* $\sigma\in[1/2,1]$, $I_1(\sigma,t)$ *increases along with the increasing of* $\theta(t)$, *at least on the interval defined by*

$$2m\pi + \pi/2 + \frac{8G_2\left(1/2, e^{w_2}\right)}{t_0 F\left(1/2, e^{2\pi/t_0}\right)} < \theta(t) < 2m\pi + 3\pi/2. \tag{47}$$

In the following, we begin with assuming $\theta(t_0) = 2m\pi - \pi/2$ and consider the case $\theta(t)\in(2m\pi - \pi/2, 2m\pi + \pi/2)$ with $m\ge 1$. To repeat the previous deduction process it results in

$$\begin{aligned} &I_1(\sigma,t) - I_1(\sigma,t_0) \\ &= -\int_{\theta(t_0)}^{\theta(t)} F\left(\sigma, x(z,t_0)\right)\cos z\mathrm{d}z + \int_{\theta(t)}^{\infty}\left[F\left(\sigma, x(z,t)\right) - F\left(\sigma, x(z,t_0)\right)\right]\cos z\mathrm{d}z \\ &\approx -\int_0^{\Delta\theta} F\left(\sigma, e^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* + \frac{4\Delta\theta}{t_0\log(t_0/2\pi)}\int_{\Delta\theta}^{\infty} G_1\left(\sigma, e^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^* \\ &\quad + \frac{2\Delta\theta}{t_0}\int_{\Delta\theta}^{\infty} G_2\left(\sigma, e^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*. \end{aligned} \tag{48}$$

Furthermore, to denote it by $R(\sigma,\Delta\theta)$, then we get $\partial R/\partial\Delta\theta < 0$ on condition that

$$\Delta\theta > \frac{8G_2\left(1/2, e^{w_2}\right)}{t_0 F\left(1/2, e^{2\pi/t_0}\right)}.$$

Hence, there is a similar result bellow:

Theorem 2. *For the case with*

$$\theta(t)=\frac{t}{4}\log\left(\frac{t}{2\pi e}\right)+\frac{\pi}{6}\in[2m\pi-\pi/2,2m\pi+\pi/2)$$

(m is a positive integer), let $\theta(t_0)=2m\pi-\pi/2$, *then for all* $\sigma\in[1/2,1]$, $I_1(\sigma,t)$ *decreases along with the decreasing of* $\theta(t)$, *at least on the interval defined by*

$$2m\pi-\pi/2+\frac{8G_2(1/2,e^{w_2})}{t_0F(1/2,e^{2\pi/t_0})}<\theta(t)<2m\pi+\pi/2.$$

These two theorems have clarified the periodic behaviors of $I_1(\sigma,t)$. It follows from Theorem 1 that, due to the existence of decay factor $1/t_0$ in Equation (47) ($1/F(1/2,e^{2\pi/t_0})$ also decays), the dominating interval of the increasing term $-\int_{\theta(t_0)}^{\theta(t)}F(\sigma,x(z,t_0))\cos zdz$ becomes wider and wider along with the increasing of t_0. In another word, the possible interval for $I_1(\sigma,t)<I_1(\sigma,t_0)$ becomes narrower and narrower. This implies that the minimum extreme point of $I_1(\sigma,t)$ get closer and closer to the left end of the interval $[2m\pi+\pi/2,2m\pi+3\pi/2)$ with respect to $\theta(t)$ as $m\to\infty$ ($m\geq 0$). Certainly, this only makes sense when the additional term is negative. In case it is positive, the thing becomes simple. The minimum value of $I_1(\sigma,t)$ is attained at the left boundary $t=t_0$ defined by $\theta(t_0)=2m\pi+\pi/2$. Furthermore, from Theorem 2 we see, in the left neighborhood of this boundary it is monotone decreasing, and the minimum extreme point is exactly at $t=t_0$ which accords with $\theta(t_0)=2m\pi+\pi/2$. Inversely, the maximum extreme point of $I_1(\sigma,t)$ get closer and closer to the left end of the interval $[2m\pi-\pi/2,2m\pi+\pi/2)$ with respect to $\theta(t)$ as $m\to\infty$. It only makes sense when the additional term is positive, and in case this term is negative the maximum extreme point is exactly at $t=t_0$ with $\theta(t_0)=2m\pi-\pi/2$ ($m\geq 1$).

Based on the above analysis we get a corollary below:

Corollary 1. *For the case with*

$$\theta(t)=\frac{t}{4}\log\left(\frac{t}{2\pi e}\right)+\frac{\pi}{6}\in[2m\pi,2(m+1)\pi]$$

(m is a non-negative integer), for every given $\sigma\in[1/2,1]$ *there is a minimum extreme point and a maximum extreme point of* $I_1(\sigma,t)$, *which lie separately in the intervals defined by*

$$2m\pi+\pi/2\leq\theta(t)\leq 2m\pi+\pi/2+\frac{8G_2(1/2,e^{w_2})}{\tilde{t}_mF(1/2,e^{2\pi/\tilde{t}_m})},$$

$$2m\pi+3\pi/2\leq\theta(t)\leq 2m\pi+3\pi/2+\frac{8G_2(1/2,e^{w_2})}{\hat{t}_mF(1/2,e^{2\pi/\hat{t}_m})}$$

with $\theta(\tilde{t}_m)=2m\pi+\pi/2$ *and* $\theta(\hat{t}_m)=2m\pi+3\pi/2$. *As* $m\to\infty$, *these two extreme points in the moving intervals tend separately to their left boundaries* $\tilde{t}_m$

and $\hat{t}_m$.

3.3.3. To Estimate the Change of $I_1(\sigma,t)/A(\sigma,t)$ with Respect to $\theta(t)$

According to the definition,

$$\sin\phi(\sigma,t)=\frac{I_1(\sigma,t)}{A(\sigma,t)}.$$

Hence, besides the estimation on the change of $I_1(\sigma,t)$, it also requires an estimation on that of $A(\sigma,t)$.

It follows from Equation (26) that $A(\sigma,t)=\sqrt{H^2(\sigma,t)+S^2(\sigma,t)}$ with

$$H(\sigma,t)=\int_0^\infty F\left(\sigma,e^{2z/t}\right)\cos(z)dz,$$

$$S(\sigma,t)=\int_0^\infty F\left(\sigma,e^{2z/t}\right)\sin(z)dz.$$

To take the partial derivative of $A(\sigma,t)$ about t, it leads to

$$A_t(\sigma,t)=\frac{H(\sigma,t)H_t(\sigma,t)+S(\sigma,t)S_t(\sigma,t)}{\sqrt{H^2(\sigma,t)+S^2(\sigma,t)}}. \tag{49}$$

Let $x(z,t)=e^{2z/t}$, then $\partial x/\partial t=-\left(2z/t^2\right)e^{2z/t}$. It follows from Equation (38) that

$$\begin{aligned}H_t(\sigma,t)&=\int_0^\infty F_t(\sigma,x(z,t))\cos(z)dz,\\&=\int_0^\infty F_x(\sigma,x)\frac{\partial x}{\partial t}\cos(z)dz,\\&=\frac{1}{t}\int_0^\infty G_1\left(\sigma,e^{2z/t}\right)\cos(z)dz.\end{aligned}$$

$$S_t(\sigma,t)=\frac{1}{t}\int_0^\infty G_1\left(\sigma,e^{2z/t}\right)\sin(z)dz.$$

To repeat the deduction processes in Equation (40) and Equation (41) we get $\left|H_t(\sigma,t)\right|\le 2G_1\left(1,e^{w_1}\right)/t$ and $\left|S_t(\sigma,t)\right|\le 2G_1\left(1,e^{w_1}\right)/t$.

For the reference point defined by $\theta(t_0)=2m\pi+\pi/2$ with $m\ge 0$, it holds $\Delta t\approx 4\Delta\theta/\log(t_0/2\pi)$ [see Equation (44)], and hence

$$\begin{aligned}|\Delta A|&=\left|A(\sigma,t)-A(\sigma,t_0)\right|\approx\left|A_t(\sigma,t_0)\right|\Delta t\\&\approx\frac{\left|H(\sigma,t_0)H_t(\sigma,t_0)+S(\sigma,t_0)S_t(\sigma,t_0)\right|}{\sqrt{H^2(\sigma,t_0)+S^2(\sigma,t_0)}}\cdot\frac{4\Delta\theta}{\log(t_0/2\pi)}\\&\le\frac{\left|H(\sigma,t_0)\right|\cdot\left|H_t(\sigma,t_0)\right|+\left|S(\sigma,t_0)\right|\cdot\left|S_t(\sigma,t_0)\right|}{\sqrt{H^2(\sigma,t_0)+S^2(\sigma,t_0)}}\cdot\frac{4\Delta\theta}{\log(t_0/2\pi)}\\&<\frac{2G_1\left(1,e^{w_1}\right)}{t_0}\frac{\left|H(\sigma,t_0)\right|+\left|S(\sigma,t_0)\right|}{\sqrt{H^2(\sigma,t_0)+S^2(\sigma,t_0)}}\cdot\frac{4\Delta\theta}{\log(t_0/2\pi)}\\&\le\frac{2G_1\left(1,e^{w_1}\right)}{t_0}\sqrt{2}\cdot\frac{4\Delta\theta}{\log(t_0/2\pi)}\approx\frac{8\sqrt{2}G_1\left(1,e^{w_1}\right)}{t_0\log(t_0/2\pi)}\Delta\theta.\end{aligned} \tag{50}$$

Due to the existence of the decay factor $1/t_0\log(t_0/2\pi)$, this estimation makes sense. For example, for the cases with $m = 1, 10$ and 100, the approximate values are about $0.1572\Delta\theta$, $0.0213\Delta\theta$ and $0.0025\Delta\theta$, respectively.

For an arbitrary given reference point $t_0 \in (2\pi, \infty)$ and $\sigma \in [1/2, 1]$ we have

$$\sin\phi(\sigma,t) = \frac{I_1(\sigma,t)}{A(\sigma,t)} = \frac{I_1(\sigma,t)}{A(\sigma,t_0)} \cdot \frac{A(\sigma,t_0)}{A(\sigma,t)}. \tag{51}$$

Since $A(\sigma,t_0) > 0$, the change characteristics of $I_1(\sigma,t)$ are maintained by $I_1(\sigma,t)/A(\sigma,t_0)$. The positive factor $A(\sigma,t_0)/A(\sigma,t)$ does not influence the sign change of $\sin\phi(\sigma,t)$. As the function $I_1(\sigma,t)/A(\sigma,t)$ concerned, for $\theta(t_0) = 2m\pi + \pi/2$, $I_1(\sigma,t_0)/A(\sigma,t_0) < 0$, and for $\theta(t_0) = 2m\pi + 3\pi/2$, $I_1(\sigma,t_0)/A(\sigma,t_0) > 0$. That is to say, the oscillation characteristic of $I_1(\sigma,t)$ is maintained by $I_1(\sigma,t)/A(\sigma,t)$. But the variation of the amplitude does make a difference, and a further estimation is needed.

Our task is to verify that the change of $I_1(\sigma,t)/A(\sigma,t)$ is mainly associated with that of $\theta(t)$, that is, the term $-\int_{\theta(t_0)}^{\theta(t)} F(\sigma, x(z,t_0))\cos z\,\mathrm{d}z$ also dominate. Since this term increases on $[2m\pi + \pi/2, 2m\pi + 3\pi/2]$ ($m \ge 0$) and decreases on $[2m\pi - \pi/2, 2m\pi + \pi/2]$ ($m \ge 1$) with respect to $\theta(t)$, in the following we still consider it in a separate way.

For a fixed σ, relative to $t = t_0$ defined by $\theta(t_0) = 2m\pi + \pi/2$, to denote $A(\sigma,t) = A(\sigma,t_0) + \Delta A$ and $I_1(\sigma,t) = I_1(\sigma,t_0) + \Delta I_1$, then

$$\begin{aligned} &\sin\phi(\sigma,t) - \sin\phi(\sigma,t_0) \\ &= \frac{I_1(\sigma,t)}{A(\sigma,t)} - \frac{I_1(\sigma,t_0)}{A(\sigma,t_0)} = \frac{A(\sigma,t_0)\Delta I_1 - I_1(\sigma,t_0)\Delta A}{[A(\sigma,t_0) + \Delta A]A(\sigma,t_0)}. \end{aligned} \tag{52}$$

Since the unchangeable factor $1/A(\sigma,t_0)$ does not impact the relative change of this increment, in the following arguments we omit it for simplicity.

Since the inequality $A(\sigma,t) \ge |I_1(\sigma,t)|$ holds for all $t > 2\pi$, there always be $A(\sigma,t_0) + \Delta A \ge |I_1(\sigma,t_0) + \Delta I_1|$, and it follows from Equation (45), Equation (46) and Equation (50) that

$$\begin{aligned} &\frac{\partial}{\partial\Delta\theta}\left[\frac{A(\sigma,t_0)\Delta I_1 - I_1(\sigma,t_0)\Delta A}{A(\sigma,t_0) + \Delta A}\right] \\ &= \frac{A}{A+\Delta A}\cdot\frac{\partial\Delta I_1}{\partial\Delta\theta} - \frac{I_1}{A+\Delta A}\cdot\frac{\partial\Delta A}{\partial\Delta\theta} - \frac{A\Delta I_1 - I_1\Delta A}{[A+\Delta A]^2}\cdot\frac{\partial\Delta A}{\partial\Delta\theta} \\ &= \frac{A}{A+\Delta A}\left[\frac{\partial\Delta I_1}{\partial\Delta\theta} - \frac{I_1+\Delta I_1}{A+\Delta A}\cdot\frac{\partial\Delta A}{\partial\Delta\theta}\right] \\ &\ge \frac{A}{A+\Delta A}\left[\frac{\partial\Delta I_1}{\partial\Delta\theta} - \left|\frac{\partial\Delta A}{\partial\Delta\theta}\right|\right] \\ &> \frac{A}{A+\Delta A}\left\{F\left(1/2, \mathrm{e}^{2\pi/t_0}\right)\Delta\theta - \frac{4}{t_0}\left[\frac{2G_1\left(1,\mathrm{e}^{w_1}\right)}{\log(t_0/2\pi)} + G_2\left(1/2,\mathrm{e}^{w_2}\right)\right]\right. \\ &\quad \left. - \frac{8\sqrt{2}G_1\left(1,\mathrm{e}^{w_1}\right)}{t_0\log(t_0/2\pi)}\right\} \end{aligned}$$

$$= \frac{A}{A+\Delta A}\left\{F\left(1/2, e^{2\pi/t_0}\right)\Delta\theta - \frac{4}{t_0}\left[\frac{2\left(\sqrt{2}+1\right)G_1\left(1, e^{w_1}\right)}{\log\left(t_0/2\pi\right)} + G_2\left(1/2, e^{w_2}\right)\right]\right\}$$
$$> \frac{A}{A+\Delta A}\left[F\left(1/2, e^{2\pi/t_0}\right)\Delta\theta - \frac{14}{t_0}G_2\left(1/2, e^{w_2}\right)\right] > 0 \quad (53)$$

provided that

$$\Delta\theta > \frac{14G_2\left(1/2, e^{w_2}\right)}{t_0 F\left(1/2, e^{2\pi/t_0}\right)}. \quad (54)$$

Here the inequality $2\left(\sqrt{2}+1\right)G_1\left(1, e^{w_1}\right)/\log\left(t_0/2\pi\right) < (5/2)G_2\left(1/2, e^{w_2}\right)$ is used. In fact, for the case with $m=0$, to recall that $G_1\left(1, e^{w_1}\right) \approx 0.6967$, $G_2\left(1/2, e^{w_2}\right) \approx 1.2304$ and $t_0 \approx 20.8747$, we get

$$\frac{2\left(\sqrt{2}+1\right)G_1\left(1, e^{w_1}\right)}{\log\left(t_0/2\pi\right)G_2\left(1/2, e^{w_2}\right)} \approx \frac{2\left(\sqrt{2}+1\right)\times 0.6967}{\log\left(20.8747/2\pi\right)\times 1.2304} \approx 2.2771 < 5/2.$$

Along with the increasing of m, t_0 increases accordingly, and this proportion will become smaller and smaller. So, the inequality always holds true.

For the case with $m=0$ and $t_0 \approx 20.8747$, the approximate value of $F\left(1/2, e^{2\pi/t_0}\right)$ is about 0.7205. At this time,

$$\frac{14G_2\left(1/2, e^{w_2}\right)}{t_0 F\left(1/2, e^{2\pi/t_0}\right)} \approx \frac{14\times 1.2304}{20.8747\times 0.7205} \approx 0.3646\pi < \frac{2}{5}\pi.$$

This indicates that, for the case with $\theta(t) \in (\pi/2, 3\pi/2)$, no matter the additional term

$$\int_{\theta(t)}^{\infty}\left[F\left(\sigma, x(z,t)\right) - F\left(\sigma, x(z,t_0)\right)\right]\cos z\mathrm{d}z$$

of $I_1(\sigma,t)$ is positive or not, the increasing term $-\int_{\theta(t_0)}^{\theta(t)} F\left(\sigma, x(z,t_0)\right)\cos z\mathrm{d}z$ of $I_1(\sigma,t)$ dominates the relative change of $I_1(\sigma,t)/A(\sigma,t)$ for all $\sigma \in [1/2, 1]$, at least, with respect to $\Delta\theta = \theta(t) - \theta(t_0)$ on the interval $(0.3646\pi, \pi)$. In another word, $\sin\phi(\sigma,t)$ increases along with the increasing of $\theta(t)$ on $(0.8646\pi, 3\pi/2)$. For the other cases with $m \geq 1$, the interval becomes wider than this.

In summery, for the case with $\theta(t) \in [2m\pi + \pi/2, 2m\pi + 3\pi/2)$ ($m \geq 0$), the results for $I_1(\sigma,t)/A(\sigma,t)$ are similar to that of $I_1(\sigma,t)$ in Theorem 1. The mimic theorem is as follows:

Theorem 3. *For the case with*

$$\theta(t) = \frac{t}{4}\log\left(\frac{t}{2\pi e}\right) + \frac{\pi}{6} \in [2m\pi + \pi/2, 2m\pi + 3\pi/2)$$

(m is a non-negative integer), let $\theta(t_0) = 2m\pi + \pi/2$, *then for all* $\sigma \in [1/2, 1]$, $\sin\phi(\sigma,t)$ *increases along with the increasing of* $\theta(t)$, *at least on the interval defined by*

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$$2m\pi+\pi/2+\frac{14G_2\left(1/2,e^{w_2}\right)}{t_0F\left(1/2,e^{2\pi/t_0}\right)}<\theta(t)<2m\pi+3\pi/2. \tag{55}$$

For another case with $\theta(t_0)=2m\pi-\pi/2$ and $\theta(t)\in[2m\pi-\pi/2,2m\pi+\pi/2)$ ($m\geq 1$), the term $-\int_{\theta(t_0)}^{\theta(t)}F(\sigma,x(z,t_0))\cos zdz$ of $I_1(\sigma,t)$ decreases and dominates, and there is a considerable dominating range. In fact, the above approach holds and it follows from Equation (48) and Equation (50) that:

$$\begin{aligned}
&\frac{\partial}{\partial\Delta\theta}\left[\frac{A(\sigma,t_0)\Delta I_1-I_1(\sigma,t_0)\Delta A}{A(\sigma,t_0)+\Delta A}\right]\\
&=\frac{A}{A+\Delta A}\left[\frac{\partial\Delta I_1}{\partial\Delta\theta}-\frac{I_1+\Delta I_1}{A+\Delta A}\cdot\frac{\partial\Delta A}{\partial\Delta\theta}\right]\\
&\leq\frac{A}{A+\Delta A}\left\{-F\left(1/2,e^{2\pi/t_0}\right)\Delta\theta+\frac{4}{t_0}\left[\frac{2G_1\left(1,e^{w_1}\right)}{\log(t_0/2\pi)}+G_2\left(1/2,e^{w_2}\right)\right]\right.\\
&\left.+\frac{8\sqrt{2}G_1\left(1,e^{w_1}\right)}{t_0\log(t_0/2\pi)}\right\}\\
&<\frac{A}{A+\Delta A}\left[-F\left(1/2,e^{2\pi/t_0}\right)\Delta\theta+\frac{14}{t_0}G_2\left(1/2,e^{w_2}\right)\right]<0
\end{aligned} \tag{56}$$

provided that

$$\Delta\theta>\frac{14G_2\left(1/2,e^{w_2}\right)}{t_0F\left(1/2,e^{2\pi/t_0}\right)}.$$

For this case, the result is as follows:

Theorem 4. *For the case with*

$$\theta(t)=\frac{t}{4}\log\left(\frac{t}{2\pi e}\right)+\frac{\pi}{6}\in[2m\pi-\pi/2,2m\pi+\pi/2)$$

(m is a positive integer), let $\theta(t_0)=2m\pi-\pi/2$, *then for all* $\sigma\in[1/2,1]$, $\sin\phi(\sigma,t)$ *decreases along with the increasing of* $\theta(t)$, *at least on the interval defined by*

$$2m\pi-\pi/2+\frac{14G_2\left(1/2,e^{w_2}\right)}{t_0F\left(1/2,e^{2\pi/t_0}\right)}<\theta(t)<2m\pi+\pi/2.$$

In the following we consider the relative change of $\sin\phi(\sigma,t)$ on the remainder interval $[(k-1/2)\pi,(k-1/2)\pi+p(t_0)]$ (k is a positive integer) with

$$p(t_0)=\frac{14G_2\left(1/2,e^{w_2}\right)}{t_0F\left(1/2,e^{2\pi/t_0}\right)},$$

where t_0 is defined by $\theta(t_0)=(k-1/2)\pi$. We firstly study the odd case with $k=2m+1$ ($m\geq 0$). At this time, the inequality $\sin\phi(\sigma,t_0)<0$ holds for all $\sigma\in[1/2,1]$. According to Equation (52),

$$\sin\phi(\sigma,t)-\sin\phi(\sigma,t_0)=\frac{1}{A(\sigma,t_0)+\Delta A}\left[\Delta I_1-\frac{I_1(\sigma,t_0)}{A(\sigma,t_0)}\Delta A\right],$$

whose sign is determined by the factor in the bracket. In the following we consider this part. With the denotations

$$P\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)=\frac{2}{\log(t_0/2\pi)}G_1\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)+G_2\left(\sigma,\mathrm{e}^{2z^*/t_0}\right),$$
$$Q(\sigma,t_0)=\frac{H(\sigma,t_0)H_t(\sigma,t_0)t_0+S(\sigma,t_0)S_t(\sigma,t_0)t_0}{\sqrt{H^2(\sigma,t_0)+S^2(\sigma,t_0)}}, \tag{57}$$

to recall the approximations in Equation (45) and Equation (50), we have

$$\begin{aligned}&\Delta I_1-\frac{I_1(\sigma,t_0)}{A(\sigma,t_0)}\Delta A\\&\approx\int_0^{\Delta\theta}F\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*-\frac{2\Delta\theta}{t_0}\int_{\Delta\theta}^{\infty}P\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*\\&\quad-\frac{I_1(\sigma,t_0)}{A(\sigma,t_0)}\cdot\frac{4\Delta\theta}{t_0\log(t_0/2\pi)}Q(\sigma,t_0)\\&=\int_0^{\Delta\theta}\left[F\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)+\frac{2\Delta\theta}{t_0}P\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)\right]\sin z^*\mathrm{d}z^*\\&\quad-\frac{2\Delta\theta}{t_0}\left[\int_0^{\infty}P\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*+\frac{2\sin\phi(\sigma,t_0)}{\log(t_0/2\pi)}Q(\sigma,t_0)\right].\end{aligned} \tag{58}$$

Notice that for $0<\Delta\theta\le p(t_0)<\pi/2$ the first term is positive, and its sign relies on the second term. Since for a fixed σ, the function

$$\Phi(\sigma,t_0)=\int_0^{\infty}P\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*+\frac{2\sin\phi(\sigma,t_0)}{\log(t_0/2\pi)}Q(\sigma,t_0) \tag{59}$$

is only associated with t_0, it can be used to make judgement. In case $\Phi(\sigma,t_0)\le 0$, Equation (58) implies that $\sin\phi(\sigma,t)>\sin\phi(\sigma,t_0)$. The thing becomes simple, and $\theta(t_0)=(2m+1/2)\pi$ accords with the minimum extreme point of $\sin\phi(\sigma,t)$. In case $\Phi(\sigma,t_0)>0$ the previous estimations make sense. To take the partial derivative about $\Delta\theta$, it reads

$$\begin{aligned}&\frac{\partial}{\partial\Delta\theta}\left[\Delta I_1-\frac{I_1(\sigma,t_0)}{A(\sigma,t_0)}\Delta A\right]\\&\approx\left[F\left(\sigma,\mathrm{e}^{2\Delta\theta/t_0}\right)+\frac{2\Delta\theta}{t_0}P\left(\sigma,\mathrm{e}^{2\Delta\theta/t_0}\right)\right]\sin\Delta\theta\\&\quad+\frac{2}{t_0}\int_0^{\Delta\theta}P\left(\sigma,\mathrm{e}^{2z^*/t_0}\right)\sin z^*\mathrm{d}z^*-\frac{2}{t_0}\Phi(\sigma,t_0).\end{aligned} \tag{60}$$

Corresponding to the case with $\Delta\theta=0$, its value is $-(2/t_0)\Phi(\sigma,t_0)<0$. On the one hand, from the previous sections we know F,G_1 and G_2 are all positive functions. On the other hand, for fixed $\sigma\in[1/2,1]$, $G_1(\sigma,\mathrm{e}^w)$ and $G_2(\sigma,\mathrm{e}^w)$ increase along with the increasing of w on $[0,w_1]$ and $[0,w_2]$ (with $w_1\approx 0.6484$, $w_2\approx 0.4746$), respectively. Notice that $0\le\Delta\theta<\pi/2$ and

$t_0 \approx 20.8747$ for the initial case with $m=0$, we have $w = 2\Delta\theta/t_0 < 2\times(\pi/2)/20.8747 \approx 0.1505 < w_1, w_2$. So $P\left(\sigma, e^{2\Delta\theta/t_0}\right)$ increases along with the increasing of $\Delta\theta$. In addition, though $F\left(\sigma, e^{2\Delta\theta/t_0}\right)$ decreases along with the increasing of $\Delta\theta$, the product of $\sin\Delta\theta$ and it increases. So the function in Equation (60) increases along with the increasing of $\Delta\theta$ on $\left[0, p(t_0)\right]$ for fixed σ and t_0. This indicates that $\Delta I_1 - I_1\Delta A/A$ decreases first and possibly increases later on this small interval. After all, $\Delta\theta = p(t_0)$ accords with the left end of the increasing interval of $\sin\phi(\sigma,t) - \sin\phi(\sigma,t_0)$.

So, on the remainder interval $\left[(2m+1/2)\pi, (2m+1/2)\pi + p(t_0)\right]$ ($m \ge 0$), the variation of $\theta(t)$ also makes significant influence on $\sin\phi(\sigma,t)$, and the minimum extreme point shifts to the right or not is determined by the sign of $\Phi(\sigma,t_0)$. Furthermore, from Equation (57) and Equation (59) we see, due to the existence of decay factor $2/\log(t_0/2\pi)$ in the other two terms,

$$\int_0^{\infty} G_2\left(\sigma, e^{2z^*/t_0}\right)\sin\left(z^*\right)dz^*$$

dominates the sign of $\Phi(\sigma,t_0)$, and influences the location-shifting of the minimum extreme points. For the even case with $k=2m$ ($m \ge 1$), to repeat the above deduction process we get a symmetric result for the maximum extreme points.

The correspondingly corollary is as follows:

Corollary 2. *For the case with $\theta(t) \in \left[2m\pi, 2(m+1)\pi\right]$ (m is a positive integer), let $\theta(\tilde{t}_m) = 2m\pi + \pi/2$ and $\theta(\hat{t}_m) = 2m\pi + 3\pi/2$, then there is a minimum extreme point and a maximum extreme point of $\sin\phi(\sigma,t)$, which lie separately in the intervals $\left[2m\pi + \pi/2, 2m\pi + \pi/2 + p(\tilde{t}_m)\right]$ and $\left[2m\pi + 3\pi/2, 2m\pi + 3\pi/2 + p(\hat{t}_m)\right]$, where the function p is defined by*

$$p(t) = \frac{14G_2\left(1/2, e^{w_2}\right)}{tF\left(1/2, e^{2\pi/t}\right)}. \tag{61}$$

As $m \to \infty$, $p(\tilde{t}_m), p(\hat{t}_m) \to 0$, and these two extreme points in the moving intervals tend to their left boundaries $\tilde{t}_m$ and $\hat{t}_m$.

3.3.4. The Explicit Periodic Form for $U(\sigma,t)$

It follows from Theorem 3 and Theorem 4 that, for a fixed $\sigma \in [1/2, 1]$, $\sin\phi(\sigma,t) = I_1(\sigma,t)/A(\sigma,t)$ increases or decreases along with the increasing of $\theta(t)$ on most of the variation range $\left[(k-1/2)\pi, (k+1/2)\pi\right)$ (k is a positive integer). In case k is odd it increases, and in case k is even it decreases. This indicates that $\theta(t)$ is the main part of $\phi(\sigma,t)$. It follows from Corollary 2 that, As $m \to \infty$, $\theta(t) = 2m\pi + \pi/2$ accords with the minimum extreme point of $\sin\phi(\sigma,t)$, and the phase angle needs to be adjusted by adding one π. In addition to this, in $\phi(\sigma,t)$ there should be also a perturbation term, say $-\varepsilon(\sigma,t)$. It reflects the contributions from other factors. We note that the negative sign here is to accord with the fact that all the location-shifts of the extreme points are to the right. Based on this, for every given $\sigma \in [1/2, 1]$ and every $t \ge t_0$ with $\theta(t_0) = \pi/2$ ($t_0 \approx 20.8747$), we have an explicit formula bellow:

$$\sin\phi(\sigma,t)=\sin\left[\theta(t)+\pi-\varepsilon(\sigma,t)\right]. \tag{62}$$

To recall the relation in Equation (32), we express the real part of $\xi(\sigma+it)$ in an explicit periodic form:

$$\begin{aligned}U(\sigma,t)&=\frac{2}{t}A(\sigma,t)\sin\left[\phi(\sigma,t)+\theta(t)\right]\\&=\frac{2}{t}A(\sigma,t)\sin\left[2\theta(t)+\pi-\varepsilon(\sigma,t)\right]\\&=\frac{2}{t}A(\sigma,t)\sin\left\{2\left[\frac{t}{4}\log\left(\frac{t}{2\pi\mathrm{e}}\right)+\frac{\pi}{6}\right]+\pi-\varepsilon(\sigma,t)\right\}\\&=A^*(\sigma,t)\sin\left[\frac{t}{2}\log\left(\frac{t}{2\pi\mathrm{e}}\right)+\frac{4}{3}\pi-\varepsilon(\sigma,t)\right]\end{aligned} \tag{63}$$

with $A^*(\sigma,t)=2A(\sigma,t)/t$, which matches exactly with the anticipation in *Section* 2 for the particular case with $\sigma=1/2$.

In the following we give some estimations to the amplitude function $A^*(\sigma,t)$ and the perturbation function $\varepsilon(\sigma,t)$.

3.3.5. To Refine the Estimation of Amplitude Function $A^*(\sigma,t)$

From Equation (31) we see $0<A(\sigma,t)<\sqrt{5}F_0(1)$. In fact, the lower bound can be refined. To denote $x(z,t)=\mathrm{e}^{2z/t}$, then it follows from Equation (24) and Equation (38) that

$$F_z(\sigma,x(z,t))=F_x(\sigma,x)\cdot\frac{2}{t}\mathrm{e}^{2z/t}=\frac{2}{t}F_x(\sigma,x)x=-\frac{2}{t}G_2\left(\sigma,\mathrm{e}^{2z/t}\right)<0.$$

Based on this, for a fixed $\sigma\in[1/2,1]$ we refine the estimation in Equation (27) as follows:

$$\begin{aligned}&\int_0^\infty F\left(\sigma,\mathrm{e}^{2z/t}\right)\sin(z)\mathrm{d}z\\&=2\sum_{k=0}^{\infty}\left[F\left(\sigma,\mathrm{e}^{2(z_k'+2k\pi)/t}\right)-F\left(\sigma,\mathrm{e}^{2(z_k'+2k\pi+\pi)/t}\right)\right]\\&=-2\sum_{k=0}^{\infty}\int_{z_k'+2k\pi}^{z_k'+(2k+1)\pi}F_z\left(\sigma,x(z,t)\right)\mathrm{d}z\\&=\frac{4}{t}\sum_{k=0}^{\infty}\int_{z_k'+2k\pi}^{z_k'+(2k+1)\pi}G_2\left(\sigma,\mathrm{e}^{2z/t}\right)\mathrm{d}z\\&=\frac{4\pi}{t}\sum_{k=0}^{\infty}G_2\left(\sigma,\mathrm{e}^{\hat{w}_k}\right),\end{aligned} \tag{64}$$

in which $0\le z_k'\le\pi$ and

$$\frac{4k\pi}{t}\le\frac{2}{t}\left(z_k'+2k\pi\right)\le\hat{w}_k\le\frac{2}{t}\left[z_k'+(2k+1)\pi\right]\le\frac{4(k+1)\pi}{t}. \tag{65}$$

In view of **Figure 5**, for a given $\sigma\in[1/2,1]$, $G_2\left(\sigma,\mathrm{e}^{w}\right)$ has a unique maximum extreme point at w_2. On $[0,w_2]$ it increases and on $[w_2,\infty)$ it decreases. Along with the increasing of σ, the location of the maximum extreme point is shifted from about $(0.4746,1.2304)$ to about $(0.4820,1.2237)$. Notice that $t\ge t_0\approx 20.8747$, $4\pi/t\le 4\pi/t_0\approx 0.6020>w_2$, it follows from Equation (26) and

Equation (64) that:

$$\begin{aligned} A(\sigma,t) &= \sqrt{\left[\int_0^{\infty} F\left(\sigma,\mathrm{e}^{2z/t}\right)\cos(z)\,\mathrm{d}z\right]^2+\left[\int_0^{\infty} F\left(\sigma,\mathrm{e}^{2z/t}\right)\sin(z)\,\mathrm{d}z\right]^2} \\ &\ge \int_0^{\infty} F\left(\sigma,\mathrm{e}^{2z/t}\right)\sin(z)\,\mathrm{d}z = \frac{4\pi}{t}\sum_{k=0}^{\infty} G_2\left(\sigma,\mathrm{e}^{\hat{w}_k}\right) \\ &> \frac{4\pi}{t} G_2\left(\sigma,\mathrm{e}^{\hat{w}_1}\right) \ge \frac{4\pi}{t} G_2\left(1/2,\mathrm{e}^{8\pi/t_0}\right) \approx \frac{4\pi}{t}\cdot 0.115 > \frac{2\pi}{5t}. \end{aligned}$$

To combine this result with that in Equation (31) we get $2\pi/5t < A(\sigma,t) < \sqrt{5}F_0(1)$ and hence

$$\frac{4\pi}{5t^2} < A^*(\sigma,t) < \frac{2\sqrt{5}F_0(1)}{t}, \tag{66}$$

which has clarified the decaying characteristics of $U(\sigma,t)$ with respect to *t*.

3.3.6. To Estimate the Perturbation Function $\varepsilon(\sigma,t)$

In the following we explore the perturbation term $\varepsilon(\sigma,t)$.

Firstly, for every given $\sigma\in[1/2,1]$, we affirm that the case with

$$\lim_{t\to\infty}\left|\varepsilon(\sigma,t)\right| = \infty$$

is impossible. On the contrary, if it is true, there must be a large enough t^* and an integer M such that $\varepsilon(\sigma,t^*) = (2M+1)\pi$. At this time, $\theta(t^*)$ lies either in $[2m\pi+\pi/2, 2m\pi+3\pi/2)$ or in $[2m\pi+3\pi/2, 2m\pi+5\pi/2)$ for some $m\ge 0$. In the following we consider the first half period. In case $\theta(t^*)$ lies in the increasing interval, that is, $2m\pi+\pi/2+p(t_0) < \theta(t^*) < 2m\pi+3\pi/2$ [here $p(t_0)$ is defined in Equation (61) with $\theta(t_0) = 2m\pi+\pi/2$], $\sin\phi(\sigma,t)$ must increases in the neighborhood of $t = t^*$. Yet at this time,

$$\sin\phi(\sigma,t^*) = \sin\left[\theta(t^*)+\pi-(2M+1)\pi\right] = \sin\theta(t^*)$$

decreases. This leads to a contradiction. In case $\theta(t^*)$ lies in the remainder interval $2m\pi+\pi/2 \le \theta(t^*) \le 2m\pi+\pi/2+p(t_0)$, there should be $\sin\phi(\sigma,t^*) = \sin\theta(t^*) > 0$. Yet, according to the previous arguments,

$$\sin\phi(\sigma,t_0) = I_1(\sigma,t_0)/A(\sigma,t_0) < 0.$$

As $t_0\to\infty$, $p(t_0)\to 0$ and in case t^* is large enough, it is sufficiently close to t_0, and it follows from the continuousness of the function that $\sin\phi(\sigma,t^*) < 0$. This is also a contradiction. On another half period with $2m\pi+3\pi/2 \le \theta(t^*) < 2m\pi+5\pi/2$, a contradiction is also met. This indicates that $\varepsilon(\sigma,t)$ is bounded.

Secondly, on every period there exists an interval on which $\left|\varepsilon(\sigma,t)\right| < \pi/2$. As the case $\theta(t_0) = 2m\pi+\pi/2$ concerned, it follows from the previous arguments that, either t_0 is exactly the minimum extreme point of $\sin\phi(\sigma,t)$, or there is a right-shifting of the minimum extreme point. For any given $\sigma\in[1/2,1]$, for the later case the extreme point accords with

$$\phi(\sigma,t_1) = \theta(t_1)+\pi-\varepsilon(\sigma,t_1) = 2m\pi+3\pi/2$$

for some $t = t_1\ (> t_0)$. According to Corollary 2, $\theta(t_1) \in \left(2m\pi + \pi/2, 2m\pi + \pi/2 + p(t_0)\right]$. Hence,

$$0 < \varepsilon(\sigma, t_1) = \theta(t_1) - \theta(t_0) \le p(t_0) = \frac{14G_2\left(1/2, e^{w_2}\right)}{t_0 F\left(1/2, e^{2\pi/t_0}\right)} < \frac{2}{5}\pi. \tag{67}$$

Due to the existence of the decay factor $1/t_0$, its value should become smaller and smaller as $m \to \infty$. This indicates that, in any given half period $[2m\pi + \pi/2, 2m\pi + 3\pi/2]$, there indeed exists some region such that $|\varepsilon(\sigma, t)| < \pi/2$. This result also holds for another half period.

Thirdly, the inequality $|\varepsilon(\sigma, t)| < \pi/2$ holds for big enough *t*.

For every given $\sigma \in [1/2, 1]$, the extreme point of $\sin\phi(\sigma, t)$ is unique with respect to $\phi(\sigma, t) \in (k\pi, (k+1)\pi)$ (k is a nonnegative integer), if and only if $\phi(\sigma, t)$ is increasing along with the increasing of *t*, that is,

$$\phi_t(\sigma, t) = \frac{1}{4}\log\left(\frac{t}{2\pi}\right) - \varepsilon_t(\sigma, t) > 0. \tag{68}$$

Since $\varepsilon(\sigma, t)$ is bounded, $\varepsilon_t(\sigma, t)$ should be also bounded, say $|\varepsilon_t(\sigma, t)| \le C_0$, where C_0 is a positive constant. Certainly, the above inequality holds for the case with $\varepsilon_t(\sigma, t) \le 0$. For another case with $\varepsilon_t(\sigma, t) > 0$, the estimation makes sense, and it holds provided that

$$t > T = 2\pi e^{4C_0}. \tag{69}$$

Under the above assumption, $\sin\phi(\sigma, t)$ has a unique negative minimum extreme point and a unique positive maximum extreme point on $[2m\pi, 2(m+1)\pi)$ with respect to $\phi(\sigma, t)$ for any big enough m. As the minimum extreme point concerned, according to Corollary 2, it lies in the interval $\left[2m\pi + \pi/2, 2m\pi + \pi/2 + p(t_0)\right]$ with respect to $\theta(t)$, where t_0 is defined by $\theta(t_0) = 2m\pi + \pi/2$. Similarly, the maximum extreme point lies in the interval $\left[2m\pi + 3\pi/2, 2m\pi + 3\pi/2 + p(t_0)\right]$ with respect to $\theta(t)$, where t_0 is defined by $\theta(t_0) = 2m\pi + 3\pi/2$.

If the assertion is false, there must be a $t^* > T$ and a $\sigma^* \in [1/2, 1]$ such that $\varepsilon(\sigma^*, t^*) = \pi/2$ or $-\pi/2$. Let t_1, t_2, t_3, t_4 and t_5 be the points which accord with $\phi(\sigma^*, t) = (2m+1)\pi$, $(2m+1)\pi + \pi/2$, $2(m+1)\pi$, $2(m+1)\pi + \pi/2$ and $(2m+3)\pi$, respectively. Firstly, $t^* \ne t_2, t_4$, since they are the extreme points with $0 < \varepsilon(\sigma, t^*) < 2\pi/5$ for all $\sigma \in [1/2, 1]$. Secondly, $t^* = t_1, t_3, t_5$ are also impossible. The reason is that, for the case with $t^* = t_3$, $\sin\phi(\sigma^*, t^*) = \sin[2(m+1)\pi] = 0$, yet $\theta(t_3) = \phi(\sigma^*, t_3) - \pi + \varepsilon(\sigma^*, t_3) = (2m+1)\pi \pm \pi/2$ accords with $\sin\phi(\sigma^*, t_3) \ne 0$ (see Figure 7). This is a contradiction. The same thing occurs for $t^* = t_1$ and t_5.

In the following we firstly consider the case with $\varepsilon(\sigma^*, t^*) = \pi/2$. In case t^* lies in the increasing interval (t_2, t_3) of $\sin\phi(\sigma^*, t)$, then

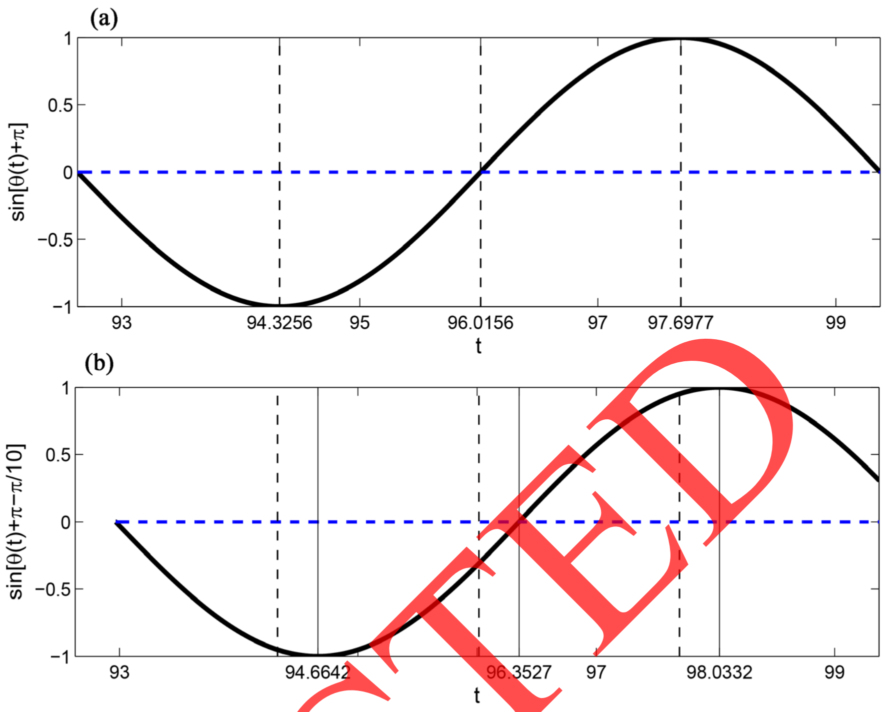

Figure 7. The variation of $\sin\left[\theta(t)+\pi\right]$ (see (a)) and $\sin\left[\theta(t)+\pi-\varepsilon(\sigma,t)\right]$ (see (b)) with respect to $\theta(t)\in[10\pi,22\pi]$ for the particular case with $\varepsilon(\sigma,t)\equiv\pi/10$.

$$\sin\phi\left(\sigma^*,t^*\right)=\sin\left[\theta\left(t^*\right)+\pi-\varepsilon\left(\sigma^*,t^*\right)\right]$$
$$=\sin\left[\left(\theta\left(t^*\right)-\pi/2\right)+\pi\right]=\sin\left[\theta\left(\tilde{t}\right)+\pi\right]$$

with $\tilde{t}\in\left(t_1,t_2\right)$. Yet from Theorem 4 we see on this interval $\sin\phi\left(\sigma^*,t\right)$ decreases along with the increasing of $\theta(t)$, that is, along with the increasing of *t*. This is a contradiction. In case $t^*\in\left(t_3,t_4\right)$, we have $\sin\phi\left(\sigma^*,t\right)>0$. Yet at this time,

$$\sin\phi\left(\sigma^*,t^*\right)=\sin\left[\left(\theta\left(t^*\right)-\pi/2\right)+\pi\right]=\sin\left[\theta\left(\tilde{t}\right)+\pi\right]<0$$

with $\tilde{t}\in\left(t_2,t_3\right)$. This is also a contradiction. In case t^* lies in the decreasing interval $\left(t_4,t_5\right)$ of $\sin\phi\left(\sigma^*,t\right)$, then it follows from Theorem 3 that, there is also a corresponding point $\tilde{t}\in\left(t_3,t_4\right)$, at which $\sin\phi\left(\sigma^*,t\right)$ increases. A contradiction is met. In case $t^*\in\left(t_1,t_2\right)$, we have $\sin\phi\left(\sigma^*,t^*\right)<0$, and there should be a corresponding point $\tilde{t}\in\left(t_0,t_1\right)$ with $\phi\left(\sigma^*,t_1\right)=(2m+1)\pi-\pi/2$, such that $\sin\phi\left(\sigma^*,t^*\right)>0$, and get a contradiction.

For the case with $\varepsilon\left(\sigma^*,t^*\right)=-\pi/2$, to repeat the above processes we get the corresponding contradictions. In summary, for all $\sigma\in[1/2,1]$ and all $t>T=2\pi e^{4C_0}$, it holds the estimation $\left|\varepsilon(\sigma,t)\right|<\pi/2$.

In addition, along with the increasing of $\theta(t)$ on $\left(\pi/2+p\left(t_0\right),3\pi/2\right)$, $\sin\phi(\sigma,t)$ increases. This indicates that $\theta(t)$ dominates and the inequality in Equation (68) holds true. Notice that $p\left(t_0\right)<2\pi/5$, to solve $\left(t^*/4\right)\log\left(t^*/2\pi e\right)+\pi/6=\pi/2+2\pi/5$, it yields $t^*\approx 26.2694$. Hence, on the inverse manner, from Equation (68) we have

$$C_0 < \frac{1}{4}\log\left(\frac{t^*}{2\pi}\right) \approx 0.3576 < \frac{2}{5}.$$

This partly indicates that, the requirement for T is not too high.

In the following we refine the estimation on $\varepsilon(\sigma,t)$ with another approach.

Firstly, as the extreme points of $\sin\phi(\sigma,t)$ concerned, for a given $\sigma \in [1/2,1]$, there must be a positive integer k and a t_k^* such that $\theta(t_k^*) + \pi - \varepsilon(\sigma,t_k^*) = k\pi + 3\pi/2$, from which we get [same to Equation (67)]

$$0 \le \varepsilon(\sigma,t_k^*) \le p(\bar{t}_k) = \frac{14G_2(1/2,e^{w_2})}{\bar{t}_k F(1/2,e^{2\pi/\bar{t}_k})} \le \frac{\bar{C}}{\bar{t}_k} \tag{70}$$

with $\bar{C} = 14G_2(1/2,e^{w_2})/F(1/2,e^{2\pi/\bar{t}_1})$, where $\bar{t}_k$ is defined by $\theta(\bar{t}_k) = k\pi + \pi/2$. Since this inequality holds for all k, we expect the estimation

$$|\varepsilon(\sigma,t)| \le \frac{C_1}{t} \tag{71}$$

holds for some finite positive constant C_1 which does not rely on σ.

On the contrary, if for any given $C > 0$ there is always a $\sigma = \sigma_0$ and a $t = T$ such that $|\varepsilon(\sigma_0,T)| > C/T > 0$, then there is an integer $K > 0$ such that $\theta(t_K^*), \theta(T) \in [K\pi + \pi/2, K\pi + 3\pi/2)$. In the following we seek for a contradiction with case by case discussion.

Case a: In case $\varepsilon(\sigma_0,t_{K+1}^*) > 0$, Since both $|\varepsilon(\sigma_0,T)|$ and $\varepsilon(\sigma_0,t_{K+1}^*)$ are positive, there must be a finite proportional coefficient $\lambda > 0$ such that $|\varepsilon(\sigma_0,T)| = \lambda\varepsilon(\sigma_0,t_{K+1}^*)$. In view of $\bar{t}_{K+1} > T$ we have

$$\frac{C}{\bar{t}_{K+1}} < \frac{C}{T} < |\varepsilon(\sigma_0,T)| = \lambda\varepsilon(\sigma_0,t_{K+1}^*) \le \lambda p(\bar{t}_{K+1}) \le \frac{\lambda\bar{C}}{\bar{t}_{K+1}}.$$

A contradiction is met only if we choose $C > \lambda\bar{C}$.

Case b: In case $\varepsilon(\sigma_0,t_{K+1}^*) = 0$, if there is an integer $n\,(>1)$ such that $\varepsilon(\sigma_0,t_{K+n}^*) > 0$, then the previous contradiction is also met by choosing $C > \lambda\bar{C}$. At this time, the inequality bellow

$$\frac{C}{\bar{t}_{K+n}} < \frac{C}{T} < |\varepsilon(\sigma_0,T)| = \lambda\varepsilon(\sigma_0,t_{K+n}^*) \le \lambda p(\bar{t}_{K+n}) \le \frac{\lambda\bar{C}}{\bar{t}_{K+n}}.$$

is contradictory.

Case c: If for all $k > K$, the equality $\varepsilon(\sigma_0,t_k^*) = 0$ always holds, furthermore, if there is a $\sigma_1 \in [1/2,1]$ and an integer $n\,(\ge 1)$ such that $\varepsilon(\sigma_1,t_{K+n}^*) > 0$, then $|\varepsilon(\sigma_0,T)| = \lambda\varepsilon(\sigma_1,t_{K+n}^*)$ also holds for some $\lambda > 0$, and the previous contradiction is also met. We note that, due to the continuousness of $\varepsilon(\sigma,t)$, the occurrence of this situation is very natural.

Case d: For all $\sigma \in [1/2,1]$ and all $k > K$, the equality $\varepsilon(\sigma,t_k^*) = 0$ always holds, then the particular point $\bar{t}_k$ with $\theta(\bar{t}_k) = k\pi + \pi/2$ is exactly the extreme point of $\sin\phi(\sigma,t)$. If that happens, the refined estimation in Equation (71) is invalid. However, the estimation $|\varepsilon(\sigma,t)| < \pi/2$ is still valid for this case.

We note that, the ideal situation in the last case hardly occurs, since it requires the inequality $\Phi(\sigma, \overline{t_k}) \le 0$ always holds. This, in turn, needs the dominating factor

$$\int_0^\infty G_2\left(\sigma, \mathrm{e}^{2z^*/\overline{t_k}}\right)\sin\left(z^*\right)\mathrm{d}z^* \le 0.$$

Yet, take the case with $\sigma = 1/2$ as an example, $G_2\left(1/2, \mathrm{e}^{2z^*/\overline{t_k}}\right)$ has peak value at about $z^* = \overline{t_k} w_2/2 \approx 0.2373\overline{t_k}$, where the ratio 0.2373 is not an integer or half integer. Furthermore, since this peak value may dominate the corresponding integral, as a whole, the variation of $\Phi(\sigma, \overline{t_k})$ is not synchronous with that of $\theta(\overline{t_k})$, and it prevents the occurrence of this ideal situation. Unfortunately, a rigid proof is unobtainable, and this case can not be excluded.

We note that, for Cases a-c, the requirement $t > T = 2\pi\mathrm{e}^{4C_0}$ for the uniqueness of extreme point can be loosened. Since it holds $|\varepsilon(\sigma,t)| \le C_1/t$, to express $\varepsilon(\sigma,t) = h(\sigma,t)/t$, then $|h(\sigma,t)| \le C_1$ and there is also a finite positive constant C_2 such that $|h_t(\sigma,t)| \le C_2$. Hence,

$$\begin{aligned}|\varepsilon_t(\sigma,t)| &= \left|\frac{\partial}{\partial t}\left[\frac{h(\sigma,t)}{t}\right]\right| = \left|\frac{h_t(\sigma,t)t - h(\sigma,t)\cdot 1}{t^2}\right| \\ &= \frac{|h_t(\sigma,t) - \varepsilon(\sigma,t)|}{t} \le \frac{C_2 + C_1}{t}.\end{aligned}$$

For this case, the inequality in Equation (68) holds on condition that $t > T$ with

$$\frac{T}{4}\log\left(\frac{T}{2\pi}\right) = C_1 + C_2.$$

Certainly, there is another understanding. For Cases a-c, the estimation $|\varepsilon(\sigma,t)| \le C_1/t$ can be used in a direct way. To ensure $|\varepsilon(\sigma,t)| < \pi/2$, it only requires $t > T = 2C_1/\pi$.

3.3.7. The Finial Results for U and the Verification of Proposition 2

As stated in the previous section, for every given $\sigma \in [1/2, 1]$, only if t is big enough, say $t > T = 2\pi\mathrm{e}^{4C_0}$, there is $\phi_t(\sigma,t) = \theta'(t) - \varepsilon_t(\sigma,t) > 0$, and on every period the uniqueness of the minimum extreme point and maximum extreme point of $\sin\phi(\sigma,t)$ is ensured. To denote

$$\begin{aligned}\vartheta(\sigma,t) &= 2\theta(t) + \frac{4}{3}\pi - \varepsilon(\sigma,t) \\ &= \frac{t}{2}\log\left(\frac{t}{2\pi\mathrm{e}}\right) + \frac{4}{3}\pi - \varepsilon(\sigma,t),\end{aligned} \tag{72}$$

then the inequality $\vartheta_t(\sigma,t) = \theta'(t) + \phi_t(\sigma,t) > 0$ holds for $t > T = 2\pi\mathrm{e}^{4C_0}$, which is bigger than 2π.

For every permitted integer k, with respect to $\vartheta(\sigma,t) \in (k\pi, (k+1)\pi)$, we verify the uniqueness of the extreme point of $U(\sigma,t) = A^*(\sigma,t)\sin\vartheta(\sigma,t)$. We note that, since $U(\sigma,t)$ is the real part of the analytic function $\xi(\sigma + it)$, both $A^*(\sigma,t)$ and $\varepsilon(\sigma,t)$ are smooth functions which have arbitrary order derivatives.

For every given $\sigma \in [1/2,1]$, the partial derivative of $U(\sigma,t)$ about t reads:

$$\begin{aligned} U_t(\sigma,t) &= \frac{\partial}{\partial t}\left[A^*(\sigma,t)\sin\vartheta(\sigma,t)\right] \\ &= A_t^*(\sigma,t)\sin\vartheta(\sigma,t) + A^*(\sigma,t)\vartheta_t(\sigma,t)\cos\vartheta(\sigma,t) \\ &= A^*(\sigma,t)\vartheta_t(\sigma,t)\sin\vartheta(\sigma,t)\left[\cot\vartheta(\sigma,t) - \frac{-A_t^*(\sigma,t)}{A^*(\sigma,t)\vartheta_t(\sigma,t)}\right] \\ &=: A^*(\sigma,t)\vartheta_t(\sigma,t)\sin\vartheta(\sigma,t)\left[\cot\vartheta(\sigma,t) - g(\sigma,t)\right]. \end{aligned} \tag{73}$$

We note that, as the extreme points concerned, the case $\sin\vartheta(\sigma,t) = 0$ is excluded since at this time $U_t(\sigma,t) \neq 0$. With this understanding, all the locations of extreme points are determined by the equation $\cot\vartheta(\sigma,t) = g(\sigma,t)$. It is easily checked that, no matter $g(\sigma,t)$ is positive or not, its curve intersects with that of $\cot\vartheta(\sigma,t)$ only once on every given period $(k\pi,(k+1)\pi)$. This indicates $U(\sigma,t)$ has a unique extreme point on $(k\pi,(k+1)\pi)$.

Furthermore, notice that for a fixed σ, $\cot\vartheta(\sigma,t) \to \infty$ as $t \to k\pi$ from the right and $\cot\vartheta(\sigma,t) \to -\infty$ as $t \to (k+1)\pi$ from the left, and $g(\sigma,t) = -A_t^*(\sigma,t)/A^*(\sigma,t)\vartheta_t(\sigma,t)$ is bounded, for all k the function $\cot\vartheta(\sigma,t) - g(\sigma,t)$ always changes from positive to negative on $(k\pi,(k+1)\pi)$. In addition, in case $k = 2m$ it holds $\sin\vartheta(\sigma,t) > 0$, yet in case $k = 2m-1$ it holds $\sin\vartheta(\sigma,t) < 0$. It follows from Equation (73) that, on $(2m\pi,(2m+1)\pi)$ the unique positive extreme point of $U(\sigma,t) = A^*(\sigma,t)\sin\vartheta(\sigma,t)$ is the maximum one, and on $((2m-1)\pi, 2m\pi)$ the unique negative extreme point is the minimum one.

Hence, on every half period, the uniqueness of the extreme point of $U(\sigma,t) = A^*(\sigma,t)\sin\vartheta(\sigma,t)$ is verified. It only requires $t > T = 2\pi e^{4C_0}$.

To summarize all the obtained results we have the final theorem:

Theorem 5. *For every given* $\sigma \in [1/2,1]$ *and every* $t \ge t_0$ *with*

$$\theta(t_0) = \frac{t_0}{4}\log\left(\frac{t_0}{2\pi e}\right) + \frac{\pi}{6} = \frac{\pi}{2}$$

($t_0 \approx 20.8747$*), the real part of* $\xi(\sigma + it)$ *is in an explicit periodic form:*

$$U(\sigma,t) = A^*(\sigma,t)\sin\left[\frac{t}{2}\log\left(\frac{t}{2\pi e}\right) + \frac{4}{3}\pi - \varepsilon(\sigma,t)\right].$$

Here the amplitude decays within

$$\frac{4\pi}{5t^2} < A^*(\sigma,t) < \frac{2\sqrt{5}F_0(1)}{t}.$$

The perturbation term $\varepsilon(\sigma,t)$ *is bounded.* $\varepsilon_t(\sigma,t)$ *is also bounded, say* $|\varepsilon_t(\sigma,t)| \le C_0$, *where* C_0 *is a positive constant. In case* $t > T = 2\pi e^{4C_0}$, *there is an estimation* $|\varepsilon(\sigma,t)| < \pi/2$, *and on every period* $U(\sigma,t)$ *has a unique positive maximum extreme point and a unique negative minimum extreme point. In addition, to define* t_k^* *by* $\theta(t_k^*) = k\pi + \pi/2$, *then except the particular case with* $\varepsilon(\sigma,t_k^*) \equiv 0$ *for all* $\sigma \in [1/2,1]$ *and all nonnegative integer k, there is a refined* *estimation:*

$$\left|\varepsilon(\sigma,t)\right| \le \frac{C_1}{t},$$

here C_1 *is a positive constant.*

These theoretical results accord well with the numerical ones in *Section* 2 (see **Figures 1-3**), and the distribution law

$$\frac{t_k}{2}\log\left(\frac{t_k}{2\pi e}\right)+\frac{4}{3}\pi = k\pi, \quad k=1,2,\cdots$$

of the zero points on the critical line $\sigma = 1/2$ is embodied perfectly. Here t_k stands for the approximate value of the k-th zero point. This indicates Proposition 2 given by Riemann is true. The explicit illustration is as follows:

Theorem 6. *For a big enough* T *which does not accord with a zero point, in the range* $0<t<T$ *the number of zero points on the critical line* $\sigma = 1/2$ *obeys the exact formula bellow (understood as taking the integer part):*

$$N(T)=\frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)+\frac{4}{3}-\frac{\varepsilon(1/2,T)}{\pi} \tag{74}$$

with $\left|\varepsilon(1/2,T)/\pi\right| < 1/2$.

4. The Finial Proof of Riemann Hypothesis

To get an intuitive understanding on the variation of $\xi(\sigma+it)$, we make numerical simulations on $U(\sigma,t)$ and $V(\sigma,t)$ according to the formulas in Equation (18) and Equation (23), that is,

$$\begin{aligned} U(\sigma,t) &= \int_1^\infty f(\sigma,x)\cos\left(\frac{t}{2}\log x\right)dx, \\ V(\sigma,t) &= \int_1^\infty g(\sigma,x)\sin\left(\frac{t}{2}\log x\right)dx \end{aligned} \tag{75}$$

with

$$f(\sigma,x)=2x^{1/4}\left(x^{\delta}+x^{-\delta}\right)\sum_{n=1}^{\infty}a_n\left(a_n x-\frac{3}{2}\right)e^{-a_n x},$$

$$g(\sigma,x)=2x^{1/4}\left(x^{\delta}-x^{-\delta}\right)\sum_{n=1}^{\infty}a_n\left(a_n x-\frac{3}{2}\right)e^{-a_n x},$$

where $\delta = (\sigma - 1/2)/2$ and $a_n = \pi n^2$.

Essentially, they are infinite integrals with infinite series. How to make objective numerical simulations for them is still a question. Our try with finite approximations yield **Figure 8** and **Figure 9** which are just for reference. These figures roughly reflect the basic variation characteristics of $U(\sigma,t)$ and $V(\sigma,t)$. We note that the simulated periodic signals are too strong, and they have depressed the random variations. There is an evidence for this, that is, the simulated zeros points of $U(1/2,t)$ does not completely coincide with the known numerical ones (mentioned in *Section* 2).

Since $\xi(\sigma+it)$ is analytic, the well-known Cauchy-Riemann conditions hold

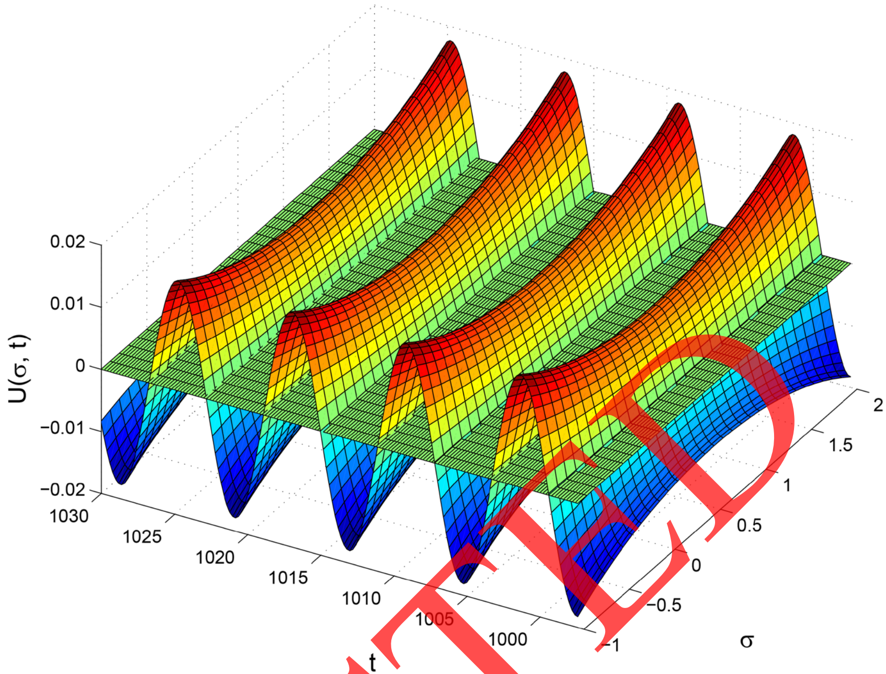

Figure 8. The numerical simulation of $U(\sigma,t)$ for the case with $-1\le\sigma\le 2$ and $997\le t\le 1030$.

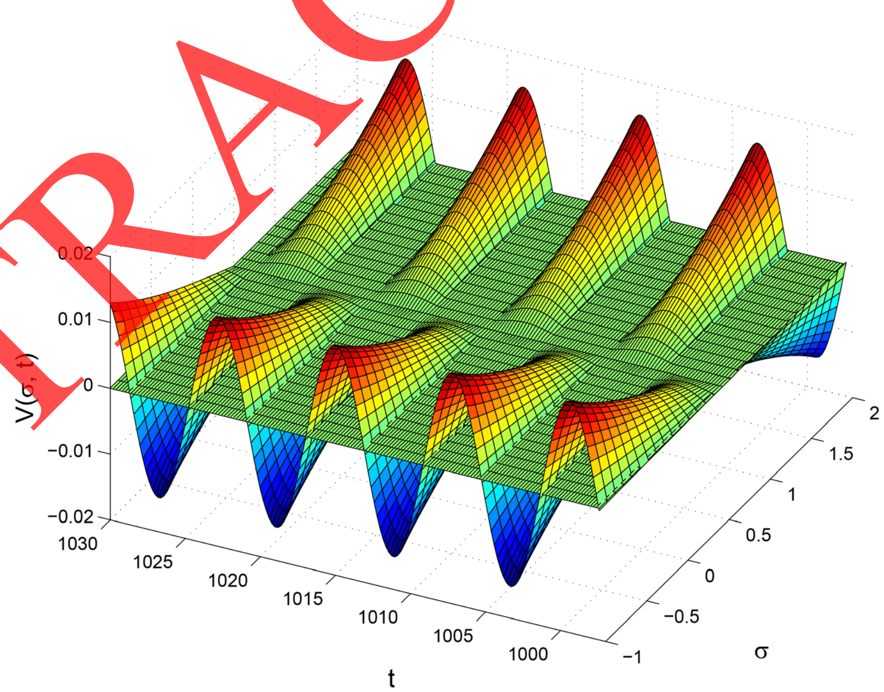

Figure 9. The numerical simulation of $V(\sigma,t)$ for the case with $-1\le\sigma\le 2$ and $997\le t\le 1030$.

for its real and imaginary parts:

$$U_{\sigma}=V_{t},\quad U_{t}=-V_{\sigma}. \tag{76}$$

One can check these with the two formulas in Equation (75) in a direct way.

Let Ω be any finite domain in two-dimensional real space $\mathbb{R}^2$. The analytic property of $\xi(\sigma+it)$ implies good smoothness of U and V on Ω. So the second-order partial derivatives of them exist and are continuous, that is, $U,V \in C^2(\Omega)$. It follows from Equation (76) that

$$U_{\sigma\sigma}+U_{tt}=0,\quad V_{\sigma\sigma}+V_{tt}=0,\quad (\sigma,t)\in\Omega.$$

These mean bother U and V satisfy the two-dimensional Laplace equation, and the result below hold for them:

Lemma 2. ***(Extremum Principle)*** [19]*: If $u(\sigma,t)$ satisfies the Laplace equation $u_{\sigma\sigma}+u_{tt}=0$ in Ω, then it has no extreme point in the interior of Ω, unless it is a constant on the entire region $\bar{\Omega}$.*

It is well-known that there is no zero point outside the critical strip $0\le\sigma\le1$ (see [2] [3] [4] [5] [6]). Notice that the zero points of $\xi(\sigma+it)$ accord with the nontrivial zero points of $\zeta(\sigma+it)$, as the real-valued problem concerned, the equivalent statement of the Riemann hypothesis is as follows:

Except on the critical line $\sigma=1/2$, $U(\sigma,t)$ and $V(\sigma,t)$ have no other mutual zero point in the region $[0,1]\times(0,\infty)$.

Just as mentioned in *Section 1*, for any given finite region bounded by $0\le\sigma\le1$ and $0<t<T$, the check can be always done. What lacks is the knowledge for the case with $t\to\infty$. Till now the known record was set by X. Gourdon in 2004 [11]. He had checked the first 10^{13} zero points and found that they all possess real part $\sigma=1/2$. To speculate with our distribution law, its upper bound is up to $T=2.4460\times10^{12}$. As reviewed in [9], another calculation had ever been done by Odlyzko in 1989 [20] in selected intervals show that the Riemann hypothesis holds for over 3×10^8 zeros at height up to $T=2\times10^{20}$. Our aim is to settle down the infinite case, and the lower bound T may be much lower than this. The following is the final theoretical result:

Theorem 7. *Except on the critical line $\sigma=1/2$, the real part $U(\sigma,t)$ and imaginary part $V(\sigma,t)$ of $\xi(\sigma+it)$ have no other mutual zero point in the region $[0,1]\times(T,\infty)\cup[0,1]\times(-\infty,-T)$ for big enough T, say $T=\max\{2\pi e^{4C_0},4C_1/\pi\}$, where C_0 is the upper bound of the bounded function $|\varepsilon_t(\sigma,t)|$, C_1 is the positive constant which ensures $|\varepsilon(\sigma,t)|\le C_1/t$.*

Proof. It follows from Lemma 1 that, $U(\sigma,t)$ is symmetric about the horizontal line $t=0$ and vertical line $\sigma=1/2$, and $V(\sigma,t)$ is anti-symmetric about the horizontal line $t=0$ and vertical line $\sigma=1/2$. Hence, as the mutual zero point of U and V concerned, it only needs to consider one quarter part of the concerned region. Here we choose the one defined by $(1/2,1]\times(T,\infty)$, which excludes the critical line $\sigma=1/2$.

Suppose $U(\sigma,t)$ and $V(\sigma,t)$ has a mutual zero point in $(1/2,1]\times(T,\infty)$, say (σ_0,t_0), then it must not be an isolated zero point of $U(\sigma,t)$ or $V(\sigma,t)$. On the contrary, if so, to draw a small circle around (σ_0,t_0), then $U(\sigma,t)$ or $V(\sigma,t)$ maintains its sign on this circle, either positive or negative. For this case, at the interior point (σ_0,t_0), $U=0$ or $V=0$, and this point must be a minimum extreme one or a maximum extreme one of $U(\sigma,t)$ or $V(\sigma,t)$ on this circular region. This contradicts with the Extremum Principle. Hence, (σ_0, t_0) is not isolated, and the 2-dimensional surfaces defined by $U(\sigma,t)$ and $V(\sigma,t)$ must intersect the zero-valued plane with two curves, say l_1 and l_2, which pass through the mutual zero point (σ_0, t_0). For convenience, we call them "zero-valued curves" of U and V.

Firstly, we assert that all the zero-valued curves of U intersect with the critical line $\sigma = 1/2$. It follows from the Extremum Principle that all the zero points of $\xi(\sigma+it)$ on $\sigma=1/2$ can not be isolated ones. Explicitly, since $V(1/2,t)\equiv 0$ holds for all $t\in(-\infty,\infty)$, this line itself is a zero-valued curve of V. Meanwhile, for every given positive integer k, let $(1/2, t_k)$ be the k-th zero point of $U(1/2,t)$, then there is also a zero-valued curve across it. This curve is also unique. In fact, it follows from Theorem 5 that $U(\sigma,t)=A^*(\sigma,t)\sin\vartheta(\sigma,t)$, and this curve is actually one of the intersecting curves of this continuous periodic surface and the zero-valued plane, which crosses the point $(1/2,t_k)$. Exactly, this zero-valued curve is determined by

$$\begin{aligned}\vartheta(\sigma,t) &= 2\theta(t)-\varepsilon(\sigma,t)\\ &= \frac{t}{2}\log\left(\frac{t}{2\pi e}\right)+\frac{4}{3}\pi-\varepsilon(\sigma,t)=k\pi\end{aligned} \tag{77}$$

with $|\varepsilon(\sigma,t)|<\pi/2$. Under the condition that $t>T\ge 2\pi e^{4C_0}$, the inequality $\vartheta_t(\sigma,t)>0$ holds, and for the case with $\sigma\in[1/2,1]$, the value of t is uniquely determined. The periodicity of $U(\sigma,t)$ ensured that there is no other zero-valued curves. Hence, the zero-valued curve l_1 of $U(\sigma,t)$, which is across (σ_0,t_0), should accord with Equation (77) with some $k=k_0$.

Secondly, there exists a horizontal strip which includes l_1, on which the sign of V_σ maintains unchanged, and a contradiction is met. In the following we make case by case discussion.

For the particular case with $\varepsilon(\sigma,t_0)\equiv 0$ for all $\sigma\in[1/2,1]$, the point t_0 accords with $\theta(t_0)=m\pi+\pi/2$ for some integer m, and hence

$$\vartheta(\sigma,t_0)=2\theta(t_0)-\varepsilon(\sigma,t_0)=2(m\pi+\pi/2)-0=(2m+1)\pi.$$

At this time, l_1 is a horizontal straight line which does not rely on σ. In view of $U(\sigma,t)=A^*(\sigma,t)\sin\vartheta(\sigma,t)$, along with the increasing of $\sigma\in[1/2,1]$, with respect this zero-valued line, the neighboring maximum extreme points and minimum extreme points are also linked into two curves, say h_1 and h_2, which satisfy

$$\begin{aligned}\vartheta(\sigma,\underline{t}) &= 2\theta(\underline{t})-\varepsilon(\sigma,\underline{t})=2m\pi+\pi/2,\\ \vartheta(\sigma,\overline{t}) &= 2\theta(\overline{t})-\varepsilon(\sigma,\overline{t})=2m\pi+3\pi/2,\end{aligned} \tag{78}$$

respectively. Since $t>T\ge 2\pi e^{4C_0}$, it follows from Theorem 5 that $|\varepsilon(\sigma,\underline{t})|<\pi/2$ and $|\varepsilon(\sigma,\overline{t})|<\pi/2$. Hence,
$2\theta(\underline{t})=2m\pi+\pi/2+\varepsilon(\sigma,\underline{t})<(2m+1)\pi$ and
$2\theta(\overline{t})=2m\pi+3\pi/2+\varepsilon(\sigma,\overline{t})>(2m+1)\pi$. This indicates that
$2\theta(\underline{t})<2\theta(t_0)<2\theta(\overline{t})$, from which we get $\underline{t}<t_0<\overline{t}$ for all $\sigma\in[1/2,1]$. Ac-

tually, h_1 and h_2 possess the coordinate $(\sigma,\underline{t})$ and $(\sigma,\bar{t})$, respectively.

Certainly, l_1 is included in the horizontal strip delimited by h_1 and h_2. If there is a zero-valued curve l_2 of $V(\sigma,t)$ which intersects with l_1 at (σ_0,t_0), then $V(1/2,t_0)=V(\sigma_0,t_0)=0$, and along l_1 there must be an extreme point $\sigma=\sigma^*$, at which $V_\sigma=0$. Yet notice that for every given $\sigma\in[1/2,1]$, with respect to $\vartheta(\sigma,t)$ on $(2m\pi+\pi/2,2m\pi+3\pi/2)$, $U(\sigma,t)=A^*(\sigma,t)\sin\vartheta(\sigma,t)$ decreases, there should be $U_t<0$ on the whole strip. Furthermore, it follows from the Cauchy-Riemann condition in Equation (76) that $V_\sigma=-U_t>0$. This leads to a contradiction.

Except the particular case $\varepsilon(\sigma,t_0)\equiv 0$, it follows from Theorem 5 that, the refined estimation $|\varepsilon(\sigma,t)|\le C_1/t$ holds for some positive constant C_1. On condition that $t>T\ge 4C_1/\pi$, we have

$$|\varepsilon(\sigma,t)|<\frac{\pi}{4}.$$

To take $2\theta(t)$ as a research object, from Equation (77) we see l_1 accords with

$$k_0\pi-\frac{\pi}{4}<2\theta(\tilde{t})=k_0\pi+\varepsilon(\sigma,\tilde{t})<k_0\pi+\frac{\pi}{4} \tag{79}$$

for some positive integer k_0. At this time, the neighboring two extreme-point curves, say h_1 and h_2, are defined as in Equation (78), which yield

$$\begin{aligned} 2\theta(\underline{t})&=k_0\pi-\pi/2+\varepsilon(\sigma,\underline{t})<k_0\pi-\frac{\pi}{4},\\ 2\theta(\bar{t})&=k_0\pi+\pi/2+\varepsilon(\sigma,\bar{t})>k_0\pi+\frac{\pi}{4}. \end{aligned} \tag{80}$$

As a default, here $\underline{t}>T$ is fulfilled. To combine the inequalities in Equation (79) we have

$$2\theta(\underline{t})<k_0\pi-\frac{\pi}{4}<2\theta(\tilde{t})<k_0\pi+\frac{\pi}{4}<2\theta(\bar{t}). \tag{81}$$

In the horizontal strip delimited by h_1 and h_2, there are two horizontal lines, say j_1 and j_2, defined by $2\theta(t)=k_0\pi-\pi/4$ and $2\theta(t)=k_0\pi+\pi/4$, respectively. If there is a zero-valued curve l_2 of $V(\sigma,t)$ which intersects with l_1 at (σ_0,t_0), then the horizontal line $t=t_0$ is between the two horizontal lines j_1 and j_2. Naturally, we have $V(1/2,t_0)=V(\sigma_0,t_0)=0$, and along the line $t=t_0$ there should be an extreme point $\sigma=\sigma^*$, at which $V_\sigma=0$. Yet in the horizontal strip delimited by h_1 and h_2, $\vartheta(\sigma,t)$ lies in the monotone interval $(k_0\pi-\pi/2,k_0\pi+\pi/2)$. In case k_0 is odd, say $k_0=2m_0+1$, for every given $\sigma\in[1/2,1]$, $U(\sigma,t)=A^*(\sigma,t)\sin\vartheta(\sigma,t)$ decreases on this interval. Particularly, between the two horizontal lines j_1 and j_2, there is $U_t<0$. Furthermore, it follows from the Cauchy-Riemann condition in Equation (76) that $V_\sigma=-U_t>0$. This leads to a contradiction. In case k_0 is even, say $k_0=2m_0$, $(k_0\pi-\pi/2,k_0\pi+\pi/2)$ is the increasing interval of $U(\sigma,t)$, and between the two horizontal lines j_1 and j_2 there must be $V_\sigma=-U_t<0$. This is also contradictory.

The contradiction above indicate that the real part $U(\sigma,t)$ and imaginary part $V(\sigma,t)$ of $\xi(\sigma+it)$ have no mutual zero point in the region $(1/2,1]\times(T,\infty)$ for big enough *T*. Furthermore, it follows from the symmetric properties of *U* and *V* that, except on the critical line $\sigma=1/2$, they have no other mutual zero point in the whole region $[0,1]\times(T,\infty)\cup[0,1]\times(-\infty,-T)$. The proof is finished.

5. Conclusions and Remarks

The Riemann hypothesis is a well-known unsolved problem. Its difficulty lies in the complexity of the Zeta function $\zeta(s)$. Essentially, it is involved in an infinite integral which includes infinite series with complex variables.

From October 2018 we had tried to detour this infinite integral, and explore an abstract proof with the "symmetry" of $\xi(s)$ [which is equivalent to $\zeta(s)$], the Extremum Principle and some new techniques. After about one year, we had found that all these tries are in vain and turned to the "monotonicity". As depicted in Figure 9, $V(\sigma,t)$ shows good horizontal monotonicity. Yet the proof is not available. In 2020 we had grasped the key point, that is, to prove the Riemann hypothesis it requires, $U(1/2,t)$ has no positive minimum extreme point and negative maximum extreme point. Yet, to prove this is like climbing the steep precipices and cliffs, there is no viable path! Until March 2021, when we began to investigate the distribution law of zero points on the critical line $\sigma=1/2$, a bold idea appeared, that is the "periodicity"! With this understanding, all the scattered results in the literatures are linked together. With three-month effort, a brand-new artwork appeared.

For every given $\sigma\in[1/2,1]$, we have found that the real part of $\xi(\sigma+it)$ is actually in an explicit periodic form:

$$U(\sigma,t)=A^{*}(\sigma,t)\sin\left[\frac{t}{2}\log\left(\frac{t}{2\pi e}\right)+\frac{4}{3}\pi-\varepsilon(\sigma,t)\right].$$

with $4\pi/5t^{2}<A^{*}(\sigma,t)<2\sqrt{5}F_{0}(1)/t$ and $|\varepsilon(\sigma,t)|<\pi/2$. Except a particular case, there is also a refined estimation $|\varepsilon(\sigma,t)|\le C_{1}/t$ for some positive constants C_{1}.

These theoretical results have verified the observed distribution law

$$\frac{t_k}{2}\log\left(\frac{t_k}{2\pi e}\right)+\frac{4}{3}\pi=k\pi,\quad k=1,2,\cdots$$

for the zero points on the critical line $\sigma=1/2$. Here t_k stands for the approximate value of the k-th zero point. This indicates Proposition 2 given by Riemann is true. Explicitly, *for a big enough T which does not accord with a zero point, in the range* $0<t<T$ *the number of zero points on the critical line* $\sigma=1/2$ *obeys the exact formula bellow (understood as taking the integer part):*

$$N(T)=\frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right)+\frac{4}{3}-\frac{\varepsilon(1/2,T)}{\pi}$$

with $|\varepsilon(1/2,T)/\pi|<1/2$.

It is well-known that there is no zero point outside the critical strip $0\le\sigma\le1$.

For any given finite region bounded by $0 \le \sigma \le 1$ and $0 < t < T$, it can be always checked that, all the zero points of $\xi(\sigma,t)$ are on the critical line $\sigma = 1/2$. What lacks is the knowledge for the case with $t \to \infty$. In 2004, Gourdon [11] had checked the first 10^{13} zero points and found the Riemann hypothesis is true. To speculate with our distribution law, its upper bound is up to $T = 2.4460 \times 10^{12}$. As reviewed in [9], another calculation had ever been done by Odlyzko in 1989 [20] in selected intervals show that the Riemann hypothesis holds for over 3 × 10^8 zeros at height up to $T = 2 \times 10^{20}$. Our request on the lower bound T may be much lower than this. The final result is as follows:

Except on the critical line $\sigma = 1/2$, *the real part* $U(\sigma,t)$ *and imaginary part* $V(\sigma,t)$ *of* $\xi(\sigma + it)$ *have no other mutual zero point in the region* $[0,1] \times (T,\infty) \cup [0,1] \times (-\infty,-T)$ *for big enough T.*

This indicates the Riemann hypothesis is true, and it is the moment to draw full stop for this suspending problem.

In addition to the "Riemann hypothesis", we had also explored another well-known problem named "P versus NP" in [21], where a brand-new fast algorithm for the "travelling salesman problem" was constructed, and a surprising result "P = NP" was obtained. Those who are interested can read it.

RETRACTED

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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