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The Number of Matching Equivalent for the Union Graph of Vertices and Cycles

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Abstract

For two graphs *G* and *H*, if *G* and *H* have the same matching polynomial, then *G* and *H* are said to be matching equivalent. We denote by $\delta(G)$, the number of the matching equivalent graphs of *G*. In this paper, we give $\delta(sK_1 \cup t_1C_9 \cup t_2C_{15})$, which is a generation of the results of in [1].

Keywords

Graph, Matching Polynomial, Matching Equivalence

1. Introduction

This paper only considers finite undirected simple graphs. Let *G* be a graph with *n* vertices. The matching of *G* means a spanning subgraph of *G*, and each of its connected branches is either an isolated vertex or an isolated edge. *t*-matching means matching that there are *t* edges. Matching polynomial of graph *G* is defined as follows in [2]:

$$\mu(G,x)=\sum_{t\ge 0}(-1)^t\alpha_t(G)x^{n-2t}, \qquad (1)$$

where $\alpha_t(G)$ is number of *t*-matching for *G*.

If graph *G* and graph *H* satisfy $\mu(G,x)=\mu(H,x)$, then we say *G* and *H* are matching equivalence, denoted as $G \sim H$.

Set $\delta(G)$ which denotes the number of matching equivalent graphs of all different isomorphisms for graph *G*, if $\delta(G)=1$, we say graph *G* is a unique matching. In [2], the authors gave some elegant properties of matching polynomials, and proved that to find the matching polynomial of a graph is an *NP*-problem. Thus, in [1], the authors studied the number of matching equivalent graphs of some vertices and some cycle-union graphs. It turns out that the problem is not simple. In this paper, we study the number of matching equivalent for the union graph of vertices and cycles, that is $\delta\left(sK_1 \cup t_1C_9 \cup t_2C_{15}\right)$.

Throughout the paper, K_1 denotes an isolated vertex, $P_n\left(n \ge 2\right)$ denotes the path including n vertices; $C_m\left(m \ge 3\right)$ denotes a cycle that including m vertices; $T_{i,j,k}$ denotes a tree that has only a 3-degree vertex, three 1-degree vertex, and the distance between this 3-degree vertex and three 1-degree vertex is i, j, k separately; $D_n\left(n \ge 4\right)$ denotes a graph that produced by bonding a vertex on a triangle to an end of a path P_{n-2}; nG denotes disjoint union of n graph G.

2. Preliminaries

Lemma 2.1 [3] Suppose graph G has k connected component: $G_1, G_2, \cdots, G_k$, then $\mu(G,x) = \prod_{i=1}^{k} \mu(G_i, x)$, the roots of matching polynomials are all real numbers, denote $M\left(G\right)$ as the maximum root of $\mu(G,x)$.

Lemma 2.2 [3] Suppose G is connected graph, then $M\left(G\right) < 2$ if and only if $G \in \Gamma = \left\{K_1, P_n, T_{1,1,n}, T_{1,2,2}, T_{1,2,3}, T_{1,2,4}, C_n, D_4\right\}$.

Lemma 2.3 [3] (i) $M\left(C_m\right) = M\left(T_{1,1,m-2}\right) = M\left(P_{2m-1}\right)$.

(ii) $M\left(C_6\right) = M\left(T_{1,1,4}\right) = M\left(T_{1,2,2}\right) = M\left(D_4\right) = M\left(P_{11}\right)$.

(iii) $M\left(C_9\right) = M\left(T_{1,1,7}\right) = M\left(T_{1,2,3}\right) = M\left(P_{17}\right)$.

(iv) $M\left(C_{15}\right) = M\left(T_{1,1,13}\right) = M\left(T_{1,2,4}\right) = M\left(P_{29}\right)$.

Lemma 2.4 [4] [5] (i) The matching equivalent graph of $K_1 \cup C_m\left(m \ne 6,9,15\right)$ is $K_1 \cup C_m, T_{1,1,m-2}$.

(ii) The matching equivalent graph of $K_1 \cup C_6$ is: $K_1 \cup C_6, T_{1,1,4}, P_3 \cup D_4$.

(iii) The matching equivalent graph of $K_1 \cup C_9$ is $K_1 \cup C_9, T_{1,1,7}, C_3 \cup T_{1,2,3}$.

(iv) The matching equivalent graph of $K_1 \cup C_{15}$ is: $K_1 \cup C_{15} \sim C_3 \cup C_5 \cup T_{1,2,4}$.

Lemma 2.5 [6] (i) $P_{2m+1} \sim P_m \cup C_{m+1}, \left(m \ge 2\right)$.

(ii) $T_{1,1n} \sim K_1 \cup C_{n+2}$.

(iii) $T_{1,2,2} \sim P_2 \cup D_4$.

(iv) $K_1 \cup C_6 \sim P_3 \cup D_4$.

(v) $K_1 \cup C_9 \sim C_3 \cup T_{1,2,3}$.

(vi) $K_1 \cup C_{15} \sim C_3 \cup C_5 \cup T_{1,2,4}$.

Lemma 2.6 [1] Suppose $G = sK_1$ or $t_1C_{m_1} \cup \cdots \cup t_kC_{m_k}$, then G is a unique matching, that is $\delta\left(G\right) = 1$.

Lemma 2.7 [1] Suppose

$$G = sK_1 \cup t_1C_{m_1} \cup t_2C_{m_2} \cup \cdots \cup t_kC_{m_k}, m_i \ne 6\left(i = 1,2,\cdots,k\right),$$

then all matching equivalent graphs of G do not contain road branches.

Lemma 2.8 [1]

(i) $G = sK_1 \cup aP_3 \cup tC_6$, then all matching equivalent graphs of G do not contain P_{11} branch and also $T_{1,2,2}$ [7],

(ii) $\delta\left(sK_1 \cup aP_3 \cup tC_6\right) = \delta\left(sK_1 \cup tC_6\right)$.

Lemma 2.9 [1] [7] If $m \ne 6,9,15$, then

$$\delta\left(sK_1 \cup tC_6\right) = \min\left\{s,t\right\} + 1. \qquad (2)$$

Lemma 2.10 [1] [8] If $m_i \ne 6,9,15\left(i = 1,2\right)$, then

$\delta\left(sK_1 \cup t_1C_{m_1} \cup t_2C_{m_2}\right) = \sum_{i=0}^{r} \min\left\{s - i, t_1\right\} + r + 1$, where

$$r=\min\{s,t_2\}. \tag{3}$$

3. Main Results

Lemma 3.1 $\delta(sK_1\cup t_1C_3\cup t_2C_5\cup t_3C_9)=\sum_{j=0}^{r}\sum_{i=j}^{r}\delta(s-i,\ t_1+(i-j),t_2)$,

where

$$r=\min\{s,t_3\}. \tag{4}$$

Proof. For simplicity, denote $\delta(sK_1\cup t_1C_3\cup t_2C_5\cup t_3C_9)=\delta(s,t_1,t_2,t_3)$.

Set $H\sim sK_1\cup t_1C_3\cup t_2C_5\cup t_3C_6$, by lemma 2.3 (iii) and lemma 2.7, we know H contains connected component $C_9,T_{1,1,7}$ or $T_{1,2,3}$.

1) If H contains C_9, by $H=C_9\cup H_2\sim sK_1\cup t_1C_3\cup t_2C_5\cup t_3C_9$, we know $H_2\sim sK_1\cup t_1C_3\cup t_2C_5\cup(t_3-1)C_9$.

Such H_2 has a total of $\delta(s,t_1,t_2,t_3-1)$.

2) If H contains $T_{1,1,7}$, by $H=T_{1,1,7}\cup H_2\sim sK_1\cup t_1C_3\cup t_2C_5\cup t_3C_9$ and lemma 2.5 (ii), we know

$$H_2\sim(s-1)K_1\cup t_1C_3\cup t_2C_5\cup(t_3-1)C_9,$$

Such H_2 has a total of $\delta(s-1,t_1,t_2,t_3-1)$.

3) If H contains $T_{1,2,3}$, by $H=T_{1,2,3}\cup H_2\sim sK_1\cup t_1C_3\cup t_2C_5\cup t_3C_9$ and lemma 2.5(v), we get

$$H_2\sim(s-1)K_1\cup(t_1+1)C_3\cup t_2C_5\cup(t_3-1)C_9,$$

Such H_2 has a total of $\delta(s-1,t_1+1,t_2,t_3-1)$.

4) If H contains C_9 and $T_{1,1,7}$ simultaneously, such H_2 has a total of $\delta(s-1,t_1,t_2,t_3-2)$.

5) If H contains C_9 and $T_{1,2,3}$ simultaneously, such H_2 has a total of $\delta(s-1,t_1+1,t_2,t_3-2)$.

6) If H contains $T_{1,1,7}$ and $T_{1,2,3}$ simultaneously, such H_2 has a total of $\delta(s-2,t_1+1,t_2,t_3-2)$.

7) If H contains C_9, $T_{1,1,7}$ and $T_{1,2,3}$ simultaneously, such H_2 has a total of $\delta(s-2,t_1+1,t_2,t_3-3)$.

Thus,

$$\begin{aligned}\delta(s,t_1,t_2,t_3)&=\delta(s,t_1,t_2,t_3-1)+\delta(s-1,t_1,t_2,t_3-1)+\delta(s-1,t_1+1,t_2,t_3-1)\\&\quad-\delta(s-1,t_1,t_2,t_3-2)-\delta(s-1,t_1+1,t_2,t_3-2)\\&\quad-\delta(s-2,t_1+1,t_2,t_3-2)+\delta(s-2,t_1+1,t_2,t_3-3)\end{aligned}$$

Then,

$$\begin{aligned}&\delta(s,t_1,t_2,t_3)-\delta(s-1,t_1,t_2,t_3-1)-\delta(s-1,t_1+1,t_2,t_3-1)\\&+\delta(s-2,t_1+1,t_2,t_3-2)\\&=\delta(s,t_1,t_2,t_3-1)-\delta(s-1,t_1,t_2,t_3-2)-\delta(s-1,t_1+1,t_2,t_3-2)\\&+\delta(s-2,t_1+1,t_2,t_3-3)\end{aligned}$$

Repeat the application of the above formula, we obtain

$$\delta(s,t_1,t_2,t_3)-\delta(s-1,t_1,t_2,t_3-1)-\delta(s-1,t_1+1,t_2,t_3-1)$$
$$+\delta(s-2,t_1+1,t_2,t_3-2)$$
$$=\delta(s,t_1,t_2,2)-\delta(s-1,t_1,t_2,1)-\delta(s-1,t_1+1,t_2,1)+\delta(s-2,t_1+1,t_2,0)$$
$$=\delta(s,t_1,t_2,1)-\delta(s-1,t_1,t_2,0)-\delta(s-1,t_1+1,t_2,0)$$
$$=\delta(s,t_1,t_2,0)=\delta(s,t_1,t_2)$$

Thus,

$$\delta(s,t_1,t_2,t_3)-\delta(s-1,t_1,t_2,t_3-1)$$
$$=\delta(s-1,t_1+1,t_2,t_3-1)-\delta(s-2,t_1+1,t_2,t_3-2)+\delta(s,t_1,t_2)$$
$$=\delta(s-2,t_1+2,t_2,t_3-2)-\delta(s-3,t_1+2,t_2,t_3-3)$$
$$+\delta(s-1,t_1+1,t_2)+\delta(s-1,t_1+1,t_2)$$
$$=\cdots=\sum_{i=0}^{r}\delta(s-i,t_1+i,t_2)$$

$$\delta(s,t_1,t_2,t_3)-\delta(s-1,t_1,t_2,t_3-1)=\sum_{i=0}^{r}\delta(s-i,t_1+i,t_2) \quad (1')$$

$$\delta(s-1,t_1,t_2,t_3-1)-\delta(s-2,t_1,t_2,t_3-2)=\sum_{i=1}^{r}\delta(s-i,t_1+(i-1),t_2) \quad (2')$$

$\cdots$

$$\delta(s-(r-1),t_1,t_2,t_3-(r-1))-\delta(s-r,t_1,t_2,t_3-r)$$
$$=\sum_{i=r-1}^{r}\delta(s-i,t_1+i-(r-1),t_2) \quad (r')$$

$$\delta(s-r,t_1,t_2,t_3-r)=\sum_{i=r}^{r}\delta(s-i,t_1+(i-r),t_2) \quad (r+1')$$

Add (1'), (2'), $\cdots$, (r+1') together, we get

$$\delta(s,t_1,t_2,t_3)=\sum_{j=0}^{r}\sum_{i=j}^{r}\delta(s-i,t_1+(i-j),t_2). \qquad \blacksquare$$

Theorem 3.1

$$\delta(sK_1\cup t_1C_9\cup t_2C_{15})=\sum_{j=0}^{r}\sum_{i=j}^{r}\delta(s-i,i-j,i-j,t_1). \quad (5)$$

Proof. For simplicity, denote

$$\delta(sK_1\cup tC_3\cup t'C_5\cup t_1C_9\cup t_2C_{15})=\delta(s,t,t't_1,t_2).$$

Suppose $H\sim G$, by lemma 2.3(iv) and lemma 2.7, we know H contains connected component $C_{15},T_{1,1,13}$ or $T_{1,2,4}$.

1) If H contains C_{15}, by $H=C_{15}\cup H_2\sim sK_1\cup t_1C_9\cup t_2C_{15}$ we know $H_2\sim sK_1\cup t_1C_9\cup(t_2-1)C_{15}$. Such H_2 has a total of $\delta(s,0,0,t_1,t_2-1)$.

2) If H contains $T_{1,1,13}$, by $H=T_{1,1,13}\cup H_2\sim G=sK_1\cup t_1C_9\cup t_2C_{15}$ and lemma 2.5(ii), we get $H_2\sim(s-1)K_1\cup t_1C_9\cup(t_2-1)C_{15}$. Such H_2 has a total of $\delta(s-1,0,0,t_1,t_2-1)$.

3) If H contains $T_{1,2,4}$, by $H=T_{1,2,4}\cup H_2\sim G=sK_1\cup t_1C_9\cup t_2C_{15}$ and lemma 2.5(vi), we get $H_2\sim(s-1)K_1\cup C_3\cup C_5\cup t_1C_9\cup(t_2-1)C_{15}$, such H_2 has a total of $\delta(s-1,1,1,t_1,t_2-1)$.

4) If H contains C_{15} and $T_{1,1,13}$, such H_2 has a total of $\delta(s-1,0,0,t_1,t_2-2)$.

5) If H contains C_{15} and $T_{1,2,4}$, such H_2 has a total of $\delta(s-1,1,1,t_1,t_2-2)$.

6) If H contains $T_{1,1,13}$ and $T_{1,2,4}$, such H_2 has a total of $\delta(s-2,1,1,t_1,t_2-2)$.

7) If H contains C_{15}, $T_{1,1,13}$ and $T_{1,2,4}$, such H_2 has a total of $\delta(s-2,1,1,t_1,t_2-3)$.

Then

$$\begin{aligned}\delta(s,0,0,t_1,t_2) = {} & \delta(s,0,0,t_1,t_2-1)+\delta(s-1,0,0,t_1,t_2-1) \\ & +\delta(s-1,1,1,t_1,t_2-1)-\delta(s-1,0,0,t_1,t_2-2) \\ & -\delta(s-1,1,1,t_1,t_2-2)-\delta(s-2,1,1,t_1,t_2-2) \\ & +\delta(s-2,1,1,t_1,t_2-3)\end{aligned}$$

Thus,

$$\begin{aligned} & \delta(s,0,0,t_1,t_2)-\delta(s-1,0,0,t_1,t_2-1)-\delta(s-1,1,1,t_1,t_2-1) \\ & +\delta(s-2,1,1,t_1,t_2-2) \\ & =\delta(s,0,0,t_1,t_2-1)-\delta(s-1,0,0,t_1,t_2-2)-\delta(s-1,1,1,t_1,t_2-2) \\ & +\delta(s-2,1,1,t_1,t_2-3)\end{aligned}$$

Repeat the application of the above formula, we obtain

$$\begin{aligned} & \delta(s,0,0,t_1,t_2)-\delta(s-1,0,0,t_1,t_2-1)-\delta(s-1,1,1,t_1,t_2-1) \\ & +\delta(s-2,1,1,t_1,t_2-2) \\ & =\delta(s,0,0,t_1,2)-\delta(s-1,0,0,t_1,1)-\delta(s-1,1,1,t_1,1)+\delta(s-2,1,1,t_1,0) \\ & =\delta(s,0,0,t_1,1)-\delta(s-1,0,0,t_1,0)-\delta(s-1,1,1,t_1,0) \\ & =\delta(s,0,0,t_1)\end{aligned}$$

So,

$$\begin{aligned} & \delta(s,0,0,t_1,t_2)-\delta(s-1,0,0,t_1,t_2-1) \\ & =\delta(s-1,1,1,t_1,t_2-1)-\delta(s-2,1,1,t_1,t_2-2)+\delta(s,0,0,t_1) \\ & =\delta(s-2,2,2,t_1,t_2-2)-\delta(s-3,2,2,t_1,t_2-3) \\ & \quad +\delta(s,0,0,t_1)+\delta(s-1,1,1,t_1) \\ & =\delta(s-3,3,3,t_1,t_2-3)-\delta(s-4,3,3,t_1,t_2-4)+\delta(s,0,0,t_1) \\ & \quad +\delta(s-1,1,1,t_1)+\delta(s-2,2,2,t_1) \\ & =\cdots=\sum_{i=0}^{r}\delta(s-i,\ i,i,t_1)\end{aligned}$$

$$\delta(s,0,0,t_1,t_2)-\delta(s-1,0,0,t_1,t_2-1)=\sum_{i=0}^{r}\delta(s-i,i,i,t_1) \quad (1')$$

$$\delta(s-1,0,0,t_1,t_2-1)-\delta(s-2,0,0,t_1,t_2-2)=\sum_{i=0}^{r}\delta(s-i,i-1,i-1,t_1) \quad (2')$$

…

$$\begin{aligned} & \delta(s-(r-1),0,0,t_1,t_2-(r-1))-\delta(s-r,0,0,t_1,t_2-r) \\ & =\sum_{i=r-1}^{r}\delta(s-i,i-(r-1),i-(r-1),t_1)\end{aligned} \quad (r')$$

$$\delta(s-r,0,0,t_1,t_2-r)=\sum_{i=r}^{r}\delta(s-i,i-r,i-r,t_1)\quad (r+1')$$

Add (1′), (2′), $\cdots$, (r+1′) together, we get:

$$\delta(s,0,0,t_1,t_2)=\sum_{j=0}^{r}\sum_{i=j}^{r}\delta(s-i,i-j,i-j,t_1).\ \blacksquare$$

Characterizing all graphs determined by a graph polynomial is an important subject in algebraic graph theory, among them, matching polynomial is considered to be a better algebraic tool. It is NP difficult to completely characterize the matched equivalent graphs of a class of graphs. In this paper, we study the number of matching equivalent graphs of some points and some cycli-union graphs, that is to say we calculate

$$\delta(sK_1\cup t_1C_9\cup t_2C_{15})=\sum_{j=0}^{r}\sum_{i=j}^{r}\delta(s-i,i-j,i-j,t_1).$$

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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