

On Symmetry Reduction of the $(1 + 3)$ -Dimensional Inhomogeneous Monge-Ampère Equation to the First-Order ODEs

Vasyl M. Fedorchuk, Volodymyr I. Fedorchuk

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, National Academy of Sciences of Ukraine, L'viv, Ukraine
Email: vased@gmail.com, volfed@gmail.com

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Abstract

We present the results obtained concerning the classification of symmetry reduction of the $(1 + 3)$ -dimensional inhomogeneous Monge-Ampère equation to first-order ODEs. Some classes of the invariant solutions are constructed.

Keywords

Symmetry Reduction, Invariant Solutions, Monge-Ampère Equation, Classification of Lie Algebras, Poincaré Group $P(1, 4)$

1. Introduction

It is well known that the symmetry reduction is one of the most universal tools for investigation of partial differential equations (PDEs). In particular, for this aim, we can use the classical Lie-Ovsiannikov method [1–4] (see, also, the references therein).

However, it turned out that in order to apply the classical Lie-Ovsiannikov method for PDEs with non-trivial symmetry groups, we had to solve a pure algebraic problem of describing all nonconjugate (nonsimilar) subalgebras of the Lie algebras of symmetry groups of the equations under investigation. More details on this theme can be found in [2, 3] (see, also, the references therein).

In 1975, Patera, Winternitz, and Zassenhaus [5] proposed a general method for describing the nonconjugate subalgebras of Lie algebras with nontrivial ideals.

In 1984, Grundland, Harnad, and Winternitz [6] pointed out that the reduced equations, obtained with the help of nonconjugate subalgebras of the same ranks of the Lie algebras of the symmetry groups of some PDEs, were of different types. They also investigated the similar

phenomenon. Some details on this theme can be found in [7–14] (see, also, the references therein).

The results obtained cannot be explained using only the rank of nonconjugate subalgebras of the Lie algebras of the symmetry groups of PDEs under investigation.

To try to explain some of the differences in the properties of the reduced equations for PDEs with nontrivial symmetry groups, we suggested to investigate the relationship between the structural properties of nonconjugate subalgebras of the same rank of the Lie algebras of the symmetry groups of those PDEs and the properties of the reduced equations corresponding with them [13].

At the present time, the relationship has been studied between the structural properties of the low-dimensional ($\dim L \leq 3$) nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group $P(1, 4)$ and the properties of the reduced equations for the Eikonal and Euler-Lagrange-Born-Infeld equations. The details on this theme can be found in [12–16].

A solution of many problems of the geometry, unified string theories, geometrical optics, elastic theories of shallow shells, optimal transportation, one-dimensional gas dynamics, meteorology and oceanography etc. is reduced to the investigation of the Monge-Ampère equations in the spaces of different dimensions and different types. Some details on this theme can be found in [17–34] (see, also, the references therein).

This paper is devoted to the study of the relationship between the structural properties of the low-dimensional ($\dim L \leq 3$) nonconjugate subalgebras of the same rank of the Lie algebra of the group $P(1, 4)$ and the properties of the reduced equations for the $(1 + 3)$ -dimensional inhomogeneous Monge-Ampère equation.

By now, the relationship has been investigated between the structural properties of the three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ and the properties of the reduced equations for the $(1 + 3)$ -dimensional inhomogeneous Monge-Ampère equation. We obtained the following types of the reduced equations:

- algebraic equations,
- the first-order linear ODEs,
- the first-order nonlinear ODEs,
- the second-order nonlinear ODEs,
- the partial differential equations.

From the invariants of some nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$, it is impossible to construct the ansatzes, which reduce the $(1 + 3)$ -dimensional inhomogeneous Monge-Ampère equation.

This paper is the first in which the relationship is studied between the structural properties of the three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ and types of reductions of the $(1 + 3)$ -dimensional inhomogeneous Monge-Ampère equation to the first-order ODEs.

Institute for Applied Problems of Mechanics and Mathematics, National Academy of Sciences of Ukraine, 3-b, Naukova Str., 79060, L'viv, Ukraine

2. Lie Algebra of the Poincaré Group $P(1, 4)$ and Its Nonconjugate Subalgebras

The group $P(1, 4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1, 4)$. It is the smallest group, which contains, as subgroups, the extended Galilei group $\tilde{G}(1, 3)$ [35] (the symmetry group of classical physics) and the Poincaré group $P(1, 3)$ (the symmetry group of relativistic physics).

Lie algebra of the group $P(1, 4)$ is generated by 15 bases elements $M_{\mu\nu} = -M_{\nu\mu}$ ($\mu, \nu = 0, 1, 2, 3, 4$) and P_μ ($\mu = 0, 1, 2, 3, 4$), which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu, \quad (1)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}, \quad (2)$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

In this paper, we consider the following representation [36] of the Lie algebra of the group $P(1, 4)$:

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3}, \quad (3)$$

$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad x_4 \equiv u. \quad (4)$$

In the following, we will use the next bases elements:

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12}, \quad (5)$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad (a = 1, 2, 3), \quad (6)$$

$$X_0 = \frac{1}{2}(P_0 - P_4), \quad X_k = P_k \quad (k = 1, 2, 3), \quad X_4 = \frac{1}{2}(P_0 + P_4). \quad (7)$$

Nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ have been described in the papers [37–39].

The Lie algebra of the extended Galilei group $\tilde{G}(1, 3)$ is generated by the following bases elements:

$$L_1, \quad L_2, \quad L_3, \quad P_1, \quad P_2, \quad P_3, \quad X_0, \quad X_1, \quad X_2, \quad X_3, \quad X_4. \quad (8)$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ of dimensions ≤ 3 was performed in [40].

3. On Symmetry Reduction of the $(1 + 3)$ -Dimensional Inhomogeneous Monge-Ampère Equation to the First-Order ODEs

In this section, we consider equation of the form:

$$\det(u_{\mu\nu}) = \lambda(1 - u_\nu u^\nu)^3, \quad \lambda \neq 0,$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1, 3)$,
 $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u^\nu = g^{\nu\alpha} u_\alpha$, $u_\alpha \equiv \frac{\partial u}{\partial x_\alpha}$,
 $g_{\mu\nu} = (1, -1, -1, -1)\delta_{\mu\nu}$, $\mu, \nu, \alpha = 0, 1, 2, 3$.

Here, $M(1, 3)$ is a four-dimensional Minkowski space.

In 1983, Fushchich and Serov [36] studied symmetry properties and constructed some classes of exact solutions for the multidimensional Monge–Ampère equation. From this work, it follows that the equation under investigation is invariant with respect to the group $P(1, 4)$.

In order to perform symmetry reduction as well as to construct classes of independent invariant solutions for the equation under consideration, we used the structural properties of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ [40].

Below, we present a short review of the results obtained.

3.1. Lie Algebras of the Type $3A_1$

Reductions to the first-order linear ODEs

Taking into account the invariants of 11 subalgebras we constructed the ansatzes, which reduced the equation under consideration to the first-order linear ODEs. Below we present some of the results obtained.

1. $\langle P_1 \rangle \oplus \langle P_2 - X_2 \rangle \oplus \langle X_3 \rangle :$

Ansatz

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega(\omega + 1)\varphi' = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = c, \quad \omega + 1 = 0, \quad \omega = 0.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge–Ampère equation

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = c, \quad x_0 + u + 1 = 0, \\ x_0 + u = 0.$$

2. $\langle P_1 \rangle \oplus \langle P_2 \rangle \oplus \langle X_3 \rangle :$

Ansatz

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c\omega.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge–Ampère equation

$$x_0^2 - x_1^2 - x_2^2 - u^2 = c(x_0 + u).$$

3. $\langle P_3 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle :$

Ansatz

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c\omega.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_0^2 - x_3^2 - u^2 = c(x_0 + u).$$

It should be noted that subalgebras (1)–(3) belong to the Lie algebra of the extended Galilei group $\tilde{G}(1, 3) \subset P(1, 4)$.

Reductions to the first-order nonlinear ODEs.

From the invariants of seven subalgebras we constructed the ansatzes, which reduced the equation under consideration to the first-order nonlinear ODEs. Below we present the result obtained.

1. $\langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle :$

Ansatz

$$\frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u = \varphi(\omega),$$

$$\omega = (x_0 + u)^2 + 4x_3.$$

Reduced equation

$$16(\varphi')^2 - \omega = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \frac{\varepsilon}{6}\omega^{3/2} + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\begin{aligned} & \frac{1}{6}(x_0 + u)^3 + x_3(x_0 + u) + x_0 - u \\ &= \frac{\varepsilon}{6}((x_0 + u)^2 + 4x_3)^{3/2} + c, \quad \varepsilon = \pm 1. \end{aligned}$$

2. $\langle P_3 - 2X_0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_4 \rangle :$

Ansatz

$$(x_0 + u)^2 + 4x_3 = \varphi(\omega), \quad \omega = x_2.$$

Reduced equation

$$(\varphi')^2 + 16 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = 4i\varepsilon\omega + c.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_0 + u)^2 + 4x_3 = 4i\varepsilon x_2 + c.$$

3. $\langle G + \alpha X_3, \alpha > 0 \rangle \oplus \langle X_1 \rangle \oplus \langle X_2 \rangle :$

Ansatz

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_0^2 - u^2.$$

Reduced equation

$$4\omega(\varphi')^2 + 4\alpha\varphi' - 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \varepsilon\sqrt{\omega + \alpha^2} + \frac{\varepsilon}{2}\alpha \ln\left(\frac{\sqrt{\alpha^2 + \omega} - \alpha}{\sqrt{\alpha^2 + \omega} + \alpha}\right) - \frac{\alpha}{2}\ln(\omega) + c.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_3 - \alpha \ln(x_0 + u) = \varepsilon\sqrt{x_0^2 - u^2 + \alpha^2} + \frac{\varepsilon}{2}\alpha \ln\left(\frac{\sqrt{x_0^2 - u^2 + \alpha^2} - \alpha}{\sqrt{x_0^2 - u^2 + \alpha^2} + \alpha}\right) - \frac{\alpha}{2}\ln(x_0^2 - u^2) + c.$$

4. $\langle L_3 \rangle \oplus \langle P_3 + C_3 \rangle \oplus \langle X_0 + X_4 \rangle :$

Ansatz

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$((\varphi')^2 + 1)\varphi = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c, \quad \varphi(\omega) = 0.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_3^2 + u^2)^{1/2} = i\varepsilon(x_1^2 + x_2^2)^{1/2} + c, \quad u^2 + x_3^2 = 0.$$

5. $\langle L_3 + \alpha(X_0 + X_4), \alpha > 0 \rangle \oplus \langle X_3 \rangle \oplus \langle X_4 \rangle :$

Ansatz

$$x_0 + u + \alpha \arctan \frac{x_2}{x_1} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + \alpha^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon(\alpha \ln(\omega) + ic).$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$u = i\varepsilon\left(\frac{\alpha}{2}\ln(x_1^2 + x_2^2) + ic\right) - \alpha \arctan \frac{x_2}{x_1} - x_0.$$

6. $\langle G \rangle \oplus \langle X_2 \rangle \oplus \langle X_1 \rangle :$

Ansatz

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3.$$

Reduced equation

$$(\varphi' - 1)(\varphi' + 1) = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_0^2 - u^2)^{1/2} = \varepsilon x_3 + c, \quad \varepsilon = \pm 1.$$

7. $\langle G \rangle \oplus \langle L_3 \rangle \oplus \langle X_3 \rangle :$

Ansatz

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$(\varphi' - 1)(\varphi' + 1) = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_0^2 - u^2)^{1/2} = \varepsilon(x_1^2 + x_2^2)^{1/2} + c, \quad \varepsilon = \pm 1.$$

3.2. Lie Algebras of the Type $A_2 \oplus A_1$

Reductions to the first-order nonlinear ODEs

Taking into account the invariants of six subalgebras we constructed the ansatzes, which reduced the equation under consideration to the first-order nonlinear ODEs. Below we present some of the result obtained.

1. $\langle -(G + \alpha X_2), X_4, \alpha > 0 \rangle \oplus \langle X_1 \rangle :$

Ansatz

$$x_2 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_3.$$

Reduced equation

$$(\varphi')^2 + 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_2 - \alpha \ln(x_0 + u) = i\varepsilon x_3 + c.$$

2. $\langle -(G + \alpha X_2), P_3, \alpha > 0 \rangle \oplus \langle X_1 \rangle :$

Ansatz

$$x_2 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_0^2 - x_3^2 - u^2)^{1/2}.$$

Reduced equation

$$\omega (\omega(\varphi')^2 + 2\alpha\varphi' - \omega) = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon\alpha \ln \left(\frac{2(\alpha\sqrt{\omega^2 + \alpha^2} + \alpha^2)}{\omega} \right) - \alpha \ln(\omega)$$

$$- \varepsilon\sqrt{\omega^2 + \alpha^2} + c, \quad \varepsilon = \pm 1; \quad \omega = 0.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_2 = \varepsilon\alpha \ln \left(\frac{2(\alpha\sqrt{x_0^2 - x_3^2 - u^2 + \alpha^2} + \alpha^2)}{\sqrt{x_0^2 - x_3^2 - u^2}} \right)$$

$$- \frac{\alpha}{2} \ln(x_0^2 - x_3^2 - u^2) - \varepsilon\sqrt{x_0^2 - x_3^2 - u^2 + \alpha^2}$$

$$+ \alpha \ln(x_0 + u) + c, \quad \varepsilon = \pm 1; \quad x_0^2 - x_3^2 - u^2 = 0.$$

3. $\langle -(G + \alpha X_3), X_4, \alpha > 0 \rangle \oplus \langle L_3 + \beta X_3, \beta > 0 \rangle :$

Ansatz

$$x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} = \varphi(\omega),$$

$$\omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + \omega^2 + \beta^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\beta \ln \left(\frac{2 \left(\beta\sqrt{\omega^2 + \beta^2} + \beta^2 \right)}{\omega} \right) - i\varepsilon\sqrt{\omega^2 + \beta^2} + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_3 - \alpha \ln(x_0 + u) + \beta \arctan \frac{x_1}{x_2} = i\varepsilon\beta \ln \left(\frac{2 \left(\beta\sqrt{x_1^2 + x_2^2 + \beta^2} + \beta^2 \right)}{\sqrt{x_1^2 + x_2^2}} \right) - i\varepsilon\sqrt{x_1^2 + x_2^2 + \beta^2} + c, \quad \varepsilon = \pm 1.$$

4. $\langle -(G + \alpha X_3), X_4, \alpha > 0 \rangle \oplus \langle L_3 \rangle :$

Ansatz

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$(\varphi')^2 + 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_3 - \alpha \ln(x_0 + u) = i\varepsilon(x_1^2 + x_2^2)^{1/2} + c, \quad \varepsilon = \pm 1.$$

5. $\langle -G, P_3 \rangle \oplus \langle X_1 \rangle :$

Ansatz

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_2.$$

Reduced equation

$$(\varphi' - 1)(\varphi' + 1)\varphi = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon\omega + c, \quad \varphi(\omega) = 0, \quad \varepsilon = \pm 1.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon x_2 + c, \quad \varepsilon = \pm 1, \quad x_0^2 - x_3^2 - u^2 = 0.$$

3.3. Lie Algebras of the Type $A_{3,1}$

Reductions to the first-order nonlinear ODEs

Taking into account the invariants of three nonconjugate subalgebras, we constructed the ansatzes, which reduced the equation under consideration to the first-order nonlinear ODEs. Below we present the results obtained.

1. $\langle 2\mu X_4, P_3 - 2X_0, X_1 + \mu X_3, \mu > 0 \rangle :$

Ansatz

$$(x_0 + u)^2 + 4x_3 - 4\mu x_1 = \varphi(\omega), \quad \omega = x_2.$$

Reduced equation

$$(\varphi')^2 + 16(\mu^2 + 1) = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = 4i\varepsilon\sqrt{\mu^2 + 1}\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_0 + u)^2 + 4x_3 - 4\mu x_1 = 4i\varepsilon\sqrt{\mu^2 + 1}x_2 + c, \quad \varepsilon = \pm 1.$$

2. $\langle 2X_4, P_3 - L_3 - 2\alpha X_0, X_3, \alpha > 0 \rangle :$

Ansatz

$$2\alpha \arctan \frac{x_1}{x_2} - x_0 - u = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + 4\alpha^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = 2i\varepsilon\alpha \ln(\omega) + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$2\alpha \arctan \frac{x_1}{x_2} - x_0 - u = i\varepsilon\alpha \ln(x_1^2 + x_2^2) + c, \quad \varepsilon = \pm 1.$$

3. $\langle -2\beta X_4, L_3 + \beta X_3, P_3 - 2X_0, \beta > 0 \rangle :$

Ansatz

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + \omega^2 + \beta^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\beta \ln \left(\frac{2(\beta\sqrt{\omega^2 + \beta^2} + \beta^2)}{\omega} \right)$$

$$- i\varepsilon\sqrt{\omega^2 + \beta^2} + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3$$

$$= i\varepsilon\beta \ln \left(\frac{2(\beta\sqrt{x_1^2 + x_2^2 + \beta^2} + \beta^2)}{\sqrt{x_1^2 + x_2^2}} \right)$$

$$- i\varepsilon\sqrt{x_1^2 + x_2^2 + \beta^2} + c, \quad \varepsilon = \pm 1.$$

3.4. Lie Algebras of the Type $A_{3,2}$

Reduction to the first-order nonlinear ODEs

From the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the equation under consideration to the first-order nonlinear ODEs. Below we present the results obtained.

1. $\langle 2\beta X_4, P_3, G + \alpha X_1 + \beta X_3, \alpha > 0, \beta > 0 \rangle :$

Ansatz

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_2.$$

Reduced equation

$$(\varphi')^2 + 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_1 - \alpha \ln(x_0 + u) = i\varepsilon x_2 + c, \quad \varepsilon = \pm 1.$$

2. $\left\langle 2\alpha X_4, \lambda_1 P_3, \frac{1}{\lambda_1} L_3 + G + \frac{\alpha}{\lambda_1} X_3, \alpha > 0, \lambda_1 > 0 \right\rangle :$

Ansatz

$$\ln(x_0 + u) + \lambda_1 \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2 (\varphi')^2 + \lambda_1^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\lambda_1 \ln(\omega) + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\ln(x_0 + u) + \lambda_1 \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda_1}{2} \ln(x_1^2 + x_2^2) + c, \quad \varepsilon = \pm 1.$$

3.5. Lie Algebras of the Type $A_{3,3}$

Reductions to the first-order nonlinear ODEs

Taking into account the invariants of two nonconjugate subalgebras, we constructed the ansatzes, which reduced the equation under consideration to the first-order nonlinear ODEs. Below we present the results obtained.

1. $\langle P_3, X_4, G + \alpha X_1, \alpha > 0 \rangle :$

Ansatz

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = x_2.$$

Reduced equation

$$(\varphi')^2 + 1 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_1 - \alpha \ln(x_0 + u) = i\varepsilon x_2 + c, \quad \varepsilon = \pm 1.$$

$$2. \left\langle P_3, X_4, \frac{1}{\lambda_1}L_3 + G, \lambda_1 > 0 \right\rangle :$$

Ansatz

$$\ln(x_0 + u) + \lambda_1 \arctan \frac{x_1}{x_2} = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\omega^2(\varphi')^2 + \lambda_1^2 = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = i\varepsilon\lambda_1 \ln(\omega) + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\ln(x_0 + u) + \lambda_1 \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda_1}{2} \ln(x_1^2 + x_2^2) + c, \quad \varepsilon = \pm 1.$$

3.6. Lie Algebras of the Type $A_{3,6}$

Reductions to the first-order linear ODEs

Taking into account the invariants of four nonconjugate subalgebras, we constructed the ansatzes, which reduced the equation under consideration to the first-order linear ODEs. Below we present some of the results obtained.

$$1. \langle X_1, -X_2, P_3 - L_3 \rangle :$$

Ansatz

$$x_0^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c\omega.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_0^2 - x_3^2 - u^2 = c(x_0 + u).$$

$$2. \langle P_1, -P_2, -(L_3 + \alpha X_3), \alpha > 0 \rangle :$$

Ansatz

$$x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega), \quad \omega = x_0 + u.$$

Reduced equation

$$\omega\varphi' - \varphi = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = c\omega.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$x_0^2 - x_1^2 - x_2^2 - u^2 = c(x_0 + u).$$

It should be noted that subalgebras (1) and (2) belong to the Lie algebra of the extended Galilei group $\tilde{G}(1, 3) \subset P(1, 4)$.

Reductions to the first-order nonlinear ODEs

From the invariants of eight subalgebras we constructed the ansatzes, which reduced the equation under consideration to the first-order nonlinear ODEs. Below we present the results obtained.

1. $\langle X_1, -X_2, P_3 - L_3 - 2\alpha X_0, \alpha > 0 \rangle :$

Ansatz

$$(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = \varphi(\omega),$$

$$\omega = (x_0 + u)^2 + 4\alpha x_3.$$

Reduced equation

$$4(\varphi')^2 - 9\omega = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \varepsilon\omega^{3/2} + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\begin{aligned} &(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) \\ &= \varepsilon((x_0 + u)^2 + 4\alpha x_3)^{3/2} + c, \quad \varepsilon = \pm 1. \end{aligned}$$

2. $\left\langle X_1, -X_2, -L_3 - \frac{1}{2}(P_3 + C_3) - \alpha(X_0 + X_4), \alpha > 0 \right\rangle :$

Ansatz

$$\alpha \arctan \frac{x_3}{u} - x_0 = \varphi(\omega), \quad \omega = (x_3^2 + u^2)^{1/2}.$$

Reduced equation

$$\omega(\omega^2(\varphi')^2 - \omega^2 + \alpha^2) = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = i\varepsilon\alpha \ln \left(\frac{2(i\alpha\sqrt{\omega^2 - \alpha^2} - \alpha^2)}{\omega} \right) - \varepsilon\sqrt{\omega^2 - \alpha^2} + c, \quad \varepsilon = \pm 1;$$

$$\omega = 0.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\begin{aligned} &\alpha \arctan \frac{x_3}{u} - x_0 = \varepsilon\sqrt{x_3^2 + u^2 - \alpha^2} \\ &- i\varepsilon\alpha \ln \left(\frac{2(i\alpha\sqrt{x_3^2 + u^2 - \alpha^2} - \alpha^2)}{\sqrt{x_3^2 + u^2}} \right) + c, \quad \varepsilon = \pm 1, \end{aligned}$$

$$x_3^2 + u^2 = 0.$$

3. $\left\langle X_1, X_2, L_3 + \frac{\lambda_1}{2}(P_3 + C_3) + \alpha(X_0 + X_4), \alpha > 0, \right.$

$$\left. 0 < \lambda_1 < 1 \right\rangle :$$

Ansatz

$$\alpha \arctan \frac{x_3}{u} - \lambda_1 x_0 = \varphi(\omega), \quad \omega = (x_3^2 + u^2)^{1/2}.$$

Reduced equation

$$\omega(\omega^2(\varphi')^2 - \lambda_1^2\omega^2 + \alpha^2) = 0.$$

Solutions of the reduced equation

$$\begin{aligned} &\varphi(\omega) = i\varepsilon\alpha \ln \left(\frac{2(i\alpha\sqrt{\lambda_1^2\omega^2 - \alpha^2} - \alpha^2)}{\omega} \right) \\ &- \varepsilon\sqrt{\lambda_1^2\omega^2 - \alpha^2} + c, \quad \varepsilon = \pm 1; \quad \omega = 0. \end{aligned}$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\alpha \arctan \frac{x_3}{u} - \lambda_1 x_0 = \varepsilon \sqrt{\lambda_1^2(x_3^2 + u^2) - \alpha^2} - i\varepsilon \alpha \ln \left(\frac{2(i\alpha \sqrt{\lambda_1^2(x_3^2 + u^2) - \alpha^2} - \alpha^2)}{\sqrt{x_3^2 + u^2}} \right) + c,$$

$$\varepsilon = \pm 1, \quad x_3^2 + u^2 = 0.$$

4. $\langle X_1, X_2, L_3 + \lambda_1 G + \alpha X_3, \alpha > 0, \lambda_1 > 0 \rangle :$

Ansatz

$$\lambda_1 x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \quad \omega = (x_0^2 - u^2)^{1/2}.$$

Reduced equation

$$\omega (\omega(\varphi')^2 + 2\alpha\varphi' - \lambda_1^2\omega) = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon \alpha \ln \left(\frac{2(\alpha \sqrt{\lambda_1^2\omega^2 + \alpha^2} + \alpha^2)}{\omega} \right) - \alpha \ln(\omega)$$

$$- \varepsilon \sqrt{\lambda_1^2\omega^2 + \alpha^2} + c, \quad \varepsilon = \pm 1, \quad \omega = 0.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$\lambda_1 x_3 - \alpha \ln(x_0 + u) = \varepsilon \sqrt{\lambda_1^2(x_0^2 - u^2) + \alpha^2} - \varepsilon \alpha \ln \left(\frac{2(\alpha \sqrt{\lambda_1^2(x_0^2 - u^2) + \alpha^2} + \alpha^2)}{\sqrt{x_0^2 - u^2}} \right) - \frac{\alpha}{2} \ln(x_0^2 - u^2) + c, \quad \varepsilon = \pm 1, \quad x_0^2 - u^2 = 0.$$

5. $\langle X_1, -X_2, -(L_3 + \alpha X_3), \alpha > 0 \rangle :$

Ansatz

$$u = \varphi(\omega), \quad \omega = x_0.$$

Reduced equation

$$(\varphi' + 1)(\varphi' - 1) = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \varepsilon \omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$u = \varepsilon x_0 + c, \quad \varepsilon = \pm 1.$$

6. $\left\langle X_1, X_2, L_3 + \frac{1}{2}(P_3 + C_3) \right\rangle :$

Ansatz

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0.$$

Reduced equation

$$(\varphi' + 1)(\varphi' - 1)\varphi = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon \omega + c, \quad \varphi(\omega) = 0, \quad \varepsilon = \pm 1.$$

Solutions of the (1+3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c, \quad x_3^2 + u^2 = 0, \quad \varepsilon = \pm 1.$$

$$7. \left\langle -X_1, X_2, -L_3 - \frac{\lambda_1}{2}(P_3 + C_3), 0 < \lambda_1 < 1 \right\rangle :$$

Ansatz

$$(x_3^2 + u^2)^{1/2} = \varphi(\omega), \quad \omega = x_0.$$

Reduced equation

$$(\varphi' + 1)(\varphi - 1)\varphi = 0.$$

Solutions of the reduced equation

$$\varphi(\omega) = \varepsilon\omega + c, \quad \varphi(\omega) = 0, \quad \varepsilon = \pm 1.$$

Solutions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c, \quad x_3^2 + u^2 = 0, \quad \varepsilon = \pm 1.$$

$$8. \langle -X_1, X_2, -(L_3 + \lambda_1 G), \lambda_1 > 0 \rangle :$$

Ansatz

$$(x_0^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3.$$

Reduced equation

$$(\varphi' + 1)(\varphi - 1) = 0.$$

Solution of the reduced equation

$$\varphi(\omega) = \varepsilon\omega + c, \quad \varepsilon = \pm 1.$$

Solution of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation

$$(x_0^2 - u^2)^{1/2} = \varepsilon x_3 + c, \quad \varepsilon = \pm 1.$$

4. Conclusions

We study the relationship between the structural properties of the low-dimensional ($\dim L \leq 3$) nonconjugate subalgebras of the same rank of the Lie algebra of the Poincaré group $P(1, 4)$ and the properties of the reduced equations for the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation.

At the present time, the connection has been investigated between the structural properties of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ and the results of symmetry reduction for the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation. We obtained different types of the reduced equations.

In this paper, we presented a short review of our results concerning the relationship between the structural properties of the three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1, 4)$ and reductions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation to the first-order ODEs.

It is known [40] that the Lie algebra of the group $P(1, 4)$ contains three-dimensional nonconjugate subalgebras of the following types: $3A_1, A_2 \oplus A_1, A_{3,1}, A_{3,2}, A_{3,3}, A_{3,4}, A_{3,6}, A_{3,7}^a, A_{3,8}, A_{3,9}$.

From the results obtained it follows that

—the reductions to the first-order linear ODEs can be obtained using some subalgebras of the following types: $3A_1, A_{3,6}$.

—the reductions to the first-order nonlinear ODEs can be obtained using some subalgebras of the following types: $3A_1, A_2 \oplus A_1, A_{3,1}, A_{3,2}, A_{3,3}, A_{3,6}$.

The future work based on our current study will be devoted to the classification of symmetry reductions of the (1 + 3)-dimensional inhomogeneous Monge-Ampère equation to algebraic equations. Some classes of the invariant solutions will also be constructed.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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