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On Multiplicative Generators of n-Dimensional Overlap Functions

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Abstract

In this paper, inspired by the multiplicative generators of overlap functions, we mainly propose the concepts of multiplicative generator pairs of n-dimensional overlap functions, in order to extend the dimensionality of overlap functions from 2 to n. We present the condition under which the pair (g, h) can multiplicatively generate an n-dimensional overlap function $O_{g,h}$. we focus on the homogeneity and idempotency property on multiplicatively generated n-dimensional overlap functions.

Keywords

n-Dimensional Overlap Functions, Overlap Functions, Multiplicative Generators

1. Introduction

Overlap functions [1] [2] and grouping functions [3] are two particular cases of bivariate continuous aggregation functions. Those two concepts have been applied to some interesting problems, for example, image processing, classification or decision making. In [4], Gómez *et al.* introduced the definition of n-dimensional overlap functions and the conditions under which n-dimensional overlap functions are migrative, homogeneous or Lipschitz continuous. In [5], Dimuro *et al.* introduced the notion of additive generator pair for overlap functions and analyzed the influence of the migrativity, homogeneity and idempotency properties in the overlap functions obtained by such distortion and their respective additive generator pairs. Qiao and Hu [6] proposed the concept of multiplicative generator pair for overlap functions and grouping functions, and investigated the migrativity, homogeneity, idempotency, Archimedean and cancellation properties

for the overlap functions and grouping functions obtained by such multiplicative generator pairs. The main purpose of [6] is to present one new way to construct overlap function and grouping function by use of multiplicative generator pairs. In a fuzzy classification system, one always need to measure the degree of overlapping of an object with more than two classes. From the theoretical and applied point of view, we need to study how to construct n-dimensional overlap function by use of multiplicative generator pairs. In this paper, we will propose the notions of multiplicative generator pairs of n-dimensional overlap functions. Furthermore, we study the homogeneity and idempotency property on multiplicatively generated n-dimensional overlap functions.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions on overlap functions and n-dimensional overlap functions. We introduce the concept of multiplicative generators of n-dimensional overlap functions in Section 3. We study the homogeneity and idempotency property on multiplicatively generated n-dimensional overlap functions in Section 4. Finally, we end this paper with some conclusions.

2. Preliminaries

In this section, we recall some basic concepts of overlap functions and n-dimensional overlap functions, which shall be needed in the sequel.

Definition 2.1. (See Bustince *et al.* [1]) A bivariate function $O:[0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions:

(O_1) O is commutative;

(O_2) $O(x, y) = 0$ iff $xy = 0$;

(O_3) $O(x, y) = 1$ iff $xy = 1$;

(O_4) O is increasing;

(O_5) O is continuous.

Definition 2.2. (See Gómez *et al.* [4]) An n-dimensional aggregation function $\boldsymbol{O}:[0,1]^n \to [0,1]$ is an n-dimensional overlap function if and only if:

($\boldsymbol{O}_1$) $\boldsymbol{O}$ is symmetric.

($\boldsymbol{O}_2$) $\boldsymbol{O}(x_1,\cdots,x_n) = 0$ if and only if $\prod_{i=1}^{n} x_i = 0$.

($\boldsymbol{O}_3$) $\boldsymbol{O}(x_1,\cdots,x_n) = 1$ if and only if $x_i = 1$ for all $i \in \{1,\cdots,n\}$.

($\boldsymbol{O}_4$) $\boldsymbol{O}$ is increasing.

($\boldsymbol{O}_5$) $\boldsymbol{O}$ is continuous.

Definition 2.3. (See Dimuro *et al.* [5]) A function $F:[0,1] \to [0,1]$ is said to be a pseudo-automorphism if the following conditions hold:

(F_1) F is increasing;

(F_2) F is continuous;

(F_3) $F(x) = 1$ iff $x = 1$;

(F_4) $F(x) = 0$ iff $x = 0$.

A function $\varphi:[0,1] \to [0,1]$ is an automorphism if it is a continuous and strictly increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$ [7]. Obviously, any automorphism is a strictly increasing pseudo-automorphism.

3. Multiplicative Generators of n-Dimensional Overlap Functions

In this section, we try to extend the notion of multiplicative generators of overlap functions to the n-dimensional case, and characterize the basic properties of multiplicative generators of n-dimensional overlap functions.

Definition 3.1. Consider two continuous and increasing functions $\mathbf{g}, \mathbf{h}: [0,1] \to [0,1]$. The n-dimensional function $\boldsymbol{O}_{\mathbf{g,h}}: [0,1]^n \to [0,1]$ given by

$$\boldsymbol{O}_{\mathbf{g,h}}(x_1, x_2, \cdots, x_n) = \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)) \tag{1}$$

If $\boldsymbol{O}_{\mathbf{g,h}}$ is an n-dimensional overlap function, then $(\mathbf{g}, \mathbf{h})$ is said to be a multiplicative generator pair of the overlap function $\boldsymbol{O}_{\mathbf{g,h}}$ and $\boldsymbol{O}_{\mathbf{g,h}}$ is called multiplicatively generated by the pair $(\mathbf{g}, \mathbf{h})$.

Proposition 3.1. Consider two continuous and increasing functions $\mathbf{g}, \mathbf{h}: [0,1] \to [0,1]$ such that

1) $\mathbf{h}(x) = 0$ iff $x = 0$;
2) $\mathbf{h}(x) = 1$ iff $x = 1$;
3) $\mathbf{g}(x) = 0$ iff $x = 0$;
4) $\mathbf{g}(x) = 1$ iff $x = 1$.

Then, the n-dimensional function $\boldsymbol{O}_{\mathbf{g,h}}: [0,1]^n \to [0,1]$, given by

$$\boldsymbol{O}_{\mathbf{g,h}}(x_1, x_2, \cdots, x_n) = \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)) \tag{2}$$

is an n-dimensional overlap function.

Proof. We check out one by one that $\boldsymbol{O}_{\mathbf{g,h}}$ satisfies the conditions of Definition 2.2 as follows.

$(\boldsymbol{O}_1)$ The commutativity is obvious by the definition of $\boldsymbol{O}_{\mathbf{g,h}}$.

$(\boldsymbol{O}_2)$ $\boldsymbol{O}_{\mathbf{g,h}}(x_1, x_2, \cdots, x_n) = 0 \iff \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)) = 0$
$\iff \mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n) = 0$ by item (3)
$\iff \mathbf{h}(x_1) = 0$ or $\mathbf{h}(x_2) = 0$ or $\cdots$ or $\mathbf{h}(x_n) = 0$
$\iff x_1 = 0$ or $x_2 = 0$ or $\cdots$ or $x_n = 0$ by item (1)
$\iff x_1 x_2 \cdots x_n = 0$.

$(\boldsymbol{O}_3)$ $\boldsymbol{O}_{\mathbf{g,h}}(x_1, x_2, \cdots, x_n) = 1 \iff \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)) = 1$
$\iff \mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n) = 1$ by item (4)
$\iff \mathbf{h}(x_1) = 1$ and $\mathbf{h}(x_2) = 1$ and $\cdots$ and $\mathbf{h}(x_n) = 1$
$\iff x_1 = 1$ and $x_2 = 1$ and $\cdots$ and $x_n = 1$ by item (2).

$(\boldsymbol{O}_4)$ By the monotonicity of $\mathbf{g}$ and $\mathbf{h}$, it is easy to get that $\boldsymbol{O}_{\mathbf{g,h}}$ is increasing.

$(\boldsymbol{O}_5)$ From the continuities of $\mathbf{g}$ and $\mathbf{h}$, the continuity can be obtained immediately.

Proposition 3.2. Consider two continuous and increasing functions $\mathbf{g}, \mathbf{h}: [0,1] \to [0,1]$ such that

1) $\mathbf{g}(x) = 0$ iff $x = 0$;
2) $\mathbf{g}(x) = 1$ iff $x = 1$;
3) $\boldsymbol{O}_{\mathbf{g,h}}(x_1, x_2, \cdots, x_n) = \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n))$ is an n-dimensional overlap function.

Then the following statements hold:

1) $\mathbf{h}(x) = 0$ iff $x = 0$;

2) $\mathbf{h}(x) = 1$ iff $x = 1$.

Proof. 1) ($\Rightarrow$) If $\mathbf{h}(x) = 0$, then $\mathbf{h}(x)\mathbf{h}(x_1)\cdots\mathbf{h}(x_{n-1}) = 0$ for any $x_i \in (0,1]$ ($i = 1,\cdots,n-1$). Furthermore, by items (1) and (3), one can get that

$$\boldsymbol{O}_{\mathbf{g},\mathbf{h}}(x, x_1, \cdots, x_{n-1}) = \mathbf{g}(\mathbf{h}(x)\mathbf{h}(x_1)\cdots\mathbf{h}(x_{n-1})) = 0.$$

Thus, it follows that $x = 0$ from ($\boldsymbol{O}_2$).

($\Leftarrow$) If $x = 0$, then, by item (3), we can obtain that

$$\mathbf{g}(\mathbf{h}(x)\mathbf{h}(x)\cdots\mathbf{h}(x)) = \boldsymbol{O}_{\mathbf{g},\mathbf{h}}(x, x, \cdots, x) = 0.$$

Furthermore, one has that $\mathbf{h}(x) = 0$ by item (1). Hence, we get that $\mathbf{h}(x) = 0$ iff $x = 0$.

2) It can be verified in a similar way as item (1).

Proposition 3.3. Consider two continuous and increasing functions $\mathbf{g}, \mathbf{h} : [0,1] \rightarrow [0,1]$ and $\mathbf{g}, \mathbf{h}$ such that

1) $\mathbf{h}(x) = 0$ iff $x = 0$;

2) $\mathbf{h}(x) = 1$ iff $x = 1$;

3) $\boldsymbol{O}_{\mathbf{g},\mathbf{h}}(x_1, x_2, \cdots, x_n) = \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n))$ is an n-dimensional overlap function.

Then the following statements hold:

1) $\mathbf{g}(x) = 0$ iff $x = 0$;

2) $\mathbf{g}(x) = 1$ iff $x = 1$.

Proof. 1) ($\Rightarrow$) Since $\mathbf{h} : [0,1] \rightarrow [0,1]$ is continuous, we have that $\mathbf{h}^n : [0,1] \rightarrow [0,1]$ is continuous, where $\mathbf{h}^n$ is defined by $\mathbf{h}^n(x) = (\mathbf{h}(x))^n$ for all $x \in [0,1]$. Furthermore, we can obtain that for all $x \in [0,1]$, there exists $x' \in [0,1]$ such that $\mathbf{h}^n(x') = x$ by items (1), (2). Thus, if $\mathbf{g}(x) = 0$, then it follows that $\mathbf{g}(\mathbf{h}^n(x')) = 0$ for $x = \mathbf{h}^n(x')$. Moreover, by item (3), one can get that

$$\boldsymbol{O}_{\mathbf{g},\mathbf{h}}(x', x', \cdots, x') = \mathbf{g}(\mathbf{h}(x')\mathbf{h}(x')\cdots\mathbf{h}(x')) = \mathbf{g}(\mathbf{h}^n(x')) = 0.$$

Thus, using item (3) again, one has that $x' = 0$. Furthermore, using item (1) again, it follows that $x = \mathbf{h}^n(x') = \mathbf{h}^n(0) = 0$.

($\Leftarrow$) If $x = 0$, then, by item (3), it follows that

$$\mathbf{g}(\mathbf{h}(x)\mathbf{h}(x)\cdots\mathbf{h}(x)) = \boldsymbol{O}_{\mathbf{g},\mathbf{h}}(x, x, \cdots, x) = 0.$$

Furthermore, by item (1), one has that

$$\mathbf{g}(x) = \mathbf{g}(0) = \mathbf{g}(\mathbf{h}(x)\mathbf{h}(x)\cdots\mathbf{h}(x)) = 0.$$

Hence, we have that $\mathbf{g}(x) = 0$ iff $x = 0$.

2) It can be proven in a similar way as item (1).

Proposition 3.4. Suppose that $F : [0,1] \rightarrow [0,1]$ is a pseudo-automorphism. Then, for any n-dimensional overlap function $\boldsymbol{O} : [0,1]^n \rightarrow [0,1]$, the n-dimensional function $\boldsymbol{O}_F : [0,1]^n \rightarrow [0,1]$, given by

$$\boldsymbol{O}_F(x_1, x_2, \cdots, x_n) = F(\boldsymbol{O}(x_1, x_2, \cdots, x_n)) \tag{3}$$

is an n-dimensional overlap function.

Proof. We verify that $\boldsymbol{O}_F$ satisfies the conditions of Definition 2.2 one by one as follows.

($\boldsymbol{O}_1$) The symmetry is obvious by the definition of $\boldsymbol{O}_F$.

($\boldsymbol{O}_2$) $\boldsymbol{O}_F(x_1, x_2, \cdots, x_n) = 0 \quad \Leftrightarrow \quad F(\boldsymbol{O}(x_1, x_2, \cdots, x_n)) = 0$

$\Leftrightarrow \quad \boldsymbol{O}(x_1, x_2, \cdots, x_n) = 0$

$\Leftrightarrow \quad x_1 x_2 \cdots x_n = 0$.

($\boldsymbol{O}_3$) $\boldsymbol{O}_F(x_1, x_2, \cdots, x_n) = 1 \quad \Leftrightarrow \quad F(\boldsymbol{O}(x_1, x_2, \cdots, x_n)) = 1$

$\Leftrightarrow \quad \boldsymbol{O}(x_1, x_2, \cdots, x_n) = 1$

$\Leftrightarrow \quad x_1 x_2 \cdots x_n = 1$.

($\boldsymbol{O}_4$) Since F is increasing, one has that $\boldsymbol{O}_F$ is increasing immediately.

($\boldsymbol{O}_5$) The continuity can be obtained immediately from the continuities of F and $\boldsymbol{O}$.

Proposition 3.5. Suppose that $F:[0,1] \rightarrow [0,1]$ is a pseudo-automorphism. If ($\mathbf{g}$, $\mathbf{h}$) is a multiplicative generator pair of n-dimensional overlap function $\boldsymbol{O}:[0,1]^n \rightarrow [0,1]$, then ($F \circ \mathbf{g}$, $\mathbf{h}$) is a multiplicative generator pair of the n-dimensional overlap function $\boldsymbol{O}_F$ given in Proposition 3.4.

Proof. Since $\boldsymbol{O}$ is multiplicatively generated by the pair ($\mathbf{g}$, $\mathbf{h}$), we have that

$$\boldsymbol{O}(x_1, x_2, \cdots, x_n) = \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)).$$

for all $x, y \in [0,1]$. Moreover, by the definition of $\boldsymbol{O}_P$, it follows that for all $x, y \in [0,1]$,

$$\begin{aligned} \boldsymbol{O}_F(x_1, x_2, \cdots, x_n) &= F(\boldsymbol{O}(x_1, x_2, \cdots, x_n)) \\ &= F(\mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n))) \\ &= (F \circ \mathbf{g})(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)) \end{aligned}$$

Thus, since F is continuous and increasing, by Definition 3.1, we conclude that ($F \circ \mathbf{g}$, $\mathbf{h}$) is a multiplicative generator pair of the overlap function $\boldsymbol{O}_F$.

Proposition 3.6. Suppose that $F:[0,1] \rightarrow [0,1]$ is a pseudo-automorphism and $\boldsymbol{O}:[0,1]^n \rightarrow [0,1]$ is an n-dimensional overlap function. If ($\mathbf{g}$, $\mathbf{h}$) is a multiplicative generator pair of n-dimensional overlap function $\boldsymbol{O}_F$ given in Proposition 3.4, then ($F^{-1} \circ \mathbf{g}$, $\mathbf{h}$) is a multiplicative generator pair of the n-dimensional overlap function O.

Proof. Since O_F is multiplicatively generated by the pair ($\mathbf{g}$, $\mathbf{h}$), one has that

$$\boldsymbol{O}_F(x_1, x_2, \cdots, x_n) = \mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)).$$

for all $x, y \in [0,1]$. Moreover, by the definition of $\boldsymbol{O}_P$, it follows that for all $x, y \in [0,1]$,

$$\begin{aligned} \boldsymbol{O}(x_1, x_2, \cdots, x_n) &= (F^{-1} \circ F)(\boldsymbol{O}(x_1, x_2, \cdots, x_n)) \\ &= F^{-1}(F(\boldsymbol{O}(x_1, x_2, \cdots, x_n))) \\ &= F^{-1}(\boldsymbol{O}_F(x_1, x_2, \cdots, x_n)) \\ &= F^{-1}(\mathbf{g}(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n))) \\ &= (F^{-1} \circ \mathbf{g})(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots\mathbf{h}(x_n)) \end{aligned}$$

Thus, since F^{-1} is continuous and strictly increasing, by Definition 3.1, we conclude that $(F^{-1} \circ \mathbf{g}, \mathbf{h})$ is a multiplicative generator pair of the n-dimensional overlap function $\boldsymbol{O}$.

4. Homogeneity and Idempotency Property on Multiplicatively Generated n-Dimensional Overlap Functions

Proposition 4.1. Suppose that $\boldsymbol{O} : [0,1]^n \to [0,1]$ is an n-dimensional overlap function multiplicatively generated by the pair $(\mathbf{g}, \mathbf{h})$. If $\mathbf{h}$ is homogeneous of order k_1 and $\mathbf{g}$ is homogeneous of order k_2, then $\boldsymbol{O}$ is homogeneous of order nk_1k_2.

Proof. If h is homogeneous of order k_1 and $\mathbf{g}$ is homogeneous of order k_2, then we can obtain that

$$
\begin{aligned}
\boldsymbol{O}(\alpha x_1, \alpha x_2, \cdots, \alpha x_n) &= \mathbf{g}\big(\mathbf{h}(\alpha x_1)\mathbf{h}(\alpha x_2)\cdots \mathbf{h}(\alpha x_n)\big) \\
&= \mathbf{g}\big(\alpha^{nk_1}\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots \mathbf{h}(x_n)\big) \\
&= \big(\alpha^{nk_1}\big)^{k_2}\mathbf{g}\big(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots \mathbf{h}(x_n)\big) \\
&= \alpha^{nk_1k_2}\mathbf{g}\big(\mathbf{h}(x_1)\mathbf{h}(x_2)\cdots \mathbf{h}(x_n)\big) \\
&= \alpha^{nk_1k_2}\boldsymbol{O}(x_1, x_2, \cdots, x_n)
\end{aligned}
$$

Hence, it follows that $\boldsymbol{O}$ is homogeneous of order nk_1k_2.

Proposition 4.2. Suppose $F : [0,1] \to [0,1]$ is a k_1-homogeneous pseudo-automorphism and $\boldsymbol{O} : [0,1]^n \to [0,1]$ is an n-dimensional overlap function. Consider the following conditions:

1) $\boldsymbol{O}$ is homogeneous of order k_2;

2) $\boldsymbol{O}_F$ is homogeneous of order k_1k_2.

Then (1) $\Rightarrow$ (2), and if F is an automorphism, then (1) $\Leftrightarrow$ (2).

Proof. (1) implies (2): If $\boldsymbol{O}$ is homogeneous of order k_2, then we get that

$$
\begin{aligned}
\boldsymbol{O}_F(\alpha x_1, \alpha x_2, \cdots, \alpha x_n) &= F\big(\boldsymbol{O}(\alpha x_1, \alpha x_2, \cdots, \alpha x_n)\big) \\
&= F\big(\alpha^{k_2}\boldsymbol{O}(x_1, x_2, \cdots, x_n)\big) \\
&= \big(\alpha^{k_2}\big)^{k_1} F\big(\boldsymbol{O}(x_1, x_2, \cdots, x_n)\big) \\
&= \alpha^{k_1k_2}\boldsymbol{O}_F(x_1, x_2, \cdots, x_n)
\end{aligned}
$$

Hence, it follows that $\boldsymbol{O}_F$ is homogeneous of order k_1k_2.

Moreover, if F is an automorphism and $\boldsymbol{O}_F$ is homogeneous of order k_1k_2, then we prove item (1) as follows.

$$
\begin{aligned}
F\big(\alpha^{k_2}\boldsymbol{O}(x_1, x_2, \cdots, x_n)\big) &= \alpha^{k_1k_2} F\big(\boldsymbol{O}(x_1, x_2, \cdots, x_n)\big) \\
&= \alpha^{k_1k_2}\boldsymbol{O}_F(x_1, x_2, \cdots, x_n) \\
&= \boldsymbol{O}_F(\alpha x_1, \alpha x_2, \cdots, \alpha x_n) \\
&= F\big(\boldsymbol{O}(\alpha x_1, \alpha x_2, \cdots, \alpha x_n)\big)
\end{aligned}
$$

Hence, it follows that $\boldsymbol{O}(\alpha x_1, \alpha x_2, \cdots, \alpha x_n) = \alpha^{k_2}\boldsymbol{O}(x_1, x_2, \cdots, x_n)$, since F is

strictly increasing. Furthermore, we conclude that $\boldsymbol{O}$ is homogeneous of order k_2.

An element $x \in [0,1]$ is said to be an idempotent element of an n-dimensional function $A : [0,1]^n \to [0,1]$ if and only if $A(x, x, \cdots, x) = x$.

Proposition 4.3. Suppose that $\boldsymbol{O} : [0,1]^n \to [0,1]$ is an n-dimensional overlap function multiplicatively generated by the pair (**g**, **h**), where $\mathbf{g} : [0,1] \to [0,1]$ is given by $\mathbf{g}(x) = \sqrt[n]{x}$ for all $x \in [0,1]$. Consider the following conditions:

1) $\mathbf{h}(x_0) = x_0$ for some $x_0 \in [0,1]$;

2) x_0 is an idempotent element of $\boldsymbol{O}$.

Then (1) $\Leftrightarrow$ (2).

Proof. 1) $\Rightarrow$ (2): If x_0 is a fixed point of h, then we can obtain

$$
\begin{aligned}
\boldsymbol{O}(x_0, x_0, \cdots, x_0) &= \mathbf{g}\big(\mathbf{h}(x_0)\mathbf{h}(x_0)\cdots\mathbf{h}(x_0)\big) \\
&= \sqrt[n]{\mathbf{h}(x_0)\mathbf{h}(x_0)\cdots\mathbf{h}(x_0)} \\
&= \mathbf{h}(x_0) \\
&= x_0
\end{aligned}
$$

Hence, it follows that x_0 is an idempotent element of $\boldsymbol{O}$.

2) $\Rightarrow$ (1): If x_0 is an idempotent element of $\boldsymbol{O}$, then one can have

$$
\begin{aligned}
x_0 &= \boldsymbol{O}(x_0, x_0, \cdots, x_0) \\
&= \mathbf{g}\big(\mathbf{h}(x_0)\mathbf{h}(x_0)\cdots\mathbf{h}(x_0)\big) \\
&= \sqrt[n]{\mathbf{h}(x_0)\mathbf{h}(x_0)\cdots\mathbf{h}(x_0)} \\
&= \mathbf{h}(x_0)
\end{aligned}
$$

Hence, it follows that x_0 is a fixed point of **h**.

Corollary 4.1. Suppose that $\boldsymbol{O} : [0,1]^n \to [0,1]$ is an n-dimensional overlap function multiplicatively generated by the pair (**g**, **h**), where $\mathbf{h} : [0,1] \to [0,1]$ is given by $\mathbf{h}(x) = \sqrt[n]{x}$ for all $x \in [0,1]$. Consider the following conditions:

1) $\mathbf{g}(x_0) = x_0$ for some $x_0 \in [0,1]$;

2) x_0 is an idempotent element of $\boldsymbol{O}$.

Then (1) $\Leftrightarrow$ (2).

Proof. It can be proven in a similar way as Proposition 4.3.

Proposition 4.4. Suppose that $\boldsymbol{O} : [0,1]^2 \to [0,1]$ is an n-dimensional overlap function and $F : [0,1] \to [0,1]$ is a pseudo-automorphism with $F(x_0) = x_0$ for some $x_0 \in [0,1]$. Consider the following conditions:

1) x_0 is an idempotent element of $\boldsymbol{O}$;

2) x_0 is an idempotent element of $\boldsymbol{O}_F$;

Then (1) $\Rightarrow$ (2), and if F is an automorphism, then (1) $\Leftrightarrow$ (2).

Proof. (1) $\Rightarrow$ (2): If x_0 is an idempotent element of $\boldsymbol{O}$, then we can obtain

$$
\begin{aligned}
\boldsymbol{O}_F(x_0, x_0, \cdots, x_0) &= F\big(\boldsymbol{O}(x_0, x_0, \cdots, x_0)\big) \\
&= F(x_0) \\
&= x_0
\end{aligned}
$$

Hence, it holds that x_0 is an idempotent element of $\boldsymbol{O}_F$.

Moreover, if F is an automorphism and x_0 is an idempotent element of $\boldsymbol{O}_P$, then we prove item (1) as follows.

$$
\begin{aligned}
F(x_0) &= x_0 \\
&= \boldsymbol{O}_F(x_0, x_0, \cdots, x_0) \\
&= F(\boldsymbol{O}(x_0, x_0, \cdots, x_0))
\end{aligned}
$$

Hence, we get that $x_0 = \boldsymbol{O}(x_0, x_0, \cdots, x_0)$, sine F is strictly increasing. Moreover, we conclude that x_0 is an idempotent element of $\boldsymbol{O}$.

5. Conclusion

In this paper, we mainly extend the notions of multiplicative generator pairs of overlap functions to n-dimensional case. We propose some basic properties on multiplicatively generated n-dimensional overlap functions, such as the homogeneity and idempotency property. In a similar way, one can also study the multiplicative generator pairs of n-dimensional grouping functions by the duality of n-dimensional overlap and grouping functions.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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