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On the Nonlinear Neutral Conformable Fractional Integral-Differential Equation

Rui Li, Wei Jiang, Jiale Sheng, Sen Wang

School of Mathematical Sciences, Anhui University, Hefei, China
Email: jiangwei@ahu.edu.cn

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Abstract

In this paper, we investigate the nonlinear neutral fractional integral-differential equation involving conformable fractional derivative and integral. First of all, we give the form of the solution by lemma. Furthermore, existence results for the solution and sufficient conditions for uniqueness solution are given by the Leray-Schauder nonlinear alternative and Banach contraction mapping principle. Finally, an example is provided to show the application of results.

Keywords

Conformable Fractional Derivative, Delay, Existence and Uniqueness, Functional Differential Equation

1. Introduction

The theory of fractional calculus has played a major role in control theory, fluid dynamics, biological systems, economics and other fields [1] [2] [3]. It serves as a valuable tool for the description of memory and hereditary properties of various materials and processes. In recent years, plenty of interesting results have been observed for the Riemann-Liouville and Caputo type fractional derivatives.

The definition of the conformable fractional derivative and integral was introduced in 2014 by Khalil *et al.* [4]. Compared with Riemann-Liouville and Caputo type fractional derivatives, the conformable fractional derivative satisfies the Leibniz rule and chain rule, and can be converted to classical derivative [5]. This is of great help to study fractional differential equations. In the past few years, the conformable fractional derivative has been used in the field of fractional newtonian mechanics, heat equation, biology and so on, and the results are abundant. The conformable fractional optimal control problems with time-delay were studied, which proved that the embedding method, embedding the admissible set into a subset of measures, can be successfully applied to nonlinear problems [6]. Alharbi *et al.* utilized the homotopy perturbation method to solve a model of Ambartsumian equation with the conformable derivative and gave the approximate solution of equation [7]. The conformable fractional derivative was utilized to solve the time-fractional Burgers type equations approximately [8].

Over these years, there has been a significant development in fractional functional differential equations. Among them, Li, Liu and Jiang gave sufficient conditions of the existence of positive solutions for a class of nonlinear fractional differential equations with caputo derivative [9]. Guo *et al.* studied fractional functional differential equation with impulsive, then they obtained existence, uniqueness, and data-dependent results of solutions to the equation [10]. The existence of positive periodic solutions was given by Zhang and Jiang, for n-dimensional impulsive periodic functional differential equations [11]. In addition, the fractional stochastic functional system driven by Rosenblatt process was investigated by Shen *et al.*, and they obtained controllability and stability results [12].

Time-delay is part of the theoretical fields investigated by many authors, including unbounded time-delay, bounded time-delay, state-dependent time-delay and others. In 2009, the existence and uniqueness of solutions of the Caputo fractional neutral differential equations with unbounded delays were discussed [13]. In 2011, Li and Zhang considered the Caputo fractional neutral integral-differential equations with unbounded delay, which used the fixed point theorem to study the existence of mild solutions of equations [14]. The caputo fractional neutral integral-differential equations with unbounded delay were discussed in 2013 [15]. The paper used Monch's fixed point theorem via measures of non-compactness to study the existence of solutions of equations.

Based on the above research background and relevant discussions, we found that few people used conformable derivative to study fractional differential equations with time-delay. In 2019, Mohamed I. Abbas gave the existence of solutions and uniqueness of solution for fractional neutral integro-differential equations by the Hadamard fractional derivative of order $\alpha \in (0,1)$ and the Riemann-Liouville integral [16]. In this paper, we will discuss the nonlinear neutral fractional integral-differential equation in the frame of the conformable derivative of order $\alpha \in (1,2)$ and the comformable integral. Then we make the condition 3 weaker to improve feasibility. Considering the following equation:

$$\begin{cases} T^{\alpha}\left[w(t) - \sum_{i=1}^{p} I^{\beta_i} u_i(t, w_t) \right] = l(t, w_t), \ t \in [0, \rho], \\ w(t) = \psi(t), \ t \in [-\upsilon, 0], \end{cases} \tag{1.1}$$

where T^{α} denotes the conformable fractional derivative of order α, $1 < \alpha < 2$, I^{β_i} denotes the conformable fractional integral with order β_i, $\beta_i \in (0,1)$, $i = 1, 2, 3, \cdots, p$, $p \in N_+$. $\rho, \upsilon > 0$ are constants. And for any $t \in [0, \rho]$, we denote by w_t the element of $C([-\upsilon, 0], R)$ and is defined by $w_t(\theta) = w(t+\theta)$, $\theta \in [-\upsilon, 0]$. Here $w_t(\cdot)$ represents the history of the state from time $t-\upsilon$ up to the present time *t*. $l, u_i : [0, \rho] \times C([-\upsilon, 0], R) \to R$ are continuous functions that satisfy some hypotheses given later, $\psi \in C([-\upsilon, 0], R)$.

The rest of this paper is organized as follows: In Section 2, we introduce the concepts and basic properties of conformable fractional integral and derivative. In Section 3, we give existence results for the solution and sufficient conditions for uniqueness solution by Leray-Schauder nonlinear alternative and Banach contraction mapping principle. In Section 4, the numerical simulation is showed to illustrate the results.

Notations: $C([0,\rho],R)$ denotes all continuous functions that mapped from $[0,\rho]$ to R and R denotes all real numbers. R^+ denotes all positive real numbers.

2. Preliminaries

In this section, we present some necessary definitions and lemmas to establish our main results.

Definition 2.1. ([5]) For a function $w:[a,+\infty)\to R$, the conformable fractional integral of order $\alpha\,(n<\alpha\le n+1, n\in N)$ of the function w is defined as follows:

$$I_a^\alpha w(t)=I_a^{n+1}\left[(t-a)^{\alpha-n-1}w(t)\right]=\frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\alpha-n-1} w(x)\mathrm{d}x.$$

If $a=0$, $I_a^\alpha w(t)$ can be written as $I^\alpha w(t)$.

Definition 2.2. ([5]) For a function $w:[a,+\infty)\to R$, the conformable fractional derivative of order $\alpha\,(n<\alpha\le n+1, n\in N)$ of the function w is defined as follows:

$$T_a^\alpha w(t)=\lim_{\varepsilon\to 0}\frac{w^{[\alpha]-1}\left(t+\varepsilon(t-a)^{[\alpha]-\alpha}\right)-w^{[\alpha]-1}(t)}{\varepsilon},$$

where $[\alpha]$ denotes the smallest integer greater than or equal to α. If $a=0$, $T_a^\alpha w(t)$ can be written as $T^\alpha w(t)$.

Lemma 2.3 ([4]) If function l is α times differential at a point $t>0$ for $n<\alpha\le n+1$, $n\in N$, then

1) $T^\alpha l(t)=0$, for all constant functions $l(t)=\lambda$;

2) l is $n+1$ times differential simultaneously, then $T^\alpha(l)(t)=t^{[\alpha]-\alpha}l^{[\alpha]}(t)$.

Lemma 2.4. ([5])

1) For a function $w:[a,+\infty)\to R$, if $w^n(t)$ is continuous, then for any $t>a$, we have

$$T_a^\alpha I_a^\alpha w(t)=w(t), \alpha\in(n,n+1];$$

2) For a function $w:[a,+\infty)\to R$, if w is $n+1$ times differentiable, then for any $t>a$, we have

$$I_a^\alpha T_a^\alpha w(t)=w(t)-\sum_{i=0}^{n}\frac{w^{(i)}(a)(t-a)^i}{i!}, \alpha\in(n,n+1].$$

Lemma 2.5. ([4]) If $\alpha\in(0,1]$, functions w_1, w_2 are α times differentiable at a point $t>0$, then

1) $T^\alpha(a_1w_1+a_2w_2)=a_1T^\alpha(w_1)+a_2T^\alpha(w_2)$;

2) $T^{\alpha}\left(t^{p}\right)=pt^{p-\alpha}$;

3) $T^{\alpha}\left(w_1w_2\right)=w_1T^{\alpha}\left(w_2\right)+w_2T^{\alpha}\left(w_1\right)$;

4) $T^{\alpha}\left(\dfrac{w_1}{w_2}\right)=\dfrac{w_2T^{\alpha}\left(w_1\right)-w_1T^{\alpha}\left(w_2\right)}{\left(w_2\right)^2}$.

Lemma 2.6. ([17]) (The nonlinear alternative of Leray-Schauder type) Let C be a Banach space, C_1 be a closed, convex subset of C, and P be an open subset of C_1, $0\in P$. If $A:\overline{P}\to C_1$ is a continuous, compact map. Then

1) A has a fixed point in P, or

2) There is a $w\in\partial P$ (the boundary of P in C_1) and $\lambda\in(0,1)$, such that $w=\lambda A(w)$.

Lemma 2.7. Let $l(t)$ be a continuous function, then the fractional differential equation

$$\begin{cases}T^{\alpha}\left[w(t)-\eta(t)\right]=l(t),t\in[0,\rho],1<\alpha<2,\\ w(0)=\psi_0,w'(0)=\psi_0',\end{cases}\tag{2.1}$$

is equivalent to the integral-differential equation

$$w(t)=\psi_0-\eta(0)+\left[\psi_0'-\eta'(0)\right]t+\eta(t)+\int_0^t(t-s)s^{\alpha-2}l(s)\mathrm{d}s,\tag{2.2}$$

where $\psi_0=w(0)$, $\psi_0'=w'(0)$.

Proof. Consider Equation (2.1), for any $t\in[0,\rho]$,

$$T^{\alpha}\left[w(t)-\eta(t)\right]=l(t).$$

Transforming α times conformable fractional integral on both sides of the equation and using Lemma 2.4, since $1<\alpha<2$, we have

$$w(t)=\psi_0-\eta(0)+\left[\psi_0'-\eta'(0)\right]t+\eta(t)+\int_0^t(t-s)s^{\alpha-2}l(s)\mathrm{d}s,$$

where $\psi_0=w(0)$, $\psi_0'=w'(0)$. The proof is completed. □

3. Main Results

In this section, we give several results about the fractional integral-differential Equation (1.1).

If $X=\left\{w\mid w\in C\left([-\upsilon,\rho],\mathbb{R}\right)\right\}$ is Banach Space, the norm is defined as $\|w\|=\sup\left\{|w(t)|,t\in[-\upsilon,\rho]\right\}$.

Let $\eta(t)=\sum_{i=1}^{p}I^{\beta_i}u_i(t,w_t)$. Giving the definition of the operator $A:X\to X$,

$$Aw(t)=\begin{cases}\psi_0+\left[\psi_0'-\eta'(0)\right]t+\sum_{i=1}^{p}I^{\beta_j}u_i(t,w_t)+\int_0^t(t-s)s^{\alpha-2}l(s,w_s)\mathrm{d}s,\ t\in[0,\rho],\\ \psi(t),\ t\in[-\upsilon,0].\end{cases}\tag{3.1}$$

where $\psi_0=w(0)$, $\psi_0'=w'(0)$, $w_0=w(0+\theta)=\psi(\theta)$, $\theta\in[-\upsilon,0]$, $\eta(0)=\left.\sum_{i=1}^{p}I^{\beta_i}u_i(t,w_t)\right|_{t=0}=0$.

It should be noticed that Equation (1.1) has solutions if and only if the operator A has fixed points. So as to achieve the desired goals, we impose the following assumptions for the Equation (1.1).

(H_1) There exist functions $\gamma(t), \delta_i(t):[0,\rho]\to R^+$ and continuous non decreasing functions $\zeta(t), \phi_i(t):[0,\infty]\to[0,\infty]$, such that, for any $(t,w_t)\in[0,\rho]\times C([-\upsilon,0],R)$,

$$|l(t,w_t)|\le\gamma(t)\zeta(\|w_t\|),$$
$$|u_i(t,w_t)|\le\delta_i(t)\phi_i(\|w_t\|).$$

(H_2) Functions $l,u_i:[0,\rho]\times C([-\upsilon,0],R)\to R$ are continuous. There exist positive functions λ_i,μ with bounds $\|\lambda_i\|,\|\mu\|$, $i=1,2,\cdots,p$, $p\in N_+$, respectively such that

$$|u_i(t,x)-u_i(t,y)|\le\lambda_i(t)\|x-y\|,$$
$$|l(t,x)-l(t,y)|\le\mu(t)\|x-y\|.$$

(H_3) There exists constant $M>0$, $\upsilon_1<\upsilon$, such that

$$\frac{M}{|\psi_0|+[\psi_0'-\eta'(0)]\rho+\sum_{i=1}^{p}\|\delta_i\|\phi_i(\upsilon_1)\frac{\rho^{\beta_i}}{\beta_i}+\|\gamma\|\zeta(\upsilon_1)\frac{\rho^\alpha}{\alpha(\alpha-1)}}>1.$$

(H_4)

$$\sum_{i=1}^{p}\|\lambda_i\|\frac{\rho^{\beta_i}}{\beta_i}+\|\mu\|\frac{\rho^\alpha}{\alpha(\alpha-1)}<1.$$

We give an existence result based on the nonlinear alternative of Leray-Schauder type applied to a completely continuous operator.

Theorem 3.1. *Suppose that the assumptions* (H_1)-(H_3) *are satisfied, then the* Equation (1.1) *has at least one solution.*

Proof. The operator A is defined as (3.1). Define

$$D=\{w\in C([-\upsilon,\rho],R):\|w_t\|\le\upsilon_1\}.$$

Firstly, we prove that operator A is uniformly bounded. For any $(t,w_t)\in[0,\rho]\times C([-\upsilon,0],R)$, $i=1,2,\cdots,p$, $p\in N_+$, and $w\in D$, by (H_1), we have

$$\begin{aligned}|Aw(t)|&=\left|\psi_0+[\psi_0'-\eta'(0)]t+\sum_{i=1}^{p}I^{\beta_i}u_i(s,w_s)+\int_0^t(t-s)s^{\alpha-2}l(s,w_s)\,ds\right|\\&\le|\psi_0|+|\psi_0'-\eta'(0)|t+\sum_{i=1}^{p}I^{\beta_i}|u_i(s,w_s)|+\int_0^t(t-s)s^{\alpha-2}|l(s,w_s)|\,ds\\&\le|\psi_0|+|\psi_0'-\eta'(0)|t+\sum_{i=1}^{p}I^{\beta_i}\delta_i(t)\phi_i(\|w_t\|)+\int_0^t(t-s)s^{\alpha-2}\gamma(t)\zeta(\|w_t\|)\,ds\\&\le|\psi_0|+|\psi_0'-\eta'(0)|t+\sum_{i=1}^{p}\|\delta_i\|\phi_i(\upsilon_1)\int_0^t s^{\beta_i-1}ds+\|\gamma\|\zeta(\upsilon_1)\int_0^t(t-s)s^{\alpha-2}ds\\&\le|\psi_0|+|\psi_0'-\eta'(0)|t+\sum_{i=1}^{p}\|\delta_i\|\phi_i(\upsilon_1)\frac{t^{\beta_i}}{\beta_i}+\|\gamma\|\zeta(\upsilon_1)\frac{t^\alpha}{\alpha(\alpha-1)}\\&\le|\psi_0|+|\psi_0'-\eta'(0)|\rho+\sum_{i=1}^{p}\|\delta_i\|\phi_i(\upsilon_1)\frac{\rho^{\beta_i}}{\beta_i}+\|\gamma\|\zeta(\upsilon_1)\frac{\rho^\alpha}{\alpha(\alpha-1)}<M.\end{aligned}$$

For any $t\in[-\upsilon,0]$ and $w\in D$, we have

$$|Aw(t)| = |\psi(t)| \le \|\psi\|.$$

Denote $M_1 = \max\{M, \|\psi\|\}$, then

$$\|Aw(t)\| \le M_1, t \in [-\upsilon, \rho], w \in D.$$

This implies that the operator A is uniformly bounded in D.

Besides, we need to prove that AD is an equicontinuous set. Let w^n be a sequence such that $\lim_{n\to\infty} w^n = w$ in D. Then, for any $t \in [0, \rho]$, we have

$$\lim_{n\to\infty} Aw^n(t) = \psi_0 + [\psi_0' - \eta'(0)]t + \lim_{n\to\infty} \sum_{i=1}^{p} \int_0^t s^{\beta_i - 1} u_i(s, w_s^n) ds + \lim_{n\to\infty} \int_0^t (t-s) s^{\alpha-2} l(s, w_s^n) ds.$$

By (H_2), functions l, u_i are uniformly continuous, thus, we have

$$\begin{aligned}\lim_{n\to\infty} Aw^n(t) &= \psi_0 + [\psi_0' - \eta'(0)]t + \sum_{i=1}^{p} \int_0^t \lim_{n\to\infty} s^{\beta_i - 1} u_i(s, w_s^n) ds \\ &\quad + \int_0^t \lim_{n\to\infty} (t-s) s^{\alpha-2} l(s, w_s^n) ds \\ &= \psi_0 + [\psi_0' - \eta'(0)]t + \sum_{i=1}^{p} \int_0^t s^{\beta_i - 1} u_i(s, w_s) ds \\ &\quad + \int_0^t (t-s) s^{\alpha-2} l(s, w_s) ds \\ &= Aw(t).\end{aligned}$$

For any $t \in [-\upsilon, 0]$, it is obvious that

$$\lim_{n\to\infty} Aw^n(t) = \psi(t) = Aw(t).$$

Therefore, $Aw(t)$ is continuous and uniformly continuous for $t \in [-\upsilon, \rho]$, which implies that $Aw(t)$ is equicontinuous for $t \in [-\upsilon, \rho]$, A is continuous in $C([-\upsilon, \rho], R)$.

Furthermore, we consider $|Aw(t_2) - Aw(t_1)|$, for any $t_1, t_2 \in [-\upsilon, \rho]$, $t_1 < t_2$.

Case 1. If $0 \le t_1 < t_2 \le \rho$, for any $(t, w_t) \in [0, \rho] \times C([-\upsilon, 0], R)$ and $w \in D$, $i = 1, 2, \cdots, p$, we have

$$\begin{aligned}&|Aw(t_2) - Aw(t_1)| \\ &= \Big|\psi_0 + [\psi_0' - \eta'(0)]t_2 + \sum_{i=1}^{p} \int_0^{t_2} s^{\beta_i - 1} u_i(s, w_s) ds \\ &\quad + \int_0^{t_2} (t_2 - s) s^{\alpha-2} l(s, w_s) ds - \psi_0 - [\psi_0' - \eta'(0)]t_1 \\ &\quad - \sum_{i=1}^{p} \int_0^{t_1} s^{\beta_i - 1} u_i(s, w_s) ds - \int_0^{t_1} (t_1 - s) s^{\alpha-2} l(s, w_s) ds\Big| \\ &\le |\psi_0' - \eta'(0)|(t_2 - t_1) + \sum_{i=1}^{p} \int_{t_1}^{t_2} s^{\beta_i - 1} |u_i(s, w_s)| ds \\ &\quad + \int_0^{t_1} (t_2 - t_1) s^{\alpha-2} |l(s, w_s)| ds + \int_{t_1}^{t_2} (t_2 - s) s^{\alpha-2} |l(s, w_s)| ds \\ &\le |\psi_0' - \eta'(0)|(t_2 - t_1) + \sum_{i=1}^{p} \int_{t_1}^{t_2} s^{\beta_i - 1} \delta_i(t) \phi_i(\|w_t\|) ds \\ &\quad + \int_0^{t_1} (t_2 - t_1) s^{\alpha-2} \gamma(t) \zeta(\|w_t\|) ds + \int_{t_1}^{t_2} (t_2 - s) s^{\alpha-2} \gamma(t) \zeta(\|w_t\|) ds\end{aligned}$$

$$\leq \left|\psi_0' - \eta'(0)\right|(t_2 - t_1) + \sum_{i=1}^{p} \|\delta_i\| \phi_i(\upsilon_1) \int_{t_1}^{t_2} s^{\beta_i - 1} \mathrm{d}s$$
$$+ \|\gamma\| \zeta(\upsilon_1) \int_0^{t_1} (t_2 - t_1) s^{\alpha-2} \mathrm{d}s + \|\gamma\| \zeta(\upsilon_1) \int_{t_1}^{t_2} (t_2 - s) s^{\alpha-2} \mathrm{d}s$$
$$\leq \left|\psi_0' - \eta'(0)\right|(t_2 - t_1) + \sum_{i=1}^{p} \|\delta_i\| \phi_i(\upsilon_1) \frac{t_2^{\beta_i} - t_1^{\beta_i}}{\beta_i} + \|\gamma\| \zeta(\upsilon_1) \frac{t_2^{\alpha} - t_1^{\alpha}}{\alpha(\alpha-1)}.$$

If $t_2 - t_1 \to 0$, then $\left|Aw(t_2) - Aw(t_1)\right| \to 0$.

Case 2. If $-\upsilon < t_1 < 0 \leq t_2 < \rho$, for any $(t, w_t) \in [0, \rho] \times C([-\upsilon, 0], R)$ and $w \in D$, $i = 1, 2, \cdots, p$, $Aw(t_1) = \psi(t_1)$ holds for any $-\upsilon < t_1 < 0$, we have

$$\left|Aw(t_2) - Aw(t_1)\right|$$
$$= \left|\psi_0 + \left[\psi_0' - \eta'(0)\right] t_2 + \sum_{i=1}^{p} \int_0^{t_2} s^{\beta_i - 1} u_i(s, w_s) \mathrm{d}s \right.$$
$$\left. + \int_0^{t_2} (t_2 - s) s^{\alpha-2} l(s, w_s) \mathrm{d}s - \psi(t_1)\right|$$
$$\leq \left[\psi_0' - \eta'(0)\right] t_2 + \left|\psi_0 - \psi(t_1)\right| + \sum_{i=1}^{p} \int_0^{t_2} s^{\beta_i - 1} \left|u_i(s, w_s)\right| \mathrm{d}s$$
$$+ \int_0^{t_2} (t_2 - s) s^{\alpha-2} \left|l(s, w_s)\right| \mathrm{d}s$$
$$\leq \left[\psi_0' - \eta'(0)\right] t_2 + \left|\psi_0 - \psi(t_1)\right| + \sum_{i=1}^{p} \|\delta_i\| \phi_i(\upsilon_1) \int_0^{t_2} s^{\beta_i - 1} \mathrm{d}s$$
$$+ \|\gamma\| \zeta(\upsilon_1) \int_0^{t_2} (t_2 - s) s^{\alpha-2} \mathrm{d}s$$
$$\leq \left[\psi_0' - \eta'(0)\right] t_2 + \left|\psi_0 - \psi(t_1)\right| + \sum_{i=1}^{p} \|\delta_i\| \phi_i(\upsilon_1) \frac{t_2^{\beta_i}}{\beta_i}$$
$$+ \|\delta\| \zeta(\upsilon_1) \frac{t_2^{\alpha}}{\alpha(\alpha-1)}.$$

Since $\psi(t)$ is a continuous function, if $t_1 \to 0$, then we have $\psi(t_1) \to \psi_0$. If $t_2 - t_1 \to 0$, then $\left|Aw(t_2) - Aw(t_1)\right| \to 0$.

Case 3. If $-\upsilon \leq t_1 < t_2 < 0$, for any $-\upsilon \leq t_1 < t_2 < 0$ and $w \in D$,

$$\left|Aw(t_2) - Aw(t_1)\right| = \left|\psi(t_2) - \psi(t_1)\right|.$$

Since $\psi(t)$ is continuous function, if $t_2 - t_1 \to 0$, then $\left|Aw(t_2) - Aw(t_1)\right| \to 0$.

From what has been discussed above, AD is equicontinuous. By Arzelá-Ascoli theorem, AN is compact, then A is completely continuous on X.

For any $(t, w_t) \in [0, \rho] \times C([-\upsilon, 0], R)$ and $t \in [0, \rho]$, $i = 1, 2, \cdots, p$, we have

$$w(t) = \psi_0 + \left[\psi_0' - \eta'(0)\right] t + \sum_{i=1}^{p} I^{\beta_i} u_i(s, w_s) + \int_0^t (t - s) s^{\alpha-2} l(s, w_s) \mathrm{d}s,$$

For any $t \in [0, \rho]$, by assumption (H_1), we have

$$\left|w(t)\right| \leq \left|\psi_0\right| + \left|\psi_0' - \eta'(0)\right| t + \sum_{i=1}^{p} I^{\beta_i} \left|u_i(s, w_s)\right| + \int_0^t (t - s) s^{\alpha-2} \left|l(s, w_s)\right| \mathrm{d}s$$
$$\leq \left|\psi_0\right| + \left|\psi_0' - \eta'(0)\right| t + \sum_{i=1}^{p} I^{\beta_i} \delta_i(t) \phi_i(\|w_t\|) + \int_0^t (t - s) s^{\alpha-2} \gamma(t) \zeta(\|w_t\|) \mathrm{d}s$$
$$\leq \left|\psi_0\right| + \left|\psi_0' - \eta'(0)\right| t + \sum_{i=1}^{p} \|\delta_i\| \phi_i(\upsilon_1) \int_0^t s^{\beta_i - 1} \mathrm{d}s + \|\gamma\| \zeta(\upsilon_1) \int_0^t (t - s) s^{\alpha-2} \mathrm{d}s$$

$$\le \left|\psi_0\right| + \left|\psi_0' - \eta'(0)\right| t + \sum_{i=1}^{p} \left\|\delta_i\right\| \phi_i(\upsilon_1) \frac{t^{\beta_i}}{\beta_i} + \left\|\gamma\right\| \zeta(\upsilon_1) \frac{t^{\alpha}}{\alpha(\alpha-1)}$$

$$\le \left|\psi_0\right| + \left|\psi_0' - \eta'(0)\right| \rho + \sum_{i=1}^{p} \left\|\delta_i\right\| \phi_i(\upsilon_1) \frac{\rho^{\beta_i}}{\beta_i} + \left\|\gamma\right\| \zeta(\upsilon_1) \frac{\rho^{\alpha}}{\alpha(\alpha-1)}.$$

Consider the assumption (H_3), there exists $M \neq \left\|w(t)\right\|$. Define set $P = \left\{w \in C([-\upsilon, \rho], R) : \left\|w(t)\right\| < M\right\}$. We can show that $A : \overline{P} \to D$ is continuous and completely continuous. Assuming there exists $w \in \partial P$ and $\lambda \in (0,1)$, such that $w = \lambda A(w)$. Then we have $\left|\lambda\right| = \frac{\|w\|}{\|A(w)\|} \ge 1$.

By Lemma 2.6, A has a fixed point in P, which implies that there exists at least one solution to the Equation (1.1). The proof is completed. □

We will give the uniqueness result of solutions of Equation (1.1):

Theorem 3.2. *Suppose that the assumptions* (H_2) *and* (H_4) *are satisfied, then the* Equation (1.1) *has a unique solution.*

Proof. The operator A is defined as (3.1). For any $t \in [0, \rho]$ and $w^1, w^2 \in C([-\upsilon, \rho], R)$, by ($H_2$), we have

$$\begin{aligned}
&\left|Aw^1(t) - Aw^2(t)\right| \\
&= \Bigg|\psi_0 + \left[\psi_0' - \eta'(0)\right] t + \sum_{i=1}^{p} I^{\beta_i} u_i\left(s, w^1 s\right) \\
&\quad + \int_0^t (t-s) s^{\alpha-2} l\left(s, w_s^1\right) \mathrm{d}s - \psi_0 - \left[\psi_0' - \eta'(0)\right] t \\
&\quad - \sum_{i=1}^{p} I^{\beta_i} u_i\left(s, w^2 s\right) - \int_0^t (t-s) s^{\alpha-2} l\left(s, w_s^2\right) \mathrm{d}s\Bigg| \\
&\le \sum_{i=1}^{p} I^{\beta_i} \left|u_i\left(s, w_s^1\right) - u_i\left(s, w_s^2\right)\right| + \int_0^t (t-s) s^{\alpha-2} \left|l\left(s, w_s^1\right) - l\left(s, w_s^2\right)\right| \mathrm{d}s \\
&\le \sum_{i=1}^{p} I^{\beta_i} \lambda_i(t) \left\|w^1 - w^2\right\| + \int_0^t (t-s) s^{\alpha-2} \mu(t) \left\|w^1 - w^2\right\| \mathrm{d}s \\
&\le \left\|w^1 - w^2\right\| \left(\sum_{i=1}^{p} \int_0^t s^{\beta_i - 1} \left\|\lambda_i\right\| \mathrm{d}s + \int_0^t (t-s) s^{\alpha-2} \left\|\mu\right\| \mathrm{d}s\right) \\
&\le \left\|w^1 - w^2\right\| \left(\sum_{i=1}^{p} \left\|\lambda_i\right\| \frac{t^{\beta_i}}{\beta_i} + \left\|\mu\right\| \frac{t^{\alpha}}{\alpha(\alpha-1)}\right) \\
&\le \left\|w^1 - w^2\right\| \left(\sum_{i=1}^{p} \left\|\lambda_i\right\| \frac{\rho^{\beta_i}}{\beta_i} + \left\|\mu\right\| \frac{\rho^{\alpha}}{\alpha(\alpha-1)}\right).
\end{aligned}$$

For any $t \in [-\upsilon, 0]$,

$$\left|Aw^1(t) - Aw^2(t)\right| = \left|\psi(t) - \psi(t)\right| = 0.$$

By (H_4), A is a contraction mapping, then Equation (1.1) has a unique solution. The proof is completed. □

4. An Illustrative Example

This section presents an example where we apply Theorems 3.1 and 3.2 to some particular cases.

Example 4.1. *Consider the fractional integral-differential equation*

$$\begin{cases} T^{\frac{3}{2}}\left[w(t)-\sum_{i=1}^{3} I^{\frac{i}{4}} \dfrac{|w_t|}{(20i+6t)(1+|w_t|)}\right]=\dfrac{1}{16-t^2}\left(\dfrac{|w_t|}{3(1+|w_t|)}+\dfrac{1}{5}\right), \ t\in[0,2], \\ w(t)=\psi(t), \ t\in\left[-\dfrac{1}{2},0\right]. \end{cases} \tag{4.1}$$

where $T^{\frac{3}{2}}$ denotes the conformable fractional derivative of order $\frac{3}{2}$, $I^{\frac{i}{4}}$ denotes the conformable fractional integral of the order $\frac{i}{4}$, $i=1,2,3$. If $w(t):\left[-\frac{1}{2},2\right]\to R$, then for any $t\in[0,2]$, we define $w_t(\theta)=w(t+\theta)$, $\theta\in\left[-\frac{1}{2},0\right]$. Functions

$$u_i(t,w_t)=\frac{|w_t|}{(20i+6t)(1+|w_t|)},$$

$$l(t,w_t)=\frac{1}{16-t^2}\left(\frac{|w_t|}{3(1+|w_t|)}+\frac{1}{5}\right).$$

The continuous function $\psi(t)$ satisfies the condition that $\psi(0)=\psi'(0)=0$.

For any $(t,w_t)\in[0,2]\times C\left(\left[-\frac{1}{2},0\right],R\right)$, $i=1,2,3$, we have

$$|u_i(t,w_t)|=\left|\frac{|w_t|}{(20i+6t)(1+|w_t|)}\right|\le\frac{1}{20i+6t}\le\frac{1}{20}.$$

For any $(t,w_t)\in[0,2]\times C\left(\left[-\frac{1}{2},0\right],R\right)$, we have

$$|l(t,w_t)|=\left|\frac{1}{16-t^2}\left(\frac{|w_t|}{3(1+|w_t|)}+\frac{1}{5}\right)\right|\le\frac{2}{45}.$$

For any $(t,w_t)\in[0,2]\times C\left(\left[-\frac{1}{2},0\right],R\right)$, $\psi(0)=\psi'(0)=\eta(0)=\eta'(0)=0$, if $M>0.659$, we have

$$\frac{M}{\sum_{i=1}^{3}\frac{1}{20}\frac{2^{\frac{i}{4}}}{\frac{i}{4}}+\frac{2}{45}\frac{2^{\frac{3}{2}}}{\frac{3}{2}\left(\frac{3}{2}-1\right)}}>1, i=1,2,3.$$

Consequently, by Theorem 3.1, the Equation (4.1) has at least one solution.

For continuous functions column $u_i(t,w_t^1)$, $u_i(t,w_t^2):[0,2]\times C\left(\left[-\frac{1}{2},0\right],R\right)$, $i=1,2,3$, we have

$$\left|u_i(t,w_t^1)-u_i(t,w_t^2)\right|=\left|\frac{|w_t^1|}{(20i+6t)(1+|w_t^1|)}-\frac{|w_t^2|}{(20i+6t)(1+|w_t^2|)}\right|$$

$$\le \frac{\left\| w^1 - w^2 \right\|}{(20i+6t)\left(1+\left|w_t^1\right|\right)\left(1+\left|w_t^2\right|\right)}$$

$$\le \frac{1}{20i+6t}\left\| w^1 - w^2 \right\|.$$

For continuous functions $l\left(t, w_t^1\right)$, $l\left(t, w_t^2\right):[0,2]\times C\left(\left[-\frac{1}{2},0\right],R\right)$, we have

$$\left| l\left(t, w_t^1\right) - l\left(t, w_t^2\right) \right| = \left| \frac{1}{16-t^2}\left(\frac{\left|w_t^1\right|}{3\left(1+\left|w_t^1\right|\right)} + \frac{1}{5} \right) - \frac{1}{16-t^2}\left(\frac{\left|w_t^2\right|}{3\left(1+\left|w_t^2\right|\right)} + \frac{1}{5} \right) \right|$$

$$\le \frac{\left\| w^1 - w^2 \right\|}{3\left(16-t^2\right)\left(1+\left|w_t^1\right|\right)\left(1+\left|w_t^2\right|\right)}$$

$$\le \frac{\left\| w^1 - w^2 \right\|}{3\left(16-t^2\right)}.$$

For any $(t, w_t) \in [0,2]\times C\left(\left[-\frac{1}{2},0\right],R\right)$, we have

$$\sum_{i=1}^{3} \frac{1}{20i} \frac{2^{\frac{i}{4}}}{\frac{i}{4}} + \frac{1}{36} \frac{2^{\frac{3}{2}}}{\frac{3}{2}\left(\frac{3}{2}-1\right)} < 0.596 < 1, i = 1,2,3.$$

Thus, by Theorem 3.2, the Equation (4.1) has a unique solution.

5. Conclusion

The conformable fractional derivative brings great convenience to the study of fractional functional differential equations due to its unique properties. This paper uses conformable derivative to study the fractional neutral integro-differential equations, and obtains the results of the existence of the solution and the sufficient conditions for the uniqueness of the solution.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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