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Partial Variable Stability for a Class of Nonlinear Systems with Time Delay

Ran Huo[1], Xiaoli Wang[2*]

[1]College of Science, Inner Mongolia Agricultural University, Huhhot, China
[2]College of Statistics and Mathematics, Inner Mongolia University of Finance and Economics, Huhhot, China
Email: huoran124@163.com, *ncdwangxiaoli@163.com

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Abstract

This article first gives a new class of integral inequalities. Then, as an application, the nonlinear neutral differential system with multiple delays is considered, and the trivial solution of the nonlinear neutral system with multiple delays is obtained. Uniform asymptotic Lipschitz stability. Obviously, the above system is a generalization of the traditional differential system. The purpose of this paper is to study the dual stability of neutral differential equations with delays, including equal asymptotically Lipschitz stability and uniformly asymptotic Lipschitz stability. The author uses the method of integral inequality to establish a double stability criterion. As a result, the local stability of differential equations is widely used in theory and practice, such as dynamic systems and control systems.

Keywords

Nonlinear Neutral Systems, Double Stability, Lipschitz Asymptotic Stability, Integral Inequality

1. Introduction

In 1892, Lyapunov, a Russian mathematician, mechanician and physicist, proposed the notion of the stability of motion. He gave the general research methods in his doctoral dissertation "The general problem of the stability of motion" [1], in which he established the foundation of the stability theory. When studying nonlinear systems, especially studying dynamic systems or control systems, we cannot study the stability of all variables because of the technology difficulties, the limitation of practical conditions, or it is not necessary to study all variables considering the actual need. As a result, studying the partial stability of differential equations becomes more important. In addition, the partial stability is widely used in science and technology. For instance the absolute stability of famous Lurie adjusting systems can be changed into a problem of partial stability. In a word, it is of practical significance to study the partial stability of differential equations.

Since Bellman created a class of integral inequalities in 1958, integral inequalities have been greatly developed. The main results are:

In 1960, Li Yuesheng gave the following inequality in [2]:

$$u(t) \le u_0 + \int_0^t g(s)u(s)\mathrm{d}s + \int_0^t f(s)u^\alpha(s)\mathrm{d}s$$

In 2005, Sligeng discussed the following inequality in [3]:

$$\begin{aligned} u_1(t) \le k_1 &+ \int_0^t h_1(s)u_1(s)\mathrm{d}s + \int_0^t h_2(s)u_2(s)\mathrm{e}^{\mu s}\mathrm{d}s \\ &+ \int_0^t \overline{h}_1(s)u_1^\alpha(s)\mathrm{e}^{-(\alpha-1)\mu s}\mathrm{d}s + \int_0^t \overline{h}_2(s)u_2^\alpha(s)\mathrm{e}^{\mu s}\mathrm{d}s \end{aligned}$$

$$\begin{aligned} u_2(t) \le k_2 &+ \int_0^t h_3(s)u_1(s)\mathrm{e}^{-\mu s}\mathrm{d}s + \int_0^t h_4(s)u_2(s)\mathrm{d}s \\ &+ \int_0^t \overline{h}_3(s)u_1^\alpha(s)\mathrm{e}^{-\alpha\mu s}\mathrm{d}s + \int_0^t \overline{h}_4(s)u_2^\alpha(s)\mathrm{d}s \end{aligned}$$

In 2009, the author discussed a new class of inequalities in [4].

Vorotnikov, V.I. [5] [6] considered the following system

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + B(t)z + Y(t,y,z) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = C(t)y + D(t)z + Z(t,y,z) \end{cases}$$

and studied the double stability as $\|y\| + \|z\| \to 0$ and $\dfrac{\|Y(t,y,z)\| + \|Z(t,y,z)\|}{\|y\| + \|z\|} \to 0$.

In 2002, Wang Feng used the differential inequality of delay in article [7] to study the following delay system:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + B(t)z + Y\big(t,y,z,y(t-\tau),z(t-\tau)\big) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = C(t)y + D(t)z + Z\big(t,y,z,y(t-\tau),z(t-\tau)\big) \end{cases}$$

In 2006, Siligeng used the integral inequality extended in [3] in [8] to discuss the double stability of the following system to some variables:

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + f_1(t,y,z) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = B(t)z + f_2(t,y,z) \end{cases}$$

In this paper the author consider a new class of the nonlinearly perturbed differential systems with time-delay

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = B(t)y + C(t)z + Y\left(s,y(s),z(s),\int_0^t h_1\big(s,y(s),z(s),y(s-\tau),\dot{z}(s-\tau)\big)\mathrm{d}s\right) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = D(t)y + E(t)z + Z\left(s,y(s),z(s),\int_0^t h_2\big(s,y(s),z(s),y(s-\tau),\dot{z}(s-\tau)\big)\mathrm{d}s\right) \end{cases}$$

It is obvious that the above system is a generalization of the systems in [5] [6] [7] [8].

The aim of this paper is to investigate the double stability of neutural differential equations, including Uniform stability and Uniform Lipschitz stability. The author uses the method of differential inequalities with time-delay and integral inequalities to establish double stability criteria.

2. Preliminaries

Consider the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(t, x(t), x(t-\tau), \dot{x}(t-\tau)\right) \tag{1}$$

where $x \in R^n$, $y = col\left(x_1, x_2, \cdots, x_m\right)$, $z = col\left(x_{m+1}, x_{m+2}, \cdots, x_n\right)$, $x = col\left(y, z\right)$, $f\left(t,0,0\right) \equiv 0$, τ is a nonnegative constant. Let $\phi(t)$ be a continuous function, for $\forall t \in E_{t_0} = \left[t_0 - \tau, t_0\right]$.

Definition 1 [9] [10] [11] The trivial solution of system (1) has uniform stability and exponential asymptotic stability with respect to y if, for $\forall \varepsilon > 0$, $\forall t_0 \in I$, $\exists \delta(\varepsilon) > 0$, and $\lambda > 0$, when $\|\phi\| < \delta$ (for $\forall t \in E_{t_0}$), such that

$$\left\|y\left(t;t_0,\phi\right)\right\| + \left\|\dot{y}\left(t;t_0,\phi\right)\right\| < \varepsilon \exp\left(-\lambda\left(t-t_0\right)\right), \forall t \ge t_0.$$

Definition 2 [9] [10] [11] The trivial solution of system (1) has Lipschitz stability with respect to y if, there exists constants $M\left(t_0\right) > 0$ and $\delta\left(t_0\right) > 0$, when $\|\varphi\| + \|\dot{\varphi}\| < \delta$ (for $\forall t \in E_{t_0}$), such that

$$\left\|y\left(t;t_0,\phi\right)\right\| + \left\|\dot{y}\left(t;t_0,\phi\right)\right\| \le M\left(t_0\right)\left(\|\phi\| + \|\dot{\phi}\|\right), \forall t \ge t_0 \ge 0.$$

Definition 3 [9] [10] [11] The trivial solution of system (1) has equi-exponential Lipschitz asymptotic stability with respect to y if, there exists $\lambda > 0$, $K\left(t_0\right) > 0$ and $\delta\left(t_0\right) > 0$, when $\|\varphi\| + \|\dot{\varphi}\| < \delta$ (for $\forall t \in E_{t_0}$), such that

$$\left\|y\left(t;t_0,\phi\right)\right\| + \left\|\dot{y}\left(t;t_0,\phi\right)\right\| \le K\left(t_0\right)\left(\|\phi\| + \|\dot{\phi}\|\right)\exp\left(-\lambda\left(t-t_0\right)\right), \forall t \ge t_0 \ge 0.$$

Definition 4 [9] [10] [11] The trivial solution of system (1) has uniform exponential Lipschitz asymptotic stability with respect to y if, K and $\delta > 0$ in definition 3 are ndependent of t_0.

Lemma 1 [4] The following conditions are established on $t \ge t_0$:

i) k_1, k_2, μ are non-negative constants;

ii) 1)

$$\begin{aligned} u_1(t) \le{} & k_1 + \int_{t_0}^{t} a(s) u_1(s)\mathrm{d}s + \int_{t_0}^{t} b(s) u_2(s) \mathrm{e}^{\mu(s-t_0)}\mathrm{d}s \\ & + \int_{t_0}^{t} \sum_{i=1}^{I} c_i(s) u_1^{\alpha_i+1}(s) \mathrm{e}^{-\alpha_i\mu(s-t_0)}\mathrm{d}s + \int_{t_0}^{t} \sum_{i=1}^{I} d_i(s) u_2^{\alpha_i+1}(s) \mathrm{e}^{\mu(s-t_0)}\mathrm{d}s \\ & + \int_{t_0}^{t} \sum_{j=1}^{J} e_j(s) \int_{t_0}^{s} f_j(\tau) u_1(\tau)\mathrm{d}\tau\mathrm{d}s \\ & + \int_{t_0}^{t} \sum_{j=1}^{J} g_j(s)\left[\int_{t_0}^{s} h_j(\tau) u_2(\tau) \mathrm{e}^{\mu(\tau-t_0)}\mathrm{d}\tau\right]\mathrm{d}s \\ & + \int_{t_0}^{t} \sum_{k=1}^{K} o_k(s) \int_{t_0}^{s} p_k(\tau) u_1^{\beta_k+1}(\tau) \mathrm{e}^{-\beta_k\mu(\tau-t_0)}\mathrm{d}\tau\mathrm{d}s \end{aligned}$$

$$+\int_{t_0}^{t}\sum_{k=1}^{K}q_k(s)\int_{t_0}^{s}r_k(\tau)u_2^{\beta_k+1}(\tau)e^{\mu(\tau-t_0)}d\tau ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}c_i(s)u_1^{\alpha_i}(s)e^{-\alpha_i\mu(s-t_0)}\int_{t_0}^{s}\sum_{l=1}^{L}v_l(\theta)\int_{t_0}^{\theta}w_l(\tau)u_1(\tau)d\tau d\theta ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}d_i(s)u_2^{\alpha_i}(s)\int_{t_0}^{s}\sum_{l=1}^{L}x_l(\theta)\int_{t_0}^{\theta}y_l(\tau)u_2(\tau)e^{\mu(\tau-t_0)}d\tau d\theta ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}c_i(s)u_1^{\alpha_i}(s)e^{-\alpha_i\mu(s-t_0)}\int_{t_0}^{s}\sum_{m=1}^{M}A_m(\theta)\int_{t_0}^{\theta}B_m(\tau)u_1^{\gamma_m+1}(\tau)e^{-\gamma_m\mu(\tau-t_0)}d\tau d\theta ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}d_i(s)u_2^{\alpha_i}(s)\int_{t_0}^{s}\sum_{m=1}^{M}D_m(\theta)\int_{t_0}^{\theta}E_m(\tau)u_2^{\gamma_m+1}(\tau)e^{\mu(\tau-t_0)}d\tau d\theta ds$$

2)

$$u_2(t)\le k_2+\int_{t_0}^{t}\bar{a}(s)u_1(s)e^{-\mu(s-t_0)}ds+\int_{t_0}^{t}\bar{b}(s)u_2(s)ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}\bar{c}_i(s)u_1^{\alpha_i+1}(s)e^{-(\alpha_i+1)\mu(s-t_0)}ds+\int_{t_0}^{t}\sum_{i=1}^{I}\bar{d}_i(s)u_2^{\alpha_i+1}(s)ds$$

$$+\int_{t_0}^{t}\sum_{j=1}^{J}\bar{e}_j(s)\int_{t_0}^{s}\bar{f}_j(\tau)u_1(\tau)e^{-\mu(\tau-t_0)}d\tau ds$$

$$+\int_{t_0}^{t}\sum_{j=1}^{J}\bar{g}_j(s)\left[\int_{t_0}^{s}\bar{h}_j(\tau)u_2(\tau)d\tau\right]ds$$

$$+\int_{t_0}^{t}\sum_{k=1}^{K}\bar{o}_k(s)\int_{t_0}^{s}\bar{p}_k(\tau)u_1^{\beta_k+1}(\tau)e^{-(\beta_k+1)\mu(\tau-t_0)}d\tau ds$$

$$+\int_{t_0}^{t}\sum_{k=1}^{K}\bar{q}_k(s)\int_{t_0}^{s}\bar{r}_k(\tau)u_2^{\beta_k+1}(\tau)d\tau ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}\bar{c}_i(s)u_1^{\alpha_i}(s)e^{-\alpha_i\mu(s-t_0)}\int_{t_0}^{s}\sum_{l=1}^{L}\bar{v}_l(\theta)\int_{t_0}^{\theta}\bar{w}_l(\tau)u_1(\tau)e^{-\mu(\tau-t_0)}d\tau d\theta ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}\bar{d}_i(s)u_2^{\alpha_i}(s)\int_{t_0}^{s}\sum_{l=1}^{L}\bar{x}_l(\theta)\int_{t_0}^{\theta}\bar{y}_l(\tau)u_2(\tau)d\tau d\theta ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}\bar{c}_i(s)u_1^{\alpha_i}(s)e^{-\alpha_i\mu(s-t_0)}\int_{t_0}^{s}\sum_{m=1}^{M}\bar{A}_m(\theta)\int_{t_0}^{\theta}\bar{B}_m(\tau)u_1^{\gamma_m+1}(\tau)e^{-(\gamma_m+1)\mu(\tau-t_0)}d\tau d\theta ds$$

$$+\int_{t_0}^{t}\sum_{i=1}^{I}\bar{d}_i(s)u_2^{\alpha_i}(s)\int_{t_0}^{s}\sum_{m=1}^{M}\bar{D}_m(\theta)\int_{t_0}^{\theta}\bar{E}_m(\tau)u_2^{\gamma_m+1}(\tau)d\tau d\theta ds$$

where: $u_1(t),u_2(t),a(t),\bar{a}(t),b(t),\bar{b}(t),c_i(t),\bar{c}_i(t),d_i(t),\bar{d}_i(t)$ $(i=1,2,\cdots,I)$, $e_j(t),\bar{e}_j(t),f_j(t),\bar{f}_j(t),g_j(t),\bar{g}_j(t),h_j(t),\bar{h}_j(t)$ $(j=1,2,\cdots,J)$, $o_k(t),\bar{o}_k(t),p_k(t),\bar{p}_k(t),q_k(t),\bar{q}_k(t),r_k(t),\bar{r}_k(t)$ $(k=1,2,\cdots,K)$, $v_l(t),\bar{v}_l(t),w_l(t),\bar{w}_l(t),x_l(t),\bar{x}_l(t),y_l(t),\bar{y}_l(t)$ $(l=1,2,\cdots,L)$, $A_m(t),\bar{A}_m(t),B_m(t),\bar{B}_m(t),D_m(t),\bar{D}_m(t),E_m(t),\bar{E}_m(t)$ $(m=1,2,\cdots,M)$ are non-negative continuous function on R_+, and: α_i $(i=1,2,\cdots,I)$, β_k $(k=1,2,\cdots,K)$, γ_m $(m=1,2,\cdots,M)$ are all constants greater than 1, and $1\le\alpha_1\le\alpha_2\le\cdots\le\alpha_I$, $1\le\beta_1\le\beta_2\le\cdots\le\beta_K$, $1\le\gamma_1\le\gamma_2\le\cdots\le\gamma_M$

set: $\bar{\alpha}=\max(\alpha_I,\beta_K,\gamma_M)$, $\underline{\alpha}=\min(\alpha_1,\beta_1,\gamma_1)$

iii) let: $k=k_1+k_2$

$$F(t)=\max\{a(t)+\bar{a}(t),b(t)+\bar{b}(t)\},\quad G_i(t)=\max\{c_i(t)+\bar{c}_i(t),d_i(t)+\bar{d}_i(t)\},$$

$$\bar{\bar{e}}_j(t)=\max\{e_j(t),\bar{e}_j(t)\},\quad \bar{\bar{g}}_j(t)=\max\{g_j(t),\bar{g}_j(t)\},$$

$$H_j(t) = \max\left\{\bar{\bar{e}}_j(t), \bar{\bar{g}}_j(t)\right\}, \quad N_j(t) = \max\left\{f_j(t) + \bar{f}_j(t), h_j(t) + \bar{h}_j(t)\right\},$$

$$\bar{\bar{o}}_k(t) = \max\left\{o_k(t), \bar{o}_k(t)\right\}, \quad \bar{\bar{q}}_k(t) = \max\left\{q_k(t), \bar{q}_k(t)\right\},$$

$$Q_k(t) = \max\left\{\bar{\bar{o}}_k(t), \bar{\bar{q}}_k(t)\right\}, \quad R_k(t) = \max\left\{p_k(t) + \bar{p}_k(t), r_k(t) + \bar{r}_k(t)\right\},$$

$$\bar{\bar{v}}_l(t) = \max\left\{v_l(t), \bar{v}_l(t)\right\}, \quad \bar{\bar{x}}_l(t) = \max\left\{x_l(t), \bar{x}_l(t)\right\},$$

$$T_l(t) = \max\left\{\bar{\bar{v}}_l(t), \bar{\bar{x}}_l(t)\right\}, \quad W_l(t) = \max\left\{w_l(t) + \bar{w}_l(t), y_l(t) + \bar{y}_l(t)\right\},$$

$$\bar{\bar{A}}_m(t) = \max\left\{A_m(t), \bar{A}_m(t)\right\}, \quad \bar{\bar{D}}_m(t) = \max\left\{D_m(t), \bar{D}_m(t)\right\},$$

$$Y_m(t) = \max\left\{\bar{\bar{A}}_m(t), \bar{\bar{D}}_m(t)\right\}, \quad Z_m(t) = \max\left\{B_m(t) + \bar{B}_m(t), E_m(t) + \bar{E}_m(t)\right\},$$

iv) Let:

$$\Delta(t) = F(t) + \sum_{i=1}^{I} G_i(t) + \sum_{j=1}^{J} H_j(t) \int_{t_0}^{t} N_j(s)\mathrm{d}s + 2\sum_{k=1}^{K} Q_k(t) \int_{t_0}^{t} R_k(s)\mathrm{d}s$$
$$+ \sum_{i=1}^{I} G_i(t) \int_{t_0}^{t} \sum_{l=1}^{L} T_l(s) \int_{t_0}^{s} W_l(\sigma)\mathrm{d}\sigma\mathrm{d}s + \sum_{i=1}^{I} G_i(t) \int_{t_0}^{t} \sum_{m=1}^{M} Y_m(s) \int_{t_0}^{s} Z_m(\sigma)\mathrm{d}\sigma\mathrm{d}s$$

$$\Phi(t) = \sum_{i=1}^{I} G_i(t) + \sum_{k=1}^{K} Q_k(t) \int_{t_0}^{t} R_k(s)\mathrm{d}s + \sum_{i=1}^{I} G_i(t) \int_{t_0}^{t} \sum_{l=1}^{L} T_l(s) \int_{t_0}^{s} W_l(\sigma)\mathrm{d}\sigma\mathrm{d}s$$
$$+ 2\sum_{i=1}^{I} G_i(t) \int_{t_0}^{t} \sum_{m=1}^{M} Y_m(s) \int_{t_0}^{s} Z_m(\sigma)\mathrm{d}\sigma\mathrm{d}s$$

$$\Gamma(t) = \int_{t_0}^{t} \sum_{m=1}^{M} Y_m(s) \int_{t_0}^{s} Z_m(\sigma)\mathrm{d}\sigma\mathrm{d}s$$

$$\Lambda_1(t) = 1 - \bar{\alpha} c^{\bar{\alpha}} \int_{t_0}^{t} \left[\Phi(\tau) + \Gamma(\tau)\right] \exp\left(\bar{\alpha} \int_{t_0}^{\tau} \Delta(\sigma)\mathrm{d}\sigma\right)\mathrm{d}\tau$$

$$\Pi(t) = F(t) + \sum_{j=1}^{J} H_j(t) \int_{t_0}^{t} N_j(t)\mathrm{d}t$$

$$\Theta(t) = \sum_{i=1}^{I} G_i(t) \text{Щ}^{\alpha_i - \underline{\alpha}} + \sum_{k=1}^{K} Q_k(t) \int_{t_0}^{t} R_k(s) \text{Щ}^{\beta_k - \underline{\alpha}}\mathrm{d}s$$
$$+ \sum_{i=1}^{I} G_i(t) \text{Щ}^{\alpha_i - \underline{\alpha}} \int_{t_0}^{t} \sum_{l=1}^{L} T_l(s) \int_{t_0}^{s} W_l(\sigma)\mathrm{d}\sigma$$

$$\Sigma(t) = \sum_{m=1}^{M} Y_m(t) \int_{t_0}^{t} Z_m(s) \text{Щ}^{\gamma_m - \underline{\alpha}}\mathrm{d}s$$

$$\Lambda_2(t) = 1 - \underline{\alpha} c^{\underline{\alpha}} \int_{t_0}^{t} \left[\Theta(\tau) + \Sigma(\tau)\right] \exp\left(\underline{\alpha} \int_{t_0}^{\tau} \Pi(\sigma)\mathrm{d}\sigma\right)\mathrm{d}\tau$$

And $\int_{t_0}^{+\infty} \Delta(s)\mathrm{d}s < +\infty$,

$$\Lambda_1(t) > 0, \quad \left[1 - (\bar{\alpha} - 1) c^{\bar{\alpha}} \int_{t_0}^{t} \Phi(s) \Lambda_1^{-\frac{1}{\bar{\alpha}}}(s) \exp\left(\bar{\alpha} \int_{t_0}^{s} \Delta(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right] > 0$$

$$\Lambda_2(t) > 0, \quad \left[1 - (\underline{\alpha} - 1) c^{\underline{\alpha}} \int_{t_0}^{t} \Theta(s) \Lambda_2^{-\frac{1}{\underline{\alpha}}}(s) \exp\left(\underline{\alpha} \int_{t_0}^{s} \Sigma(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right] > 0$$

v) Assume:

$$\Omega(t)\le k\exp\left(\int_{t_0}^{t}\Pi(s)\mathrm{d}s\right)\cdot\left[1-(\underline{\alpha}-1)k^{\underline{\alpha}}\int_{t_0}^{t}\Theta(s)\Lambda_2^{-\frac{1}{\underline{\alpha}}}(s)\exp\left(\underline{\alpha}\int_{t_0}^{s}\Pi(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]^{-\frac{1}{\underline{\alpha}-1}}$$

then: $u_1(t)\le\Omega(t)\mathrm{e}^{\mu(t-t_0)}$, $u_2(t)\le\Omega(t)$.

3. Main Results

Consider the following system

$$\begin{cases}\dfrac{\mathrm{d}y}{\mathrm{d}t}=B(t)y+C(t)z\\ \qquad +Y\left(s,y(s),z(s),\int_0^t h_1\left(s,y(s),z(s),y(s-\tau),\dot{z}(s-\tau)\right)\mathrm{d}s\right)\\ \dfrac{\mathrm{d}z}{\mathrm{d}t}=D(t)y+E(t)z\\ \qquad +Z\left(s,y(s),z(s),\int_0^t h_2\left(s,y(s),z(s),y(s-\tau),\dot{z}(s-\tau)\right)\mathrm{d}s\right)\end{cases}\tag{2}$$

where $\tau\ge 0$ is a constant, initial condition is:

$$x(t)=\phi(t),\ \dot{x}(t)=\dot{\phi}(t),\ t_0-\tau\le t\le t_0,$$

$B(t)$ is an $m\times m$ matrix,
$Y\left(s,y(s),z(s),\int_0^t h_1\left(s,y(s),z(s),y(s-\tau),\dot{z}(s-\tau)\right)\mathrm{d}s\right)$ is an $m\times 1$ matrix,
$Z\left(s,y(s),z(s),\int_0^t h_2\left(s,y(s),z(s),y(s-\tau),\dot{z}(s-\tau)\right)\mathrm{d}s\right)$ is an $(n-m)\times 1$ matrix, they are all continuous for $t\in I$ and satisfy the condition of existence and uniqueness theorem.

Set $Y(t,s)$ and $Z(t,s)$ satisfied:

$$\begin{cases}\dfrac{\partial Y(t,s)}{\partial t}=B(t)Y(t,s)\\ Y(s,s)=I\end{cases},$$

$$\begin{cases}\dfrac{\partial Z(t,s)}{\partial t}=E(t)Z(t,s)\\ Z(s,s)=I\end{cases}$$

Theorem If (2) satisfies the following conditions:

i) $\|Y(t,s)\|\le m_1\mathrm{e}^{-\lambda(t-s)}$, $\|Z(t,s)\|\le m_2$;

ii)

$$\begin{aligned}&\left\|Y\left(t,y,z,y(t-\tau),z(t-\tau),\int_0^t h_1\left(s,y,z,y(s-\tau),\dot{z}(s-\tau)\right)\mathrm{d}s\right)\right\|\\ &\le a(t)\left(\|y(t-\delta(t))\|+\|\dot{y}(t-\delta(t))\|\right)\\ &+b(t)\left(\|z(t-\delta(t))\|+\|\dot{z}(t-\delta(t))\|\right)\mathrm{e}^{-\varepsilon(t-t_0)}\\ &+\sum_{i=1}^{I}c_i(t)\left(\|y(t-\delta(t))\|^{\alpha_i+1}+\|\dot{y}(t-\delta(t))\|^{\alpha_i+1}\right)\mathrm{e}^{\alpha_i\varepsilon(t-t_0)}\\ &+\sum_{i=1}^{I}d_i(t)\left(\|z(t-\delta(t))\|^{\alpha_i+1}+\|\dot{z}(t-\delta(t))\|^{\alpha_i+1}\right)\mathrm{e}^{-\varepsilon(t-t_0)}\end{aligned}$$

$$+\sum_{j=1}^{J}e_j(t)\int_{t_0}^{t}f_j(s)\left(\left\|y(s-\delta(s))\right\|+\left\|\dot{y}(s-\delta(s))\right\|\right)\mathrm{d}s$$

$$+\sum_{j=1}^{J}g_j(t)\int_{t_0}^{t}h_j(s)\left(\left\|z(s-\delta(s))\right\|+\left\|\dot{z}(s-\delta(s))\right\|\right)\mathrm{e}^{-\varepsilon(s-t_0)}\mathrm{d}s$$

$$+\sum_{k=1}^{K}o_k(t)\int_{t_0}^{t}p_k(s)\left(\left\|y(s-\delta(s))\right\|^{\beta_k+1}+\left\|\dot{y}(s-\delta(s))\right\|^{\beta_k+1}\right)\mathrm{e}^{\beta_k\varepsilon(s-t_0)}\mathrm{d}s$$

$$+\sum_{k=1}^{K}q_k(t)\int_{t_0}^{t}r_k(s)\left(\left\|z(s-\delta(s))\right\|^{\beta_k+1}+\left\|\dot{z}(s-\delta(s))\right\|^{\beta_k+1}\right)\mathrm{e}^{-\varepsilon(s-t_1)}\mathrm{d}s$$

$$+\sum_{i=1}^{I}c_i(t)\left(\left\|y(t-\delta(t))\right\|^{\alpha_i}+\left\|\dot{y}(t-\delta(t))\right\|^{\alpha_i}\right)\mathrm{e}^{\alpha_i\varepsilon(t-t_0)}$$

$$\cdot\int_{t_0}^{t}\sum_{l=1}^{L}v_l(s)\int_{t_0}^{s}w_l(\theta)\left(\left\|y(\theta-\delta(\theta))\right\|+\left\|\dot{y}(\theta-\delta(\theta))\right\|\right)\mathrm{d}\theta\mathrm{d}s$$

$$+\sum_{i=1}^{I}d_i(t)\left(\left\|z(t-\delta(t))\right\|^{\alpha_i}+\left\|\dot{z}(t-\delta(t))\right\|^{\alpha_i}\right)$$

$$\cdot\int_{t_0}^{t}\sum_{l=1}^{L}x_l(s)\int_{t_0}^{s}y_l(\theta)\left(\left\|z(\theta-\delta(\theta))\right\|+\left\|\dot{z}(\theta-\delta(\theta))\right\|\right)\mathrm{e}^{-\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$

$$+\sum_{i=1}^{I}c_i(t)\left(\left\|y(t-\delta(t))\right\|^{\alpha_i}+\left\|\dot{y}(t-\delta(t))\right\|^{\alpha_i}\right)\mathrm{e}^{\alpha_i\varepsilon(t-t_0)}$$

$$\cdot\int_{t_0}^{t}\sum_{m=1}^{M}A_m(s)\int_{t_0}^{s}B_m(\theta)\left(\left\|y(\theta-\delta(\theta))\right\|^{\gamma_m+1}+\left\|\dot{y}(\theta-\delta(\theta))\right\|^{\gamma_m+1}\right)\mathrm{e}^{\gamma_m\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$

$$+\sum_{i=1}^{I}d_i(t)\left(\left\|z(t-\delta(t))\right\|^{\alpha_i}+\left\|\dot{z}(t-\delta(t))\right\|^{\alpha_i}\right)$$

$$\cdot\int_{t_0}^{t}\sum_{m=1}^{M}D_m(s)\int_{t_0}^{s}E_m(\theta)\left(\left\|z(\theta-\delta(\theta))\right\|^{\gamma_m+1}+\left\|\dot{z}(\theta-\delta(\theta))\right\|^{\gamma_m+1}\right)\mathrm{e}^{-\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$

$$\left\|Z\left(t,y,z,y(t-\tau),z(t-\tau),\int_0^t h_2(s,y,z,y(s-\tau),\dot{z}(s-\tau))\mathrm{d}s\right)\right\|$$

$$\le\bar{a}(t)\left(\left\|y(t-\delta(t))\right\|+\left\|\dot{y}(t-\delta(t))\right\|\right)\mathrm{e}^{\varepsilon(t-t_0)}$$

iv) $$+\bar{b}(t)\left(\left\|z(t-\delta(t))\right\|+\left\|\dot{z}(t-\delta(t))\right\|\right)$$

$$+\sum_{i=1}^{I}\bar{c}_i(t)\left(\left\|y(t-\delta(t))\right\|^{\alpha_i+1}+\left\|\dot{y}(t-\delta(t))\right\|^{\alpha_i+1}\right)\mathrm{e}^{(\alpha_i+1)\varepsilon(t-t_0)}$$

$$+\sum_{i=1}^{I}\bar{d}_i(t)\left(\left\|z(t-\delta(t))\right\|^{\alpha_i+1}+\left\|\dot{z}(t-\delta(t))\right\|^{\alpha_i+1}\right)$$

$$+\sum_{j=1}^{J}\bar{e}_j(t)\int_{t_0}^{t}\bar{f}_j(s)\left(\left\|y(s-\delta(s))\right\|+\left\|\dot{y}(s-\delta(s))\right\|\right)\mathrm{e}^{\varepsilon(s-t_0)}\mathrm{d}s$$

$$+\sum_{j=1}^{J}\bar{g}_j(t)\int_{t_0}^{t}\bar{h}_j(s)\left(\left\|z(s-\delta(s))\right\|+\left\|\dot{z}(s-\delta(s))\right\|\right)\mathrm{d}s$$

$$+\sum_{k=1}^{K}\bar{o}_k(t)\int_{t_0}^{t}\bar{p}_k(s)\left(\left\|y(s-\delta(s))\right\|^{\beta_k+1}+\left\|\dot{y}(s-\delta(s))\right\|^{\beta_k+1}\right)\mathrm{e}^{(\beta_k+1)\varepsilon(s-t_0)}\mathrm{d}s$$

$$+\sum_{k=1}^{K}\bar{q}_k(t)\int_{t_0}^{t}\bar{r}_k(s)\left(\left\|z(s-\delta(s))\right\|^{\beta_k+1}+\left\|\dot{z}(s-\delta(s))\right\|^{\beta_k+1}\right)\mathrm{d}s$$

$$+\sum_{i=1}^{I}\overline{c}_i(t)\left(\left\|y\left(t-\delta(t)\right)\right\|^{\alpha_i}+\left\|\dot{y}\left(t-\delta(t)\right)\right\|^{\alpha_i}\right)\mathrm{e}^{\alpha_i\varepsilon(t-t_0)}$$
$$\cdot\int_{t_0}^{t}\sum_{l=1}^{L}v_l(s)\int_{t_0}^{s}w_l(\theta)\left(\left\|y\left(\theta-\delta(\theta)\right)\right\|+\left\|\dot{y}\left(\theta-\delta(\theta)\right)\right\|\right)\mathrm{e}^{\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$
$$+\sum_{i=1}^{I}\overline{d}_i(t)\left(\left\|z\left(t-\delta(t)\right)\right\|^{\alpha_i}+\left\|\dot{z}\left(t-\delta(t)\right)\right\|^{\alpha_i}\right)$$
$$\cdot\int_{t_0}^{t}\sum_{l=1}^{L}\overline{x}_l(s)\int_{t_0}^{s}\overline{y}_l(\theta)\left(\left\|z\left(\theta-\delta(\theta)\right)\right\|+\left\|\dot{z}\left(\theta-\delta(\theta)\right)\right\|\right)\mathrm{d}\theta\mathrm{d}s$$

$$+\sum_{i=1}^{I}\overline{c}_i(t)\left(\left\|y\left(t-\delta(t)\right)\right\|^{\alpha_i}+\left\|\dot{y}\left(t-\delta(t)\right)\right\|^{\alpha_i}\right)\mathrm{e}^{\alpha_i\varepsilon(t-t_0)}$$
$$\cdot\int_{t_0}^{t}\sum_{m=1}^{M}\overline{A}_m(s)\int_{t_0}^{s}\overline{B}_m(\theta)\left(\left\|y\left(\theta-\delta(\theta)\right)\right\|^{\gamma_m+1}+\left\|\dot{y}\left(\theta-\delta(\theta)\right)\right\|^{\gamma_m+1}\right)\mathrm{e}^{(\gamma_m+1)\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$
$$+\sum_{i=1}^{I}\overline{d}_i(t)\left(\left\|z\left(t-\delta(t)\right)\right\|^{\alpha_i}+\left\|\dot{z}\left(t-\delta(t)\right)\right\|^{\alpha_i}\right)$$
$$\cdot\int_{t_0}^{t}\sum_{m=1}^{M}\overline{D}_m(s)\int_{t_0}^{s}\overline{E}_m(\theta)\left(\left\|z\left(\theta-\delta(\theta)\right)\right\|^{\gamma_m+1}+\left\|\dot{z}\left(\theta-\delta(\theta)\right)\right\|^{\gamma_m+1}\right)\mathrm{d}\theta\mathrm{d}s$$

where: $a(t),\overline{a}(t),b(t),\overline{b}(t),c_i(t),\overline{c}_i(t),d_i(t),\overline{d}_i(t)\ (i=1,2,\cdots,I)$, $e_j(t),\overline{e}_j(t),f_j(t),\overline{f}_j(t),g_j(t),\overline{g}_j(t),h_j(t),\overline{h}_j(t)\ (j=1,2,\cdots,J)$, $o_k(t),\overline{o}_k(t),p_k(t),\overline{p}_k(t),q_k(t),\overline{q}_k(t),r_k(t),\overline{r}_k(t)\ (k=1,2,\cdots,K)$, $v_l(t),\overline{v}_l(t),w_l(t),\overline{w}_l(t),x_l(t),\overline{x}_l(t),y_l(t),\overline{y}_l(t)\ (l=1,2,\cdots,L)$, $A_m(t),\overline{A}_m(t),B_m(t),\overline{B}_m(t),D_m(t),\overline{D}_m(t),E_m(t),\overline{E}_m(t)\ (m=1,2,\cdots,M)$ are non-negative continuous monotonic non-increasing functions on R_+, and: $\alpha_i\ (i=1,2,\cdots,I)$, $\beta_k\ (k=1,2,\cdots,K)$, $\gamma_m\ (m=1,2,\cdots,M)$ are all constants greater than 1, $1\le\alpha_1\le\alpha_2\le\cdots\le\alpha_I$, $1\le\beta_1\le\beta_2\le\cdots\le\beta_K$, $1\le\gamma_1\le\gamma_2\le\cdots\le\gamma_M$ let: $\overline{\alpha}=\max(\alpha_I,\beta_K,\gamma_M)$, $\underline{\alpha}=\min(\alpha_1,\beta_1,\gamma_1)$

iv) Set:

$$\Delta(t)=F(t)+\sum_{i=1}^{I}G_i(t)+\sum_{j=1}^{J}H_j(t)\int_{t_0}^{t}N_j(s)\mathrm{d}s+2\sum_{k=1}^{K}Q_k(t)\int_{t_0}^{t}R_k(s)\mathrm{d}s$$
$$+\sum_{i=1}^{I}G_i(t)\int_{t_0}^{t}\sum_{l=1}^{L}T_l(s)\int_{t_0}^{s}W_l(\sigma)\mathrm{d}\sigma\mathrm{d}s+\sum_{i=1}^{I}G_i(t)\int_{t_0}^{t}\sum_{m=1}^{M}Y_m(s)\int_{t_0}^{s}Z_m(\sigma)\mathrm{d}\sigma\mathrm{d}s$$

$$\Phi(t)=\sum_{i=1}^{I}G_i(t)+\sum_{k=1}^{K}Q_k(t)\int_{t_0}^{t}R_k(s)\mathrm{d}s+\sum_{i=1}^{I}G_i(t)\int_{t_0}^{t}\sum_{l=1}^{L}T_l(s)\int_{t_0}^{s}W_l(\sigma)\mathrm{d}\sigma\mathrm{d}s$$
$$+2\sum_{i=1}^{I}G_i(t)\int_{t_0}^{t}\sum_{m=1}^{M}Y_m(s)\int_{t_0}^{s}Z_m(\sigma)\mathrm{d}\sigma\mathrm{d}s$$

$$\Gamma(t)=\int_{t_0}^{t}\sum_{m=1}^{M}Y_m(s)\int_{t_0}^{s}Z_m(\sigma)\mathrm{d}\sigma\mathrm{d}s$$

$$\Lambda_1(t)=1-\overline{\alpha}c^{\overline{\alpha}}\int_{t_0}^{t}\left[\Phi(\tau)+\Gamma(\tau)\right]\exp\left(\overline{\alpha}\int_{t_0}^{\tau}\Delta(\sigma)\mathrm{d}\sigma\right)\mathrm{d}\tau$$

$$\Pi(t)=F(t)+\sum_{j=1}^{J}H_j(t)\int_{t_0}^{t}N_j(t)\mathrm{d}t$$

$$\Theta(t)=\sum_{i=1}^{I}G_i(t)\text{Щ}^{\alpha_i-\underline{\alpha}}+\sum_{k=1}^{K}Q_k(t)\int_{t_0}^{t}R_k(s)\text{Щ}^{\beta_k-\underline{\alpha}}\mathrm{d}s$$
$$+\sum_{i=1}^{I}G_i(t)\text{Щ}^{\alpha_i-\underline{\alpha}}\int_{t_0}^{t}\sum_{l=1}^{L}T_l(s)\int_{t_0}^{s}W_l(\sigma)\mathrm{d}\sigma$$

$$\Sigma(t)=\sum_{m=1}^{M}Y_m(t)\int_{t_0}^{t}Z_m(s)Щ^{\gamma_m-\underline{\alpha}}\mathrm{d}s$$

$$\Lambda_2(t)=1-\underline{\alpha}c^{\underline{\alpha}}\int_{t_0}^{t}\left[\Theta(\tau)+\Sigma(\tau)\right]\exp\left(\underline{\alpha}\int_{t_0}^{\tau}\Pi(\sigma)\mathrm{d}\sigma\right)\mathrm{d}\tau$$

and $\int_{t_0}^{+\infty}\Delta(s)\mathrm{d}s<+\infty$,

$$\Lambda_1(t)>0,\ \left[1-(\bar{\alpha}-1)c^{\bar{\alpha}}\int_{t_0}^{t}\Phi(s)\Lambda_1^{-\frac{1}{\bar{\alpha}}}(s)\exp\left(\bar{\alpha}\int_{t_0}^{s}\Delta(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]>0$$

$$\Lambda_2(t)>0,\ \left[1-(\underline{\alpha}-1)c^{\underline{\alpha}}\int_{t_0}^{t}\Theta(s)\Lambda_2^{-\frac{1}{\underline{\alpha}}}(s)\exp\left(\underline{\alpha}\int_{t_0}^{s}\Sigma(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]>0$$

v) Set:

$$\Omega(t)=k\exp\left(\int_0^t\Pi(s)\mathrm{d}s\right)\cdot\left[1-(\underline{\alpha}-1)k^{\underline{\alpha}}\int_0^t\Theta(s)\Lambda_2^{-\frac{1}{\underline{\alpha}}}(s)\exp\left(\underline{\alpha}\int_0^s\Pi(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]^{-\frac{1}{\underline{\alpha}-1}}$$

then:

1) when $\lambda>\varepsilon$, the trivial solution of (2) is LS, GE_qELAS with respect to y;

2) when $\lambda=\varepsilon$, the trivial solution of (2) is, $GUELAS$ with respect to.

Proof Apply constant variation method to system (2), it can be deduced that:

$$y(t)=Y(t,t_0)y_0+\int_{t_0}^{t}Y(t,s)F_1\left(s,x(s-\delta(s)),\int_{t_0}^{s}h_1(\tau,x(\tau-\delta(\tau)))\mathrm{d}\tau\right)\mathrm{d}s \tag{3}$$

$$z(t)=Z(t,t_0)z_0+\int_{t_0}^{t}Z(t,s)F_2\left(s,x(s-\delta(s)),\int_{t_0}^{s}h_2(\tau,x(\tau-\delta(\tau)))\mathrm{d}\tau\right)\mathrm{d}s \tag{4}$$

By the condition of the theory, available from (3):

$$\begin{aligned}
\|y(t)\|\le{}& m_1\mathrm{e}^{-\lambda(t-t_0)}\|\varphi_1\|+\int_{t_0}^{t}m_1\mathrm{e}^{-\lambda(t-s)}\Big\{a(s)\|y(s-\delta(s))\|+b(s)\|z(s-\delta(s))\|\mathrm{e}^{-\varepsilon(s-t_0)}\\
&+\sum_{i=1}^{I}c_i(s)\|y(s-\delta(s))\|^{\alpha_i+1}\mathrm{e}^{\alpha_i\varepsilon(s-t_0)}+\sum_{i=1}^{I}d_i(s)\|z(s-\delta(s))\|^{\alpha_i+1}\mathrm{e}^{-\varepsilon(s-t_0)}\\
&+\sum_{j=1}^{J}e_j(s)\int_{t_0}^{s}f_j(\theta)\|y(\theta-\delta(\theta))\|\mathrm{d}\theta+\sum_{j=1}^{J}g_j(s)\int_{t_0}^{s}h_j(\theta)\|z(\theta-\delta(\theta))\|\mathrm{e}^{-\varepsilon(\theta-t_0)}\mathrm{d}\theta\\
&+\sum_{k=1}^{K}o_k(s)\int_{t_0}^{s}p_k(\theta)\|y(\theta-\delta(\theta))\|^{\beta_k+1}\mathrm{e}^{\beta_k\varepsilon(\theta-t_0)}\mathrm{d}\theta\\
&+\sum_{k=1}^{K}q_k(s)\int_{t_0}^{s}r_k(\theta)\|z(\theta-\delta(\theta))\|^{\beta_k+1}\mathrm{e}^{-\varepsilon(\theta-t_0)}\mathrm{d}\theta\\
&+\sum_{i=1}^{I}c_i(s)\|y(s-\delta(s))\|^{\alpha_i}\mathrm{e}^{\alpha_i\varepsilon(s-t_0)}\int_{t_0}^{s}\sum_{l=1}^{L}v_l(\sigma)\int_{t_0}^{\sigma}w_l(\theta)\|y(\theta-\delta(\theta))\|\mathrm{d}\theta\mathrm{d}\sigma\\
&+\sum_{i=1}^{I}d_i(s)\|z(s-\delta(s))\|^{\alpha_i}\int_{t_0}^{s}\sum_{l=1}^{L}x_l(\sigma)\int_{t_0}^{\sigma}y_l(\theta)\|z(\theta-\delta(\theta))\|\mathrm{e}^{-\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\\
&+\sum_{i=1}^{I}c_i(s)\|y(s-\delta(s))\|^{\alpha_i}\mathrm{e}^{\alpha_i\varepsilon(s-t_0)}\int_{t_0}^{s}\sum_{m=1}^{M}A_m(\sigma)\int_{t_0}^{\sigma}B_m(\theta)\|y(\theta-\delta(\theta))\|^{\gamma_m+1}\mathrm{e}^{\gamma_m\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}v\\
&+\sum_{i=1}^{I}d_i(s)\|z(s-\delta(s))\|^{\alpha_i}\int_{t_0}^{s}\sum_{m=1}^{M}D_m(\sigma)\int_{t_0}^{\sigma}E_m(\theta)\|z(\theta-\delta(\theta))\|^{\gamma_m+1}\mathrm{e}^{-\varepsilon(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\Big\}\mathrm{d}s
\end{aligned}\tag{5}$$

however, set

$$u_1(t)=\left(\|y(t)\|+\|\dot{y}(t)\|\right)e^{\lambda(t-t_0)},\quad u_2(t)=\|z(t)\|+\|\dot{z}(t)\|$$

$$\varphi_1^{(1)}=\sup_{-\delta\le t\le 0}\|\varphi_1(t)\|,\quad \varphi_2^{(1)}=\sup_{-\delta\le t\le 0}\|\varphi_1(t)\|^{\alpha_i+1}$$

$$\varphi_3^{(1)}=\sup_{-\delta\le t\le 0}\|\varphi_1(t)\|^{\beta_k+1},\quad \varphi_4^{(1)}=\sup_{-\delta\le t\le 0}\|\varphi_1(t)\|^{\gamma_m+1}$$

$$\varphi^{(1)}=\max\left\{\varphi_1^{(1)},\varphi_2^{(1)},\varphi_3^{(1)},\varphi_4^{(1)},\varphi_2^{(1)}\varphi_4^{(1)}\right\}$$

$$\varphi_1^{(2)}=\sup_{-\delta\le t\le 0}\|\varphi_2(t)\|,\quad \varphi_2^{(2)}=\sup_{-\delta\le t\le 0}\|\varphi_2(t)\|^{\alpha_i+1}$$

$$\varphi_3^{(2)}=\sup_{-\delta\le t\le 0}\|\varphi_2(t)\|^{\beta_k+1},\quad \varphi_4^{(2)}=\sup_{-\delta\le t\le 0}\|\varphi_2(t)\|^{\gamma_m+1}$$

$$\varphi^{(2)}=\max\left\{\varphi_1^{(2)},\varphi_2^{(2)},\varphi_3^{(2)},\varphi_4^{(2)},\varphi_2^{(2)}\varphi_4^{(2)}\right\}$$

Set:

$$\|u_1(t)\|=\begin{cases}\max\left\{\varphi_1^{(1)},\max\left(u(\xi)\right)\right\}, & 0\le\xi\le t\\ \varphi_1^{(1)}, & -\delta\le t\le 0\end{cases}$$

$$\|u_2(t)\|=\begin{cases}\max\left\{\varphi_1^{(2)},\max\left(u(\xi)\right)\right\}, & 0\le\xi\le t\\ \varphi_1^{(2)}, & -\delta\le t\le 0\end{cases}$$

Obviously, $\|u_1(t)\|$, $\|u_2(t)\|$ are monotonous, and defined by, $u_1(t)$, $u_2(t)$, we have

$$u_1(t-\delta(t))\le\varphi_1^{(1)},$$

$$u_2(t-\delta(t))\le\varphi_1^{(2)}$$

So substituting (5) into (2) gives:

$$\begin{aligned}
&u_1(t)\\
&\le\varphi^{(1)}+M_2\int_{t_0}^{t}a(s)\|u_1(s)\|ds+M_2\int_{t_0}^{t}b(s)\|u_2(s)\|e^{(\lambda-\varepsilon)(s-t_0)}ds\\
&+M_2\int_{t_0}^{t}\sum_{i=1}^{I}c_i(s)\|u_1(s)\|^{\alpha_i+1}e^{-\alpha_i(\lambda-\varepsilon)(s-t_0)}ds+M_2\int_{t_0}^{t}\sum_{i=1}^{I}d_i(s)\|u_2(s)\|^{\alpha_i+1}e^{(\lambda-\varepsilon)(\theta-t_0)}ds\\
&+M_2\int_{t_0}^{t}\sum_{j=1}^{J}e_j(s)\int_{t_0}^{s}f_j(\theta)\|u_1(\theta)\|d\theta ds\\
&+M_2\int_{t_0}^{t}\sum_{j=1}^{J}g_j(s)\int_{t_0}^{s}h_j(\theta)\|u_2(\theta)\|e^{(\lambda-\varepsilon)(\theta-t_0)}d\theta ds\\
&+M_2\int_{t_0}^{t}\sum_{k=1}^{K}o_k(s)\int_{t_0}^{s}p_k(\theta)\|u_1(\theta)\|^{\beta_k+1}e^{-\beta_k(\lambda-\varepsilon)(\theta-t_0)}d\theta ds
\end{aligned}$$

$$+M_2\int_{t_0}^{t}\sum_{k=1}^{K}q_k(s)\int_{t_0}^{s}r_k(\theta)\|u_2(\theta)\|^{\beta_k+1}\mathrm{e}^{(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}c_i(s)\|u_1(s)\|^{\alpha_i}\mathrm{e}^{-\alpha_i(\lambda-\varepsilon)(s-t_0)}\int_{t_0}^{s}\sum_{l=1}^{L}v_l(\sigma)\int_{t_0}^{\sigma}w_l(\theta)\|u_1(\theta)\|\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}d_i(s)\|u_2(s)\|^{\alpha_i}\int_{t_0}^{s}\sum_{l=1}^{L}x_l(\sigma)\int_{t_0}^{\sigma}y_l(\theta)\|u_2(\theta)\|\mathrm{e}^{(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}c_i(s)\|u_2(s)\|^{\alpha_i}\mathrm{e}^{-\alpha_i(\lambda-\varepsilon)(s-t_0)}\int_{t_0}^{s}\sum_{m=1}^{M}A_m(\sigma)\int_{t_0}^{\sigma}B_m(\theta)\|u_1(\theta)\|^{\gamma_m+1}\mathrm{e}^{-\gamma_m(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}d_i(s)\|u_2(s)\|^{\alpha_i}\int_{t_0}^{s}\sum_{m=1}^{M}D_m(\sigma)\int_{t_0}^{\sigma}E_m(\theta)\|u_2(\theta)\|^{\gamma_m+1}\mathrm{e}^{(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

Similarly available:

$$u_2(t)\le\varphi^{(2)}+M_2\int_{t_0}^{t}\bar{a}(s)\|u_1(s)\|\mathrm{e}^{-(\lambda-\varepsilon)(s-t_0)}\mathrm{d}s+M_2\int_{t_0}^{t}\bar{b}(s)\|u_2(s)\|\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}\bar{c}_i(s)\|u_1(s)\|^{\alpha_i+1}\mathrm{e}^{-(\alpha_i+1)(\lambda-\varepsilon)(s-t_0)}\mathrm{d}s+M_2\int_{t_0}^{t}\sum_{i=1}^{I}\bar{d}_i(s)\|u_2(s)\|^{\alpha_i+1}\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{j=1}^{J}\bar{e}_j(s)\int_{t_0}^{s}\bar{f}_j(\theta)\|u_1(\theta)\|\mathrm{e}^{-(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{j=1}^{J}\bar{g}_j(s)\int_{t_0}^{s}\bar{h}_j(\theta)\|u_2(\theta)\|\mathrm{d}\theta\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{k=1}^{K}\bar{o}_k(s)\int_{t_0}^{s}\bar{p}_k(\theta)\|u_1(\theta)\|^{\beta_k+1}\mathrm{e}^{-(\beta_k+1)(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{k=1}^{K}\bar{q}_k(s)\int_{t_0}^{s}\bar{r}_k(\theta)\|u_2(\theta)\|^{\beta_k+1}\mathrm{d}\theta\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}\bar{c}_i(s)\|u_1(s)\|^{\alpha_i}\mathrm{e}^{-\alpha_i(\lambda-\varepsilon)(s-t_0)}\int_{t_0}^{s}\sum_{l=1}^{L}\bar{v}_l(\sigma)\int_{t_0}^{\sigma}\bar{w}_l(\theta)\|u_1(\theta)\|\mathrm{e}^{-(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}\bar{d}_i(s)\|u_2(s)\|^{\alpha_i}\int_{t_0}^{s}\sum_{l=1}^{L}\bar{x}_l(\sigma)\int_{t_0}^{\sigma}\bar{y}_l(\theta)\|u_2(\theta)\|\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}\bar{c}_i(s)\|u_1(s)\|^{\alpha_i}\mathrm{e}^{-\alpha_i(\lambda-\varepsilon)(s-t_0)}\int_{t_0}^{s}\sum_{m=1}^{M}\bar{A}_m(\sigma)\int_{t_0}^{\sigma}\bar{B}_m(\theta)\|u_1(\theta)\|^{\gamma_m+1}\mathrm{e}^{-(\gamma_m+1)(\lambda-\varepsilon)(\theta-t_0)}\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

$$+M_2\int_{t_0}^{t}\sum_{i=1}^{I}\bar{d}_i(s)\|u_2(s)\|^{\alpha_i}\int_{t_0}^{s}\sum_{m=1}^{M}\bar{D}_m(\sigma)\int_{t_0}^{\sigma}\bar{E}_m(\theta)\|u_2(\theta)\|^{\gamma_m+1}\mathrm{d}\theta\mathrm{d}\sigma\mathrm{d}s$$

where $M_2=\max\left\{M_1,M_1^{\alpha_i+1},M_1^{\beta_k+1},M_1^{\gamma_m+1},M_1^{\alpha_i+\gamma_m+1}\right\}$.

So it can be obtained from Lemma: $u_1(t)\le k\mathrm{e}^{(\lambda-\varepsilon)(t-t_0)}\Omega(t)$, $u_2(t)\le k\Omega(t)$ here $k=\mathrm{e}^{\lambda t_0}\phi$, $\phi=\max\left(\varphi^{(1)},\varphi^{(2)}\right)$, $\Omega(t)$ As the lemma states, then:

$$\|y(t)\|+\|\dot{y}(t)\|\le M_3\varphi\mathrm{e}^{-\varepsilon(t-t_0)}\ (\lambda>\varepsilon),\quad \|y(t)\|+\|\dot{y}(t)\|\le M_4\varphi\mathrm{e}^{-\varepsilon(t-t_0)}\ (\lambda=\varepsilon)$$

$$\|z(t)\|+\|\dot{z}(t)\|\le M_5\varphi \tag{6}$$

here: $M_3=\mathrm{e}^{(\lambda-\varepsilon)t_0}\Omega(t)$; $M_4=\Omega(t)$; $M_5=\mathrm{e}^{\lambda t_0}\Omega(t)$.

Notice the theorem conditions, we have M_1 is a constant that has nothing to do with t_0, M_2 and M_3 are constants that has nothing to do with t_0.

Therefore, when $\lambda>\varepsilon$, (6) means the trivial solution of (2) is LS, GE_qELAS with respect to *y*; when $\lambda=\varepsilon$, (6) means the trivial solution of (2) is LS, $GUELAS$ with respect to *y*.

Note: The differential system discussed in this paper is the time-differential form of the ordinary differential system in [5] [6]. The time-differential system in [7] is generalized to a neutral system, and the Lipschitz stability in [8] is further extended to equi-exponential Lipschitz asymptotic stability and uniform exponential Lipschitz asymptotic stability and added global results

4. Conclusion

In this paper, we use the method of integral inequalities to establish double stability criteria. As a result, studying the partial stability of differential equations becomes more important. In addition, the partial stability of differential equations is widely used in science and technology.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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